OPEN SETS AND THE THEORY OF CONTENT

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1. *Introductory.*

In the present paper, I begin by enunciating and proving a number of theorems which will, I think, be found to be of considerable importance in the theory of open sets, more especially in connection with what we may popularly express as "the room they fill up" (*Raumerfüllung*).

I have purposely, at the risk of some apparent repetition, considered separately the case where the points fill up sets of intervals: the gain in clearness seems to me considerable. Moreover this special case is of very great importance in itself. Thus a small portion of Theorem 4 is Arzelà's *lemma fondamentale*, from which he deduces many of his most interesting results. The proofs so far given of this lemma are all due to Arzelà; the only one I have examined with care is that in his memoir entitled "Sulle Serie di Funzioni, Parte prima" (1899—Arzelà's second proof): this proof is, as I here* show, incomplete.

Since the presentation of the present paper, I see that M. Borel has pointed out the importance in the theory of functions of a theorem including Arzelà's lemma and included in Theorem 4 of the present paper. His note is printed in the *Comptes Rendus*, December, 1903; no proof is indicated, nor is any reference given either to the work of Arzelà, or to my paper on "Closed Sets of Points defined as the Limit of Closed Sets of Points," which contains the germ of my own work on the subject.

The analogy between sets of points and sets of intervals brought out in the earlier part of the present paper leads naturally to the consideration, in the latter part, of the difficult question of the content of open sets: this question is here discussed at some length. The theory to which I have independently been led coincides in some of its main features with that developed by M. Lebesgue† in his very important

* See below, p. 22.

† *Annali di Matematica* (1902). [My attention was first called to this memoir after the presentation of the present paper. In consequence I have added references to the work of M. Lebesgue, wherever it seemed desirable, and I have partially adopted his nomenclature. The only other alterations made since the presentation of the paper consist in the insertion of one or two additional theorems in the section which deals with the (outer) content.—*March 16th*, 1904.]

memoir entitled "Integrale, Longueur, Aire"; and my results confirm his. This is, perhaps, not without interest, as in one or two places M. Lebesgue's treatment is rather suggestive than detailed, and the assumption that the region of space considered is finite underlies not only his definitions, but also his proofs. Whereas, moreover, the work of M. Lebesgue on this subject consists of a discussion of the properties of the contents of "measurable sets" in combination with each other, I have in what follows, regarding the matter from a somewhat more general standpoint, investigated the relations of what I call "additive sets"—a class which includes all the sets actually shown by M. Lebesgue to be measurable—to sets in general.

It has not been shown that sets which are not measurable do not exist, and it is possible that such sets do exist. This has led M. Lebesgue to adopt the terms *mesure intérieure*, and *mesure extérieure*, though he only considers those sets for which these agree. Corresponding to these terms I use the expressions "(inner)" and "(outer)" content, and in its proper place I go shortly into the details of M. Lebesgue's theory.

The definitions given of the (inner) and (outer) content of an open set are found to simplify materially the statement of a number of the properties of open sets, and are indeed suggested by them. The question then arises whether the contents so defined obey the law of addition; whether, in fact, the sum of the contents, whether (inner) or (outer), of two non-overlapping sets is equal to the (inner) or (outer) content of their sum. This is found to be the case, provided at least one of the two components is closed, or belongs to a very extended class of open sets. I have not succeeded in proving the theorem (or disproving it) in its complete generality. We here knock up against that barrier of imperfect acquaintance with open sets which is responsible for the non-determination of the question whether or no sets of points exist whose potencies lie between that of the natural numbers and that of the continuum.

If, as is possible, the addition theorem is not true for all open sets, the extended class of additive sets for which it holds possesses a peculiar interest of its own. It appears from the results of the paper that the class forms a corpus; all known operations performed on members of the corpus lead to members of the corpus. From this point of view the paper may perhaps be regarded as making a contribution of some interest to the classification of open sets, and I have availed myself of the opportunity of stating and proving several theorems which bear on this question, and which are, I believe, new.

PART I.—SETS OF INTERVALS.

2. *Finite Sets of Intervals.*

THEOREM 1.—*Given a countably infinite series $D_1, D_2, \ldots$ of sets of intervals, each of which contains only a finite number of intervals such that each interval of D_{n+1} is contained in an interval of D_n (with possibly one or both end points common), there is at least one point common to an interval from each set; and the common points form a closed set.*—For, since the number of intervals in D_n is finite, the internal and end points form a closed set, and, by hypothesis, the closed set of points D_{n+1} is a component of the closed set D_n; hence, by Cantor's Theorem of Deduction,* the first part of the conclusion follows; the second statement is also the direct consequence of a well-known extension of that theorem.

THEOREM 2. — *If to the hypothesis of Theorem 1 we add that the content*† *of each D_n is greater than some positive quantity $\geqslant g$, the common points form a closed set of points D' of content $\geqslant g$, so that they have the potency c.* For, if possible, let the content be less than g, and let the difference be greater than e. Then we can enclose all the points in a finite set of intervals of content less than $(g-e)$. Out of the set D_n let us cut those parts which are common to D_n and the intervals just constructed: there remain over a finite number of intervals of content greater than e. The intervals so constructed for successive values of n satisfy the requirements of Theorem 1; so that there is at least one point common to them, and therefore to the original sets D_n, contrary to the assumption that all the common points had been cut out. The assumption was then inadmissible that the common points could be enclosed in a finite set of intervals of content less than g. Q. E. D.

3. *Infinite Sets of Intervals.*

If we remove the restriction that the number of intervals in D_n is finite, these conclusions are inadmissible, since the points of D_n do not then form a closed set.‡ The following simple example proves this.

Example 1.—Let D_{m+1} consist of all the abutting intervals between the points whose numbers in the binary scale are 1^n $(n \geqslant m)$ (Fig. 1).

* See Part II., Theorem 1.

† That is, the content of the equivalent set of non-overlapping intervals. *Proc. London Math. Soc.*, Vol. xxxv., p. 386, § 4.

‡ Cf. *Quarterly Journal of Pure and Applied Mathematics*, No. 138 (1903), p. 110.

Here the only limiting point of intervals one from each set is the point unity, and is external to the intervals of every set.

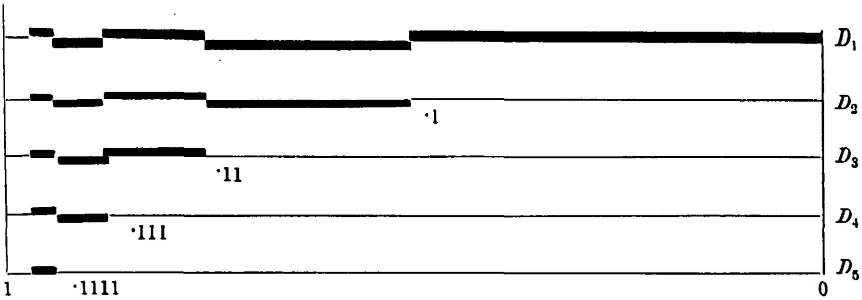

FIG. 1.

In this example the content of the sets decreases without limit; the theorem proved by me in a paper entitled "On Closed Sets of Points defined as the Limit of a Sequence of Closed Sets of Points" * shows that when the sets of intervals have no point common this must always be the case. As the application of Theorem 1 *supra* considerably simplifies the proof originally given of that theorem, I might have been tempted to repeat the theorem here. As a matter of fact, however, subsequent investigations † showed that the enunciation of that theorem might be further extended, and the result stated with greater precision, without complicating the proof. The following theorem is therefore substituted for it, and includes it as a special case of its first part (*a*):—

THEOREM 3.—*Given a countably infinite series* $D_1, D_2, \ldots$ *of sets of intervals such that* (1) *each interval of* D_{n+1} *is contained in an interval of* D_n *for every value of* n, *and* (2) *the content* I *of each set* D_n *is greater than some positive quantity* g, *then* (*a*) *there is a set of points such that each is internal to an interval of* D_n *for every value of* n, *and* (*b*) *it contains closed components of content* $> g-e$, *where* e *is as small as we please; so that the potency* ‡ *of these points is* c.

Let the non-overlapping intervals which have the same internal points as D_n be arranged in countable order, and denoted by $D_{n,r}$, for successive integral values of r.

Let us determine a finite number of the intervals $D_{1,r}$ such that the content of the remaining is less than $\frac{e}{2^{1+1}}$; and from each end of each

* *Proc. London Math. Soc.*, Vol. xxxv., p. 282.

† *Ibid.*, p. 284.

‡ Some other mathematicians have used the terms (1) "cardinal number," (2) "power," for this concept.

of these intervals (in finite number) $D_{1,r}$ let us cut off a fraction $\frac{1}{2^{1+3}}, \frac{e}{I_1}$ of its length. The sum of these pieces is less than $\frac{e}{2^{1+1}}$, and therefore the finite number of curtailed intervals, which we denote by D'_1, has content $> g-\frac{1}{2}e$.

The parts of intervals $D_{2,r}$ which lie inside D'_1 evidently have content $> I_2-\frac{1}{2}e$: choosing out a finite number of these so that the content of the remainder may be less than $\frac{e}{2^{2+1}}$, and curtailing them at each end by a fraction $\frac{1}{2^{2+3}}, \frac{e}{I_2}$ of their length, we get a finite set of intervals D'_2 of content $> g-\frac{e}{2}-\frac{e}{2^2}$ lying inside the intervals D'_1.

Proceeding thus with each successive set of intervals $D_{n,r}$, we obtain from it a finite set D'_n of content $> g-\frac{e}{2}-\frac{e}{2^2}-\ldots-\frac{e}{2^n}$, *a fortiori* $> g-e$, lying inside the finite set D'_{n-1}, for every value of n. Applying Theorem 2 to these sets, we deduce that they have in common a closed set of points of content $> g-e$. By construction these points are internal to the original intervals; which proves the theorem.

THEOREM 4.—*Given an infinite number of sets of intervals, in a finite segment* (A, B) *of length* L, *such that the content of each set of intervals is greater than some positive quantity* g, *then a set of points of potency* c *exists, which is internal to an infinite series of these sets of intervals, and contains closed components of content* $> g-e$, *where* e *is as small as we please.*

For consider the non-overlapping intervals having the same internal points as any one of the sets D_1. Their content $> g$, and therefore we can choose out a finite number of them whose content is greater than g. Suppose this done for all the sets: then in each set we have only a finite number of non-overlapping intervals.

Let the integer m be determined so that

$$mg \leqslant L < (m+1)g. \tag{1}$$

Let us consider a group of n of the sets, where n is a sufficiently large integer, later to be more particularly specified.

The parts of (A, B), if any, which are covered by these n sets doubly form a finite number of intervals, possibly overlapping, whose content we denote by $I_{1,n}$. The parts which are simply covered, therefore, form a finite set of non-overlapping intervals of content $> n(g-I_{1,n})$, whence

$$n(g-I_{1,n}) < (m+1)g;$$

therefore $$I_{1,n} > \{1-(m+1)/n\}g. \tag{2}$$

Let us choose an integer n' so that $(m+1)/n' < \frac{1}{2}e$, that is,

$$n' > 2(m+1)/e; \tag{3}$$

then $$I_{1,n'} > (1-\tfrac{1}{2}e)g. \tag{4}$$

Grouping the given sets together in distinct groups of n', and taking the corresponding sets of double non-overlapping intervals, we have conditions exactly similar to those with which we started, only that, instead of g, we have $(1-\frac{1}{2}e)g$.

To these new sets we apply the same reasoning as before, taking, however, $\frac{1}{2}e$ instead of e, and substituting for n an integer n'' such that $n'' > 4(m+1)/e$ and grouping the sets of double intervals in groups of n''. The content of the double parts corresponding to any such group being denoted by $I_{2,n''}$, it follows that

$$I_{2,n''} > (1-\tfrac{1}{2}e)(1-\tfrac{1}{4}e)g > (1-\tfrac{1}{2}e-\tfrac{1}{4}e)g > (1-e)g. \tag{5}$$

There will, therefore, certainly be such parts for every one of the groups, and they will, by the construction, be at least quadruply covered by the original sets.

In this way we can always proceed a stage further: the sets of intervals which we construct at each successive stage always have content $> (1-e)g$. Returning to the equation (2), we see that, since the whole set of intervals* in (A, B) which are covered at least doubly by the given sets has a content I_1' greater than or equal to $I_{1,n}$ for all values of n, $I_1' \geqslant g$. Similarly, denoting by I_2' the content of the set of intervals in (A, B) which are covered at least quadruply by the given intervals, $I_2' \geqslant g$; and, generally, $I_n' \geqslant g$, where I_n' is the content of the set of intervals which are covered by at least 2^n of the given sets.

Now, since the intervals corresponding to the content I_n' certainly lie inside those of content I_{n-1}', we can apply to this series Theorem 3, since the content of each is certainly greater than $g-e$, which proves the theorem.

* There might, of course, be points of (A, B) external to these intervals which belong to a finite or infinite number of given sets, but they do not affect the argument.

This theorem includes Arzelà's *lemma fondamentale*, the enunciation of which is as follows:—

Sia y_0 un punto limite per un gruppo qualsiasi di numeri (y); e indichi $G_0 = (y_0, y_2, \ldots)$ una successione, comunque scelta, di numeri (y) tendenti al limite y_0. Assumendo le variabili come coordinate ortogonali di un punto nel piano, si consideri il gruppo delle rette $y = y_1$, $y = y_2, \ldots$; nell' intervallo $a \ldots b$ sopra ciascuna si segnino dei tratticelli distinti l'uno dall' altro, in numero finito che può variare da retta a retta e anche crescere indefinitamente via via che y_s si approssima a y_0. La somma dei tratticelli $\delta_{1,s}, \delta_{2,s}, \ldots, \delta_{n,s}$ segnati sulla $y = y_s$ sia d_s. *Se per ogni valore* $s = 1, 2, \ldots$ *si ha sempre* $d > g$; g *numero determinato positivo, necessariamente esiste tra* a, *e*, b *almeno un punto* x_0 *tale che la retta* $x = x_0$ *incontra un numero infinito di tratti* δ.

In other words, assuming that the sets of intervals in the enunciation of Theorem 4 are finite (a restriction which is subsequently removed), Arzelà asserts that there is at least one point x_0 common to intervals of every set, *i.e.*, either an internal or end point of such intervals.

Arzelà's first proof, which dates from the year 1885,* and occupies four pages royal octavo, involves the consideration, *seriatim*, of a number of different possibilities. The complicated character of this first proof, and perhaps also the fundamental nature of the result, induced him to attempt to give an alternative proof of a simpler character in 1899.† The line of argument commences much in the same way as that employed by myself. By taking the sets of intervals in groups of $m+2$ he obtains the equation (2) for the particular value $n = m+2$, viz.,

$$I_{1,\,m+2} > g/(m+2).$$

Since, as he says, the process can now be repeated indefinitely, Arzelà infers the existence of a sequence of sets of intervals, each set contained in the preceding set. He then asserts that a sequence of single intervals exists, one from each set, and each interval contained in the preceding interval of the sequence. As the sets of intervals are no longer finite, such an assumption would need proof even if it appeared that their contents had a positive lower limit. There is, however, nothing in Arzelà's argument to show that the contents do not diminish without limit, and the example on p. 19 of the present paper shows that, should this happen, the required conclusion would be illegitimate.

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* *Loc. cit.*

† "Sulle Serie di Funzioni," Parte prima, *R. Acc. d. Sc. di Bologna*, 1899.

PART II.—SETS OF POINTS.

5.

I now proceed to show how to replace the intervals of the preceding part of the paper by closed sets of points of positive content. Our sets of intervals, when infinite in number, will then become open sets of points. The theorems I am about to obtain will therefore contain the earlier ones as special cases.

That we can do this is very instructive, and suggests at once the possibility of extending the theory of content to open sets. From our present point of view, we may say that we have indeed already ascribed content to an open set of points, viz., the points of an infinite set of intervals.*

The definition already adopted of the content of a set of intervals is the most natural one, and is indeed the only one of any conceivable use; it would certainly not be reasonable to substitute for it the content of the set of points got by closing it, which may be the whole continuum even when the content of the intervals is as small as we please. There would appear therefore to be no sufficient reason for defining the content of an open set to be that of the set got by closing it. Moreover, in the important special case in which the open set is expressible as the limit of the sum of n closed sets when n is infinite, we are led to define its content as the limit of the content of the sum.

Whether such a definition is logically valid, and whether it agrees with our previous notions of the properties of content as gained from the study of closed sets, requires of course discussion, as also the further question whether it is possible to extend the definition so as to embrace other kinds of open sets. I shall return to the subject in the third part of the paper. The theorems about to be proved will then be required.

6.

LEMMA 1.—*If G_1 and G_2 be two closed sets of points having no point common, the set consisting of G_1 and G_2 together is a closed set of content equal to the sum of their contents.*

This follows at once from the fact that the points of G_2 must, in this case, be internal to a finite number of the black intervals† of G_1.

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* *Loc. cit.*

† That is, the intervals free of points of G_1, except that their end points belong to G_1.

LEMMA 2.—*If a closed set of content I contain a closed component of content J, it contains a closed component of content $I-J-\epsilon$ (where ϵ is as small as we please), having no point common with the former component.*

By the preceding lemma no closed component could have content greater than $I-J$.

Let e be any assumed small quantity, and let us shut up all the points of the given closed component in a finite number of intervals of content lying between J and $J+e$. The points of the given set which are not *internal* to these intervals form, as is easily seen, a closed set; if the content of this latter set were less than $I-J-e$, we could enclose all its points in a finite number of intervals of content less than $I-J-e$, which together with the intervals first described would form a set of a finite number of intervals of content less than I, enclosing all the points of a closed set of content I; which is impossible. Thus the content of the closed component in question is not less than $I-J-e$; which proves the lemma.

LEMMA 3.—*If G_1 and G_2 be two closed sets of points of content I_1 and I_2, (a) the set consisting of all the points common to G_1 and G_2 is a closed set, say G' of content I', and (b) the set consisting of all points belonging to one or both of G_1 and G_2 is a closed set, say G of content I. Further, (c) $I_1+I_2 = I+I'$.*

For (*a*), if P be a limiting point of G', it is a limiting point both of G_1 and G_2, and therefore a point of both, that is a point of G'; so that G' is closed.

(*b*) If P be a limiting point of G, it must be a limiting point of one at least of G_1 and G_2, and is therefore a point of that one, and therefore a point of G; so that G is closed.

(*c*) By Lemma 2, G consists of the closed set G_1 and a complementary component containing closed sets of content as near $I-I_1$ as we please, but not any whose content exceeds $I-I_1$. Since this complementary component is also the complementary component of G' with respect to G_2, by the same lemma, it contains closed sets of content as near as we please to I_2-I', but none whose content exceeds I_2-I'. Hence $I-I_1 = I_2-I'$, which is equivalent to the statement to be proved.

In the proofs of the above lemmas I have for convenience employed the term "interval"; a moment's consideration, however, shows us that nowhere has the assumption been made that the sets of points are linear. In other words, the lemmas are true for closed sets of points in space of any number of dimensions. It may be remarked at once that in space of more than one dimension there is no gain in simplicity in considering sets of regions: in such a space the conception of a set of points replaces with advantage that of a set of regions.

7.

I now proceed to give for sets of points the theorems analogous to those proved in Part I. for sets of intervals.

THEOREM 1′.—*Given a countably infinite series of closed sets of points,* $G_1, G_2, \ldots,$ *such that each point of* G_{n+1} *is also a point of* G_n, *there is at least one point common to all the sets, and the common points form a closed set.*

THEOREM 2′.—*If to the hypothesis of Theorem* 1′ *we add that the content of each* G_n *is greater than some positive quantity* g, *the common points form a closed set* G' *of content* $\geqslant g$; *so that they have the potency* c.

As already remarked, Theorem 1′ is a known theorem. The proof of Theorem 2′ is as follows:—

If possible, let the content be I', where I' is less than g. Denote by $I_1, I_2, \ldots$ the contents of $G_1, G_2, \ldots$. By Lemma 2 we can find a closed component of G_1, all of whose points are distinct from those of G', and whose content is $I_1 - I' - e$, where e is a positive quantity, smaller than some assigned quantity. This set has in common with G_2 a closed set, whose content, by Lemma 3, is equal to $I_1 - I' - e + I_2 - K$, where K is the content of the closed set constituted by G_2 and the closed component of G_1 above found, and is certainly less than I_1. The content of this component of G_2 is therefore greater than $I_2 - I' - e$; *a fortiori*, greater than $g - I' - e$.

In other words, we have found a component of G_2 which is closed and has no points in common with G', whose content is greater than $g - I' - e$.

We can therefore repeat the argument, and obtain in each succeeding set such a closed component, each component lying inside the one previously obtained. It follows then, by Theorem 1, that there are points other than G' common to all the given sets, contrary to the hypothesis. Therefore, &c. Q. E. D.

8. *Open Sets.*

THEOREM 3′.—*Given a countably infinite series* $G_1, G_2, \ldots$ *of sets of points such that the upper limit* I_n *of the content of closed components in* G_n *is greater than a positive quantity* g, *the same for all values of* n, *each set G_n being contained in the foregoing G_{n-1}, then a set of points exists of potency c, common to all the sets, and this set contains closed components of content greater than $g-e$, where e is as small as we please.*

By the definition of I_n, we can find a closed component G_1' of G_1 such that, its content being denoted by I_1', $I_1 - \frac{1}{2}e < I_1 \leqslant I_1$; and, for all values of n greater than 1, we can, in like manner, find a closed component G_n'' of G_n such that, its content being denoted by I_n'',

$$I_n - \frac{1}{2^n} e < I_n'' \leqslant I_n.$$

Those points of G_2'' which belong to G_1' form a closed set, whose content is greater than $I_2 - \frac{1}{2}e - \frac{1}{4}e$ [since the set consisting of all the points belonging to one or both of the sets G_1' and G_2'' is a component of G_1, so that its content is not greater than I_1, by Lemma 3; therefore, the content of the set common to G_1' and G_2'' is greater than

$$I_1 - \tfrac{1}{2}e + I_2 - \tfrac{1}{4}e - I_1 \quad \text{or} \quad I_2 - \tfrac{1}{2}e - \tfrac{1}{4}e].$$

Let us denote this closed component of G_2 by G_2'. *Then G_2' is contained in G_1', and has content greater than $g-e$.*

Similarly we can determine a closed component G_3' of G_3'' and G_2', of content greater than $g-e$; generally we determine successively closed components of each G_n'' and G_{n-1}', of content greater than $g-e$.

Applying Theorem 2′ to these sets G_n', the result follows.

THEOREM 4′.—*Given an infinite number of sets of points $G_1, G_2, \ldots$, components of a closed set of finite content* L, such that the upper limit of the contents of the closed components of G_n is greater than some positive quantity g, the same for all values of n, then an infinite series of these sets exists, having in common a set of points of potency c, the content of whose closed components has an upper limit $\geqslant g$.*

Let us choose out a closed component of each set of content greater than g, and let these be denoted by $G_1', G_2', \ldots$. Let e be any small positive quantity, and let the integer m be determined such that

$$mg \leqslant L < (m+1)\,g. \qquad (1)$$

* It will be seen that it is sufficient if L is the upper limit of the content of closed sets in the whole set, which does not need to be closed; this is brought out in the re-statement of this theorem as Theorem 7.

Let us consider a group of n of the closed sets G', where n is a sufficiently large integer, subsequently to be further specified.

The points common to any particular pair of the sets of the group form a closed set of points (Lemma 3); therefore, since the sum of any finite number of closed sets is a closed set, the points common to at least two of the sets of the group form a closed set: let us denote it by $G_{1,n}$, and its content by $I_{1,n}$.

The points of $G_{1,n}$ which belong to any particular set of the group G' form a closed component of $G_{1,n}$, whose content is therefore less than or equal to $I_{1,n}$; by Lemma 2, therefore, there is a closed set of content greater than $g-I_{1,n}$, consisting entirely of points belonging to no set of the group, except G'. Corresponding to each of the n sets G' we can find such a closed component, and they will have no common points; so that they form a closed set of content greater than $n(g-I_{1,n})$, by Lemma 3. Hence, by (1), $n(g-I_{1,n}) < (m+1)g$; and therefore

$$I_{1,n} > \{1-(m+1)/n\}\, g. \tag{2}$$

Thus the set $G_{1,n}$ certainly exists, and has the potency c, for all values of n greater than $m+1$.

Let us determine an integer n' such that $(m+1)/n' < \tfrac{1}{2}e$, that is, $n' > 2(m+1)/e$. Then

$$I_{1,n'} > (1-\tfrac{1}{2}e)\, g. \tag{3}$$

Grouping our sets G' together in distinct groups of n' sets, and taking the sets of points belonging to at least two sets of each in turn of these groups, say G_1'', G_2'', ..., we have the same conditions as before, only the content of each closed set is now greater than $(1-\tfrac{1}{2}e)\,g$, instead of g.

Repeating on these sets the process just gone through, we obtain sets of quadruple points of the original sets whose content $I_{2,n}$ satisfies the inequality

$$I_{2,n} > \{1-(m+1)/n\}(1-\tfrac{1}{2}e)\, g. \tag{4}$$

Thus sets $G_{2,n}$, consisting of points common to at least 2^2 of the given sets, certainly exist, and have the potency c, for all values of n greater than $m+1$.

As before (using $\tfrac{1}{2}e$ instead of e), we can then determine n'' so that

$$I_{2,n''} > (1-\tfrac{1}{4}e)(1-\tfrac{1}{2}e)\, g > (1-\tfrac{1}{2}e-\tfrac{1}{4}e)\, g > (1-e)\, g. \tag{5}$$

This process can be continued *ad infinitum*, and at each stage we see that *there are sets $G_{r,n}$ (consisting of points common to at least 2^r of the given sets), of potency c, and of content greater than $(1-e)\,g$, where e is as small as we please.*

Now the set, in general open, consisting of *all* the points belonging to at least 2^r of the given sets is certainly contained in the set consisting of all the points common to at least 2^{r-1} of the given sets, and, by the above, these sets satisfy the other condition of Theorem 3, ($g-e$ being substituted for g). Hence, by Theorem 3, the result follows.

PART III.—ON THE GENERAL THEORY OF CONTENT.

9.

We have seen that, in the case of an open set, the upper limit of the content of closed components plays a most important *rôle.* In the lemmas and theorems relating to open sets, enunciated and proved, this concept has to them precisely the relation that content itself has to closed sets. With Lebesgue, I shall call it *the inner measure of the content* or briefly *(inner) content of the open set.*

DEFINITION.—*The (inner) content of a set is defined to be the upper limit of the content of its closed components.*

The introduction of this term simplifies the statements of the lemmas and theorems of Part II.: thus Lemmas 1 and 2 can be replaced by the following simple proposition :—

THEOREM 5.—*If a closed set G be the sum of two non-overlapping sets, one at least of which is closed, the content of G is the sum of the (inner) contents of the components.*

Theorems 2′ and 3′ are replaced by the following :—

THEOREM 6.—*Given a countably infinite series of sets of points, whose (inner) contents have a positive lower limit g, such that each set is contained in the preceding set, then a set of points of potency c exists, common to all the sets, and the (inner) content of this set is g.*

Theorem 4′ is replaced by the following :—

THEOREM 7.—*Given an infinite number of sets of points, components of a set of finite (inner) content, the (inner) contents of these sets having a positive lower limit g, then an infinite number of these sets exists, having in common a set of (inner) content* $\geqslant g$.

10.

The (inner) content, so defined, is certainly a magnitude, and, in the case of a closed set, the (inner) content is the content itself. The question arises whether the (inner) content possesses all the properties which we are accustomed to associate with the term "content" as long as this term was confined to closed sets. First, we ask, *is the (inner) content of the sum of two non-overlapping sets always equal to the sum of their (inner) contents ?*

All that has been proved in the preceding sections is that this is the case provided the sum of the two sets as well as one of the components is closed. We can, however, at once extend the result to the case when the sum is open. In other words—*Even if the sum of two non-overlapping sets be open, its (inner) content is the sum of their (inner) contents, provided one at least of the components is closed.**

For, if the content of the closed component be a, and the (inner) content of the sum $a+b$, we can, by the definition, find a closed component of content $a+b-e$, where e is as small as we please. The part common to these two closed components must have content $\geqslant a-e$, and $\leqslant a$ [since, otherwise, the remaining component of the first closed component would have (inner) content $> e$, and we could therefore find in it a closed component having no point common with that of content $a+b-e$, and these two together would form a closed component of the whole set of content $> a+b$].

In the closed component of content $a+b-e$ there must then, by Theorem 5, be another distinct component of (inner) content $\geqslant b-e$ and $\leqslant b$. This being true for all values of e, it follows that the (inner) content of the original open component is not less than b. But it cannot be greater than b, since otherwise we could find a closed component which with the first given component would form a closed set of content greater than $a+b$. Thus the second component has (inner) content b; which proves the theorem.

Summing up the result so far, we have the following theorem :—

The (inner) content of the sum of two non-overlapping sets, one of which is closed, is the sum of their (inner) contents.

11.

Two cases remain :—The sum of two non-overlapping open sets is (1) closed ; or (2) open.

If we assume that what we may, for shortness, call the (inner) addition theorem holds in Case (1), it is easy to deduce that it holds in Case (2).

For, if I be the (inner) content of the sum, and a and b of the com-

* This theorem takes us at once beyond the range of Lebesgue's investigations.

ponents, we can find a closed component of the sum, of content $I-e$, and this cuts out of the components two (open) sets, whose contents, by the same argument as before, lie between a and $a-e$, and b and $b-e$, respectively. The sum of these two (open) sets, being closed, has, under the supposition that the (inner) addition theorem holds in Case (1), content lying between $a+b-2e$ and $a+b$, but must be equal to $I-e$. Since this is true for all values of e, the result follows.

12.

We are now left with the discussion of Case (1). By means of the theorems of the present paper, we can reduce the problem of determining whether in this case the (inner) addition theorem holds to the following:—*Can the sum of two open sets, each of (inner) content zero, be a closed set of positive content?*

To show this we proceed as follows:—

Suppose, if possible, we have a closed set of content $a+b+c$, and it can be divided into two open components, whose (inner) contents are b and c respectively. In these open components there exist closed components of content $b-e$ and $c-e$ respectively, where e is as small as we please; the content of their sum is then $b+c-2e$. The remaining points of the whole set form a set, in general open, whose (inner) content, by Lemma 2, is $a+2e$, and which is the sum of two non-overlapping sets, the (inner) content of each of which is, by what has been proved, not greater than e. Hence, by the usual argument, we can find a *closed* component of the whole set of content $a+e$, which is the sum of two non-overlapping sets, the sum of whose (inner) contents is not greater than e. With respect to these sets we can now repeat the argument, using $\frac{1}{2}e$ instead of e, and so on. Ultimately, by Lemmas 2′ and 3′, we shall determine a closed set whose content is a, divided into two non-overlapping components, both of whose (inner) contents are zero.

With our present imperfect knowledge of open sets, it seems to me impossible to assert definitely that such a case could not arise. I can only say that I do not know of any such case.

In the next section it is shown that the (inner) addition theorem holds when one of the components is any set whatever and the other component is any one of a large class of open sets.

13.

We begin with a few preliminary definitions.

DEFINITION 1.—*If G_1, G_2, ... be a series of sets of points such that, for all values of n, G_n is contained in G_{n+1}, and G be the set such that* *every set G_n is contained in G, while every point of G belongs to some definite G_n, G is said to be the (generalised) outer limiting set of the series.*

DEFINITION 1′.—If G_1, G_2, ... be all *closed* sets, G is said to be an *ordinary outer* limiting set.

In the case of such an ordinary outer limiting set,* the upper limit of the content of its closed components is the limit of the content of G_n. Thus, with our present definition, *the (inner) content of an ordinary outer limiting set is the limit of the content of G_n.*

We shall see that the word " (inner) " is here superfluous.

DEFINITION 2.—*If, instead of being contained in G_{n+1}, G_n contains G_{n+1}, the set consisting of the points common to all the sets G_n is called a "generalised inner limiting set."*

DEFINITION 2′.—*If each set G_n consists of all the points of a set of open or closed intervals, G is called an " ordinary inner limiting set."*

I may here call attention to the fact that in my former investigations I used the term "inner limiting set" only for the case where G_n is a set of *open* intervals. It is easily proved that, if the intervals be taken closed, at most a countably infinite set of new points are introduced.†

By Theorem 3′ or 6 *the (inner) content of a generalised inner limiting set is the limit of the (inner) content of G_n*; and, since the content of a set of intervals is the same whether the intervals be closed or not, an ordinary limiting set evidently has the same content whether the intervals be closed or not. Here again the term "(inner)" will be seen to be superfluous.

The process of forming a generalised inner limiting set from the defining series I have elsewhere called *deduction.*

THEOREM 8.—*If the (inner) addition theorem for the (inner) contents holds when one of the two components is a set of a certain type, it is also true when one of the components is the outer limit of sets of that type.*

In other words, if, for every value of n, the addition theorem holds for G_n and *any* other set O_n, it holds for G and any other set O. For let the sum of G and O be H, and let O_n be the set which added to G_n

* *Quart. Jour. of Pure and Applied Math.*, No. 138 (1903), p. 191.

† This was proved in my first paper on the subject, " Zur Lehre der nicht abgeschlossenen Punktmengen," *Ber. d. K. Sachs. Ges. d. Wiss. zu Leipzig*, August 1, 1903, p. 290.

makes up H. Then, using the letters indiscriminately for the sets and their (inner) contents, $G_n + O_n = H$.

Now, as G_n increases towards G, O_n diminishes towards O, each O_n lying in the preceding O_{n-1}. Therefore, by Theorem 3′, the (inner) content O is itself the limit of the (inner) content O_n. Also $\mathrm{Lt}\, G_n + \mathrm{Lt}\, O_n = H$; therefore $\mathrm{Lt}\, G_n + O = H$.

Now the (inner) content of G is evidently not less than the limit of G_n. Therefore $G+O \geqslant H$, the letters denoting (inner) contents. But, evidently, $G+O \leqslant H$; therefore $G+O = H$, the letters denoting either sets or their contents.

COR. 1.—*The (inner) addition theorem holds when one of the components is an ordinary outer limiting set, by the conclusion of Art.* 10.

COR. 2.—*An outer limiting set which is the limit of a sequence of sets each of which has the property that the (inner) addition theorem holds for it and any other set whatever has for (inner) content the limit of the (inner) contents of the sets of the sequence.*

COR. 3.—The theorem

$$\mathrm{Lt}\,(\text{inner})\ \text{content} = (\text{inner})\ \text{content of Lt}$$

is true for a sequence of expanding open sets when the expansion is due *to the increase of a component for which the (inner) addition theorem holds.*

THEOREM 9. — *If the (inner) addition theorem is true when one of the two components is a set of a certain type, it is also true when one of the components is an inner limiting set deduced from an infinite series of sets of this type.*

In other words, if, for all values of n, $G_n + O = H_n$ (the letters being used indiscriminately for a set and its content), and each G_n is contained in the preceding set G_{n-1}, then $G+O = H$. In fact, H is itself an inner limiting set, and therefore its (inner) content is

$$\mathrm{Lt}\, H_n = \mathrm{Lt}\, G_n + O = G + O.$$

COR.—*The (inner) addition theorem holds if one of the components is an ordinary inner limiting set.*

Proof.—An infinite set of closed intervals is a special case of an ordinary outer limiting set, and therefore, by Cor. 1 to Theorem 8, the (inner) addition theorem applies when one of the components consists of the points belonging to such a set of intervals. Hence, applying our present theorem, it holds for the deduced set of a sequence of such sets of intervals. Q. E. D.

Applying the results of this section, we see that, if we keep applying in any order Theorems 8 and 9 to any series of ordinary outer or inner limiting sets, the sets so obtained must always have the property in question.

We have thus already obtained a large class of open sets possessing the property in question; we can, however, extend this class still further.

THEOREM 10.—*If each of two sets which do not overlap belong to this class, their sum also possesses the property in question.*

Let G_1 and G_2 be two such sets, and H any other set whatever. Also let G be the sum of the two sets. Let the (inner) contents of G_1, G_2, and G be denoted by I_1, I_2, and I, and that of H by J. Then, since G_1 belongs to the class, we have at once $I_1+I_2 = I$. For the same reason I_1+J is the (inner) content of (G_1+H). Hence also, since G_2 belongs to the class, the (inner) content of $(G_1+H)+G_2$ is I_1+J+I_2, *i.e.*, it is $I+J$. In other words, the (inner) content of $G+H$ is $I+J$. Therefore, &c.,

Q. E. D.

THEOREM 11.—*If each of two sets one of which is a component of the other belong to the class, so does their difference.*

Use the same notation as in the preceding theorem, G denoting the larger of the two sets, and G_1 say, the component belonging to the class. As before, $I = I_1+I_2$. We have to prove that G_2 belongs to the class. Suppose this is not the case; then the (inner) content of G_2+H must be greater than I_2+J, say I_2+J+k. But, by hypothesis, G_1 belongs to the class; hence the (inner) content of $G_1+(G_2+H)$ is $I_2+J+k+I_1$, *i.e.* it is $I+k+J$. But G_1+G_2 is G, and G belongs to the class; therefore the (inner) content of G_1+G_2+H is $I+J$; therefore k must be zero. Therefore, &c. Q. E. D.

THEOREM 12.—*If a set belonging to this class be divided into two components the sum of whose inner contents is equal to that of the original set, each of the components belongs to the class.*

Let G be the set, G_1 and G_2 the components, H any other set whatever. Denote the corresponding (inner) contents by I, I_1, I_2 and J. Suppose, if possible, that the (inner) content of G_1+H be not I_1+J; then it must be greater than I_1+J. Add the set G_2 to the set G_1+H. Then the (inner) content of the set of $(G_1+H)+G_2$ would be greater than I_1+I_2+J. But, by hypothesis, $I_1+I_2 = I$; therefore the (inner) content of the set G_1+H+G_2 would be greater than $I+J$; that is, the (inner) content of the set $G+H$ would be greater than $I+J$. But G belongs to the class in question; therefore the (inner) content of $G+H$ is equal to $I+J$. Therefore, &c. Q. E. D.

THEOREM 13.—*If G_1 and G_2 be two sets of this class, of (inner) content I_1 and I_2, (a) the set consisting of all the points common to G_1 and G_2 is a set of this class, say G', of (inner) content I'; and (b) the set consisting of all the points belonging to one or both of G_1 and G_2 is a set of this class, say G, of (inner) content I; further, (c) $I_1+I_2 = I+I'$.*

For suppose the (inner) contents of the parts of G_1 and G_2 which are not common to be I_1-x and I_2-y respectively. Then, since the (inner) addition theorem holds for G_2, $I_2+(I_1-x) = I$. Similarly, since the (inner) addition theorem holds for G_1, $I_1+(I_2-y) = I$; whence

$$x = y = I_1+I_2-I.$$

Also $I'+(I_1-x) \leqslant I_1$; therefore $I' \leqslant x$, that is,

$$I' \leqslant I_1+I_2-I. \tag{1}$$

Again, take in each component a closed set of content greater than I_1-e, I_2-e respectively. Then the common part of these closed sets lies in G', and has therefore content $\leqslant I'$. The sets of points belonging to one or both of these closed sets lies in G, and has therefore content $\leqslant I$. Then, by Lemma 3, $(I_1-e)+(I_2-e) < I+I'$, however small e may be, that is,

$$I' \geqslant I_1+I_2-I. \tag{2}$$

Comparing (1) and (2), we have

$$I_1+I_2 = I+I'.$$ Q. E. D.

Again, *the (inner) contents of the parts of G_1 and G_2 which are not common are I_1-I' and I_2-I'.* In fact, from the result just obtained, we have $x = I'$. It at once follows, by Theorem 12, *that the sets G, G', G_1-G', G_2-G', all belong to the class in question.* Q. E. D.

The theorems which we have obtained enable us, starting from closed sets, to build up a very extended class of open sets, possessing the property that the (inner) addition theorem holds for any one of them in combination with any set whatever. The great generality of the class obtained suggests the possibility that the (inner) addition theorem holds for all sets without exception. We must be careful, however, not to jump to this conclusion. We have, at most, shown that all known open sets belong to the class in question. All the known operations employed on members of the class lead to members of the same class: in modern phraseology, they form a *corpus*. If we could assert that there were no other open sets than those formed from closed sets by these processes, we should have settled, once for all, the difficult question of the classification of open sets.

14.

In connection with the class of operations made use of in the last section, the following theorems, which bear also on the classification of open sets, will be of interest, and are needed in what follows.

THEOREM 14.—*An inner limiting set of a sequence of inner limiting sets is an ordinary inner limiting set.*

THEOREM 15.—*An outer limiting set of ordinary outer limiting sets is an ordinary outer limiting set.*

Proof.—Let the sets defining G_n be $G_{n,1}$, $G_{n,2}$, ..., for all values of n. For shortness, let me use the symbol $<$ to mean "is contained in," and $>$ to mean "contains."

Then, when G is a generalised inner limiting set, $G_1 > G_2$. Hence, if $G_{1,r} < G_{2,r}$, we can remedy this by taking, instead of $G_{2,r}$, the common part of $G_{1,r}$ and $G_{2,r}$, which is also a closed set and contains G_2. Doing this for all values of r, $G_{2,r} > G_{2,r+1}$, and $G_{1,r} > G_{2,r}$.

Doing this in succession for the sets defining G_3, G_4, ..., we get the following table:—

$$\begin{array}{ccccccccccc}
G_{1,1} & > & G_{1,2} & > & G_{1,3} & > & G_{1,4} & > & \ldots\ldots & > & G_1 \\
\vee & & \vee & & \vee & & \vee & & & & \vee \\
G_{2,1} & > & G_{2,2} & > & G_{2,3} & > & G_{2,4} & > & \ldots\ldots & > & G_2 \\
\vee & & \vee & & \vee & & \vee & & & & \vee \\
G_{3,1} & > & G_{3,2} & > & G_{3,3} & > & G_{3,4} & > & \ldots\ldots & > & G_3 \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \vee \\
 & & & & & & & & & & \vdots \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & & G.
\end{array}$$

This being so, consider the sequence of closed sets $G_{1,1}$, $G_{2,2}$, $G_{3,3}$, ..., and let their inner limiting set be denoted by G'.

If P be a point of G', it belongs to $G_{m,m}$, for all values of m, and therefore to $G_{m,n}$, for all values of $n > m$, and therefore to G_m, for all values of m; that is, P is a point of G.

If, on the other hand, P is a point of G, we can assign an integer m such that P is a point of G_m, and therefore of $G_{m,m}$, for all values of m;

that is, P is a point of G'. *Thus G is identical with G', and is, as was asserted, an ordinary inner limiting set.*

Next, if G be a (generalised) outer limiting set of outer limiting sets, $G_1 < G_2$. If $G_{1,r} > G_{2,r}$, we can remedy this by taking, instead of $G_{2,r}$, the set consisting of all points belonging to one or both of $G_{1,r}$ and $G_{2,r}$, which is also a closed set and contained in G_2. Doing this for all values of r, $G_{2,r} < G_{2,r+1}$ and $G_{1,r} < G_{2,r}$.

Doing this in succession for all the sets defining $G_3, G_4, \ldots$, we get the following table :—

$$\begin{array}{ccccccccc}
G_{1,1} & < & G_{1,2} & < & G_{1,3} & < & G_{1,4} & < \ldots\ldots < & G_1 \\
\wedge & & \wedge & & \wedge & & \wedge & & \wedge \\
G_{2,1} & < & G_{2,2} & < & G_{2,3} & < & G_{2,4} & < \ldots\ldots < & G_2 \\
\wedge & & \wedge & & \wedge & & \wedge & & \wedge \\
G_{3,1} & < & G_{3,2} & < & G_{3,3} & < & G_{3,4} & < \ldots\ldots\ldots & G_3 \\
\wedge & & \wedge & & \wedge & & \wedge & & \wedge \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \wedge \\
 & & & & & & & & \vdots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & G.
\end{array}$$

This being so, consider the sequence of closed sets $G_{11}, G_{22}, G_{33}, \ldots$, and let their outer limiting set be denoted by G'. If P be any point of G', we can assign an integer m such that P is a point of $G_{n,n}$, and therefore of G_n, for all values of $n > m$; that is, P is a point of G.

If, on the other hand, P be any point of G, we can assign an integer m such that P is a point of G_m. Then, since G_m is an outer limiting set, we can assign an integer r such that P is a point of $G_{m,r}$. If now $m \geqslant r$, P is a point of $G_{m,m}$; but, if $m < r$, P is a point of $G_{r,r}$: in either case, P is a point of G'.

Thus G is identical with G', and is, as was asserted, an ordinary outer limiting set.

These theorems belong to a class of theorems of the same kind, bearing on the question of the classification of open sets. I content myself here with giving the following additional theorems.

THEOREM 16.—*The difference of two closed sets is both an ordinary outer and an ordinary inner limiting set.*

First, to prove it is an ordinary outer limiting set. Enclose the smaller closed set in a finite number of open intervals each of length less than e. The points of the larger closed set left over form a closed set. This closed set, as e decreases without limit, generates the difference of the two given closed sets. Q. E. D.

Next to prove that it is an ordinary inner limiting set. Enclose the larger closed set in intervals each of length less than e. These cover up a finite number of non-overlapping segments. Let d be any one of these segments: then the points of the smaller closed set which lie in d form a closed set, *inside* the black intervals of which lie all the points of the set in question which lie inside d. Taking all such black intervals in all the segments d, we have a set of intervals containing the whole set in question. As we diminish e, we get a series of sets of intervals each lying inside the preceding, and each containing the set in question. The inner limiting set of this series will therefore certainly contain the set in question; but, since each such set of intervals lies inside the corresponding finite number of segments, this inner limiting set is a component of the larger closed set, and contains no point of the smaller closed set; so that the set in question contains this inner limiting set. Thus the set in question is none other than this ordinary inner limiting set. Q. E. D.

THEOREM 17.—*If we subtract a closed set from either an ordinary outer or an ordinary inner limiting set, we still get an ordinary outer or an ordinary inner limiting set.*

In the former case the theorem is a direct consequence of Theorems 15 and 16. In the latter case the difference of the two sets is the ordinary inner limiting set of the parts of the defining intervals of the ordinary inner limiting set that are internal to the black intervals of the closed set.

THEOREM 18.—*If we subtract an ordinary outer limiting set from an ordinary inner limiting set containing it, the difference is an ordinary inner limiting set; and, if we subtract an ordinary inner limiting set from an ordinary outer limiting set containing it, the difference is an ordinary outer limiting set.*

The first part of the theorem is proved in precisely the same way as the second part of the preceding theorem, only that, instead of a single closed set, we have a sequence of closed sets each containing the preceding, and therefore a sequence of sets of black intervals each containing the succeeding.

To prove the second part we proceed as follows:—

Let D_1, D_2, ... denote the successive sets of intervals defining the inner limiting set D, and let P_1, P_2, ... denote the closed sets of which

$D_1, D_2, \ldots$ are the black intervals; also let $G_1, G_2, \ldots$ denote the closed sets defining the outer limiting set G. The points common to G_n and P_n form a closed set, say K_n, contained in G and having no point common with D; further, given any point of G not belonging to D, we can assign an integer m such that, for all integers n greater than m, that point is a point of P_n (since it is not a point of D), and an integer m' such that, for all integers n greater than m', it is a point of G_n (since it is a point of G); therefore, if m'' denote the larger of m and m', the point is a point of K_n, for all integers n greater than m''. Thus the outer limiting set of the series of closed sets K_n, each one of which evidently contains the succeeding, is the difference $G-D$. Q. E. D.

15.

In Art. 12 I showed that, in the discussion of the question whether, or no, the inner addition theorem holds always, we might confine our attention to sets of zero (inner) content. We may remark that *the general problem of classifying open sets may be reduced to the corresponding problem for sets of zero (inner) content.*

In fact, if we take any open set of (inner) content a, two cases at most can present themselves: either it contains a closed set of content a or it contains closed sets of content as near a as we please. In the former case the given set is the sum of a closed set of content a and an open set of (inner) content zero; in the latter case we may first subtract a closed set of content $a-e$, and so obtain an open set of content e; in this latter set we may subtract a closed set of content e', where e' is as small as we please; and so on. We thus get, by successive subtraction of closed sets, a series of open sets, each lying inside the preceding and having zero for the lower limit of their contents; their deduced set is therefore either altogether absent or has content zero. In the former case the given open set is an ordinary outer limiting set; in the latter case it is the sum of an ordinary outer limiting set and a set of zero (inner) content. In other words, we have the following theorem :—

THEOREM 19.—*Every set of (inner) content a is either a closed set or an ordinary outer limiting set, or is equal to the sum of one or other of these and of a set of zero (inner) content.** As the properties of an ordinary outer limiting set may be regarded as known, this theorem confirms the statement made above as to the classification of open sets.

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* Cp. Lebesgue, *loc. cit.*, Art. 7.

16.

The definition adopted makes the (inner) content of an open set depend on that universally adopted for a closed set; moreover, we get as the (inner) content the content [for we shall see that we can here suppress the term (inner)] of a certain ordinary outer limiting set contained in it. If we attempt to give a definition of content equally applicable to all sets of points, we are met at once by difficulties which might seem to be insuperable.

The ordinary definition of the content of a closed set is as follows :— Describe little intervals of constant length e round the points of the set: these fill up a finite set of intervals the content of which is, in the limit, when e is indefinitely diminished, the content of the closed set.

If this definition be applied to an open set, it, of course, gives us the same content as that of the set got by closing it, and thus fails to distinguish between the set and its component.

In the definition given of the content of a closed set it is, however, unnecessary to take the intervals all of the same length: not only so; it is not necessary to specify that they have a positive lower limit In fact, if round every point of a closed set we describe a little interval, say $< e$, according to any law, it follows by the extension of the Heine-Borel theorem, since the set is closed, that it will be internal to a finite number of these intervals. The equivalent non-overlapping set will also consist of a finite number of intervals only, and its content, when e is indefinitely diminished, will give us the same quantity as before.

If we try to apply this modified form of the definition of the content of a closed set to open sets in general, we are met by a similar difficulty to that which occurred before. Whereas in the case of a closed set no other points are left in ultimately, when e is indefinitely diminished, this is not true of open† sets, unless they belong to the class of what we have called "ordinary inner limiting sets." Thus, if it be legitimate to ascribe content to an ordinary inner limiting set and to define it in this manner, the process in question, when applied to an open set in general, would gives us the content of an ordinary inner limiting set of which it is a component. With Lebesgue, I shall call the content defined in this manner the outer measure of the content, or, briefly, the "(outer) content."

DEFINITION.—*Round every point of the set G describe a little interval;*

* For simplicity of explanation I confine myself to linear sets of points; it is not difficult to make the necessary modifications of language in the general case.

† For a discussion of the points which must come in, see "On Sequences of Sets of Intervals containing a given Set of Points," *Proc. London Math. Soc.*, Ser. 2, Vol. 1, Part 4, p. 262.

find the content of the set of intervals so formed; this content has a lower limit for the various possible modes of construction; this lower limit is called the "(outer) content of the sets of points."

17. *Measurable Sets.*

For closed sets we know that (inner) and (outer) content are merely different aspects of the same thing, the content of the closed set. Lebesgue uses the term *measurable set* for a set for which the (inner) and (outer) contents coincide; for such a set we may, without scruple, use the term "content."

It is evident that any definition of the content which agrees in the least with our fundamental ideas must make the content of a set greater than, or at least equal to, that of any of its components; so that, if the (outer) and (inner) contents ever do not coincide, the former gives us an upper limit and the latter a lower limit for the content. Thus, in the case of measurable sets no other definition of the content is possible.

Lebesgue proves the following properties of measurable sets in a *finite* segment of the straight line:—(1) The set consisting of all the points belonging to one or more of a finite or countably infinite number of measurable sets is itself measurable; (2) the set consisting of all the points common to a finite or countably infinite number of measurable sets is itself measurable; (3) the contents of measurable sets in combination with one another obey the law of addition; (4) the content of an inner or outer limiting set of measurable sets is the limit of the content of the defining sets; (5) the class of measurable sets has in any finite segment the potency of all possible sets and includes all ordinary sets.

I do not propose to assume any of these results, firstly, because the theorems I require, in so far as they could be deduced from theorems of Lebesgue's, are capable of a direct proof of a simple character; but, secondly, because I have not found it necessary to assume that the region of operation is finite, an assumption without which Lebesgue's proofs* would not be valid; so that the doubt arises whether his results can be assumed to hold when the region of operation is the whole of space or a more than finite portion of it.

From the point of view of an exhaustive classification of open sets, these results of Lebesgue's are not sufficient, unless it can be shown that none but measurable sets exist. This point is still open to question. If there are other sets, then, as will be shown, all the ordinary sets

* See, for example, Lebesgue, p. 239, line 3.

enumerated and indicated in Lebesgue's paper are included in a class which is included in the class of measurable sets, but may consist of only a part of it: this class has itself the potency of all sets in any segment finite or infinite, and, from the point of view of content, possesses most important characteristics; this is none other than the class of sets which in combination with *any other set whatever* are such that the sum of the (inner) contents is the (inner) content of the sum, and the sum of the (outer) contents is the (outer) content of the sum. It will be noticed that all that Lebesgue has proved for measurable sets is that this is true of measurable sets in combination with other measurable sets. I shall, for definiteness, allude to the class of sets for which the (inner) addition theorem holds as the (inner) additive class, and that for which the (outer) addition theorem holds as the (outer) additive class; the class above referred to will then belong to both these classes, and I call it *the additive class*.

Theorem 3 of the first part of this paper shows that for an ordinary inner limiting set the (outer) content coincides with the (inner) content; it shows, moreover, that, in the case of an ordinary inner limiting set, however we construct the intervals round the points of that set, the content of those intervals always approaches the same limit when the intervals are decreased without limit, viz., the content of the ordinary inner limiting set, provided ultimately no points are left in except those of the given inner limiting set.

In the case of a set which is not an inner limiting set we cannot so construct the intervals that no other points are left in, and there might seem to be a certain degree of arbitrariness in the selection of those points which are to be admitted.

According to the law of construction adopted, we may, as the length of the separate intervals is indefinitely decreased, approach the actual lower limit, that is the (outer) content, or some other quantity lying between this and the content of the set got by closing the given set.

If I be the (inner) content of a set, it is evident that the set cannot be enclosed in a set of intervals of content less than I; thus the defining property of measurable sets may be expressed by saying that *a set of (inner) content I is measurable if, and only if, it can be enclosed in a set of intervals of content $I+e$, where e is as small as we please.* This property is, as we saw, possessed *par excellence* by ordinary inner limiting sets. It is remarkable that it is also possessed by ordinary outer limiting sets, though, except in particular cases, an ordinary outer limiting set cannot be defined as the inner limiting set of a sequence of sets of intervals.

To prove the property in question, we begin by proving it for the special case when the ordinary outer limiting set is the difference of two closed sets.

Let the contents of the two sets be I_1 and I_2; so that the (inner) content of their difference is $I_1 - I_2$.

First, let us enclose the larger set in a finite number r of intervals, whose sum is $I_1 + e$. Let any one of these intervals be denoted by d.

The points of G_2 which lie in d form a closed set: let it be denoted by G'_2, and its content by I'_2; so that $\Sigma G'_2 = G_2$ and $\Sigma I'_2 = I_2$. The black intervals of G'_2 inside d have content $d - I'_2$, and *inside* these lie all those points of $G_1 - G_2$ which lie in d. Thus all the points of $G_1 - G_2$ lie inside all these intervals in the r intervals d, whose sum is $\Sigma d - \Sigma I'_2 = I_1 + e - I_2$, which proves the theorem in this case.

To deduce the theorem in the general case we proceed as follows:— Suppose the set to be the limit of G_n, when n is infinite. Let the content of G_n be I_n. Shut up G_1 in a finite number of intervals of sum $I_1 + \frac{e}{2}$; $G_1 - G_2$ in a set whose content is $I_2 - I_1 - \frac{e}{2^2}$; and so on. Evidently, in this way, we get an infinite set of intervals, in general overlapping, containing all the points of the set G, whose content is therefore certainly not greater than $\mathrm{Lt}\, I_n + e$, that is $I + e$; so that G is measurable. Thus we have the theorem:

An ordinary outer or inner limiting set is measurable, that is, if its content be I, it can be shut up in an infinite set of intervals whose content lies between I and $I+e$, and it contains closed components of content lying between $I-e$ and I, where e is as small as we please.

We might consider in detail all the sets obtained from open sets by means of the processes of Art. 15, and prove that they all possess this property. The following theorem, however, proves not only this, but that all sets belonging to what I have called the inner additive class possess this property.

THEOREM 20.—*If a set is such that when added to any other set whatever which has no points in common with it the sum of the (inner) contents is the (inner) content of the sum, the set in question is measurable.*

Let I_1 be the (inner) content of the set, and I_2 be that of the set of points required to close it, and I that of the whole set so obtained; then, by hypothesis, $I = I_1 + I_2$. As usual, let the sets whose contents are I, I_1, and I_2 respectively be denoted by G, G_1, and G_2.

Take a closed component G_2' of content $> I_2 - \frac{1}{2}e$ in G_2. The set G_1 lies, of course, in the black intervals of this set. Next shut up the set G in a finite number of intervals $d_1, d_2, \ldots, d_n$, of content $< I + \frac{1}{2}e$.

In any one of these intervals d_r, the points of G_2' form a closed set, of content I_r' say, where $\sum_1^n I_r' > I_2 - \frac{1}{2}e$.

The points of G_1 which lie in d_r lie in the black intervals of this closed component of G_2', that is, in intervals whose sum is $d_r - I_r'$. Thus *all* the points of G_1 are enclosed in a set of intervals whose sum is

$$\sum_1^n \{d_r - I_r'\} < I + \tfrac{1}{2}e - I_2 + \tfrac{1}{2}e < I_1 + e.$$

This, therefore, proves the theorem.

It is easy to see that, if a set does not belong to the (inner) additive class, we can no longer assert that it possesses the property in question. Take, for example, a closed set of content a, and suppose it, if possible, divided into two components which do not belong to the (inner) additive class, so that the sum of their (inner) contents is less than a. Then, if both these components have the property in question, we could enclose the closed set in an infinite set of intervals whose sum is less than a, and therefore in a finite number of these intervals; which is impossible. Thus at least one of the components cannot have the property in question.

We have not, however, proved that, if there are sets which do not belong to the (inner) additive class, they may not be further sub-divided into those which are and those which are not measurable.

18. *The (Outer) Content.*

The properties which we have found for the (inner) content have their exact counterparts for the (outer) content; so that we cannot say that either concept seems more fundamental than the other.

A set of (outer) content J is evidently measurable if, and only if, it contains closed components of content $J - e$, where e is as small as we please.

That this is the case when the set belongs to what I called the (outer) additive class is shown as follows; the theorem is the counterpart to Theorem 20.

THEOREM 21.—*If a set be such that, when added to any set whatever* *having no point common with it, the sum of the (outer) contents is the (outer) content of the sum, the set in question is measurable.*

As before, let G_1 be the set, G_2 the set required to close it, and G the sum of G_1 and G_2, and let the corresponding (outer) contents be J_1, J_2, and J.

Let us enclose G in a finite number of intervals of content lying between J and $J+e$, and G_2 in a set of intervals of content lying between J_2 and J_2+e.

The points of the former intervals which are not internal to the latter intervals form a closed set of content lying between $J-J_2-e$ and $J-J_2$; that is, between J_1-e and J_1, by the hypothesis. The points of this closed set which also belong to the closed set G form a closed component of G, which, since it has no point common with G_2, is also a closed component of G_1. Let its content be denoted by K; then we can enclose it in a finite number of intervals of content less than $K+e$, and these, together with the intervals constructed round G_2 contain all the points of G; hence $K+J_2+2e \geqslant J_1+J_2$, that is $K \geqslant J_1-2e$, which proves the theorem.

COR.—The sets of the additive class are all measurable.

It is easily seen that Theorem 5 holds if for (inner) we substitute (outer). Corresponding to Theorem 6 we have the following:—

THEOREM 6′.—*The (outer) content of a generalised outer limiting set is the limit of the (outer) content of the defining set G_n.*

Let J_n be the (outer) content of G_n and J of the outer limiting set G, and let us denote the limit of J_n when n is indefinitely increased by j. It is evident that, as each G_n is contained in the following G_{n+1}, the quantities J_n never decrease, and j is their upper limit.

Let us commence at such a set G_1 that, e being any small positive quantity, $j-e \leqslant J_n \leqslant j$, for all values of n, and let $e_1+e_2+\ldots < e$. Enclose G_n in a set of intervals of content less than J_n+e_n, for all values of n.

Then the parts common to the $(n-1)$-th and n-th sets of intervals contain G_{n-1}, and must therefore have content $\geqslant J_{n-1}$. Thus, if we take all the intervals together which we have constructed, we have a set of overlapping intervals containing every point of G, and their content is less than or equal to $(J_1+e_1)+(J_2-J_1+e_2)+\ldots+(J_n-J_{n-1}+e_n)+\ldots$, that is, less than $j+e$. Thus $J < j+e$. But J cannot be less than j; for otherwise we could enclose G in a set of intervals of content less than j, which is evidently impossible. Thus $J = j$. Q. E. D.

COR.—From Theorems 6 and 6′ the theorem follows that *an outer or inner limiting set of measurable sets is measurable and has for content the limit of the contents of the defining sets.*

Corresponding to Theorem 7 we have the following:—

THEOREM 7′.—*Given an infinite number of sets of points, components of a set of finite (outer) content L, the (outer) contents of these sets having a positive upper limit g, then an infinite number of these sets exists, which can all be enclosed simultaneously in a set of intervals of content $< g+e$, where e is as small as we please.*

If more than a finite number of the sets have zero (outer) content, the theorem is obviously true; we assume therefore that this is not the case; then there is certainly at least one proper upper limit $g' \leqslant g$ such that, for all values of e, there are a more than finite number of the sets whose (outer) contents lie between $g'-e$ and g', both inclusive.

This being so, let us replace the sets by ordinary inner limiting sets containing them, having the same (outer) content and contained in an outer limiting set of content L,* and let $G_1, G_2, G_3, \ldots$ be a countable set of these ordinary inner limiting sets such that, if the content of G_n be denoted by I_n, $g' \geqslant I_n > g' - \frac{e^n}{2^{n+1}}$.

Then, since an ordinary inner limiting set has the same (inner) and (outer) content, we can, since they are all contained in a set of content L, and have content $> g' - \frac{e}{2^2}$, apply to these sets the result of Theorem 4′, that is, there must be a countable number of them, say, in order, $G'_1, G'_2, G'_3, \ldots$, having in common a set of (inner) content $\geqslant g' - \frac{e}{2^2}$, and therefore containing an ordinary outer limiting set of content $\geqslant g' - \frac{e}{2^2}$. Let us denote this latter by C_1.

Similarly, there must be a countable number of the sets $G'_1, G'_2, \ldots$, whose contents are greater than $g' - \frac{e}{2^3}$, and among these we can find a countably infinite set $G''_2, G''_3, \ldots$, having in common a set of (inner) content $\geqslant g' - \frac{e}{2^3}$, and therefore containing an ordinary outer limiting

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* It is easy to see how to do this; we can enclose each of the sets in a set of intervals of content within e of its content, and the whole set in a set of intervals of content lying between L and $L+e$; if we now omit any parts of the former intervals external to the latter intervals, and let e describe a sequence having zero as limit, we get the sets above referred to.

set of content $\geqslant g' - \frac{e}{2^3}$. Let us call this C_2. In this way we obtain a series of the sets $G'_1, G''_2, G'''_3, \ldots$, and a corresponding series of ordinary outer limiting sets $C_1, C_2, C_3, \ldots$, such that C_1 is contained in all the sets $G'_1, G''_2, \ldots, C_2$ in all but the first, C_3 in all but the two first, and so on.

By Theorem 15 the outer limiting set of $C_1, C_2, \ldots$ is an ordinary outer limiting set—let us call it C—and its content is the limit of the content of C_n, that is g'.

Now, since G'_1 and C_1 are both additive sets, their difference has content $\leqslant \frac{e}{2^2}$. Similarly, the difference between G''_2 and C_2 has content $\leqslant \frac{e}{2^3}$, and so on. Thus, if we enclose C in a set of intervals of content $< g' + \frac{1}{4}e$, we shall be able to enclose the remaining points of G'_1 in a set of intervals of content $< \frac{e}{2^2} + \frac{e}{2^3}$, and the remaining points of G''_2 in a set of intervals of content $< \frac{e}{2^3} + \frac{e}{2^4}$, and so on. In this way we enclose simultaneously $G'_1, G''_2, G'''_3, \ldots$ in a set of intervals of content $< g' + e$. These intervals, of course, contain the original sets from which we obtained $G'_1, G''_2, G'''_3, \ldots$; so that this proves the theorem.

19. *The (Outer) Additive Class.*

It is not difficult to show that all closed sets belong to the outer additive class. That the (outer) content of the sum G of two non-overlapping sets G_1 and G_2 is the sum of their (outer) contents, provided both G and G_1 are closed, has already been pointed out as the correlative to Theorem 5; that this is still the case if G is open can be shown as follows.

Let G' be an ordinary inner limiting set containing G and having as content the (outer) content of G, that is I. G' contains G_1 (the closed set), and the other component (which contains G_2), is, by Theorem 17, an ordinary inner limiting set, and has therefore, by what has been proved for the (inner) content, content $I - I_1$; therefore $I_2 \leqslant I - I_1$; but, since G can certainly be enclosed in a set of intervals of content as near as we please to $I_1 + I_2$, we cannot have $I_1 + I_2 < I$; therefore $I_1 + I_2 = I$.

Thus we have the theorem:—

The (outer) content of the sum of two sets which do not overlap is the sum of their (outer) contents, provided one of the component sets is closed.

It does not follow that, if the (outer) addition theorem holds when the sum is closed, it holds generally. Instead of this, however, if we could assume that it holds when the sum consists of all the points of an interval, we could, as in § 11, show that the theorem would be true generally.

The sum of the (outer) contents of two non-overlapping sets is evidently not less than the (outer) content of the sum; thus the question corresponding to that asked on p. 30 is the following:—

Can a segment of length a be divided into two sets of points the sum of whose (outer) contents is greater than a ?

By applying Theorem 6′, we can, precisely as in the corresponding discussion of the (inner) additive class, prove the following:—

THEOREM 8′.—*The (outer) addition theorem holds for an inner limiting set of sets of the (outer) additive class.*

COR. 1.—*The (outer) additive class includes all ordinary inner limiting sets.*

THEOREM 9′.—*The (outer) additive class includes all the outer limiting sets of sets of that class.*

COR.—*This class includes all ordinary outer limiting sets.*

The proof given of Theorem 10 serves, with the mere alteration of the word "(inner)" into "(outer)" to prove the corresponding theorem, viz.:—

THEOREM 10′.—*If each of two sets which do not overlap belong to the (outer) additive class, their sum also belongs to that class.*

Similarly, with the same alteration, and writing "less than" for "greater than" and $-k$ for k, the next proof can be applied, and we get the following:—

THEOREM 11′.—*If each of two sets one of which is a component of the other belong to the (outer) additive class, so does their difference.*

Similarly,

THEOREM 12′.—*If a set belonging to the (outer) additive class be divided into two components the sum of whose (outer) contents is equal to that of the original set, each of the components belongs to that class.*

The proof of the next theorem requires a few more alterations, and is therefore given here at length.

THEOREM 13′.—*If G_1 and G_2 be two sets of the (outer) additive class of (outer) content I_1 and I_2, (a) the set consisting of all the points common to G_1 and G_2 is a set of this class, say G', of outer content I', and (b) the set* *consisting of all the points belonging to one or both of G_1 and G_2 is a set of the class, say G of (outer) content I; further (c) $I_1+I_2 = I+I'$.*

For suppose the (outer) contents of the parts of G_1 and G_2 which are not common to be I_1-x and I_2-y respectively. Then, since the (outer) addition theorem holds for G_2, $I_2+(I_1-x) = I$; similarly, since the (outer) addition theorem holds for G_1, $I_1+(I_2-y) = I$; whence

$$x = y = I_1+I_2-I.$$

Also $I'+(I_1-x) \geqslant I_1$; therefore $I' \geqslant x$, that is,

$$I' \geqslant I_1+I_2-I. \tag{1}$$

Again, take inner limiting sets of content I_1 and I_2 respectively containing G_1 and G_2. The common part of these contains G', and has therefore content $\geqslant I'$. The set of points belonging to one or both contains G and has therefore content $\geqslant I$. Therefore, by Theorem 13,

$$I_1+I_2 \geqslant I+I'. \tag{2}$$

Comparing (1) and (2), we have

$$I_1+I_2 = I+I'.$$

Q. E. D.

Again, the (outer) contents of the parts of G_1 and G_2 which are not common are I_1-I' and I_2-I', since, by the above, $x = y = I'$.

It follows, by Theorem 12′, that the sets G, G', G_1-G', G_2-G' all belong to the class in question. Q. E. D.

20. *The Additive Class.*

The theorems proved enable us without further proof to sum up the chief properties of the additive class.

DEFINITION.—*The additive class consists of all sets which have the property that, if one of them be added to any other set, having no point common with it, the sum of the contents, whether (inner) or (outer), is the corresponding content of the sum.*

(1) The additive class consists entirely of measurable sets, that is, the (inner) and (outer) contents are the same; so that we may properly speak of the content of any additive set.

(2) The additive class includes all closed sets, and ordinary inner and outer limiting sets.

(3) The additive class includes all inner and outer limiting sets of additive sets.

(4) The additive class includes the sum and difference of any two additive sets.

(5) If G_1 and G_2 be two sets of the additive class, their common component G' and the set G, consisting of all the points belonging to one or both of them, both belong to the additive class, and the sum of the contents of the two former sets is the same as the sum of the contents of the two latter sets.

(6) The additive class includes all sets of (outer) content zero or (inner) content infinity, and has therefore in any portion of the straight line the potency of all possible sets.

This last property requires proof.

If E be a set of infinite (inner) content, it is evident that the outer content will also be infinite, and that the sum of E and any other set will contain closed components of content as large as we please, and cannot be enclosed in a set of intervals of finite content; thus E belongs to the additive class. Next, let E be a set of (outer) content zero; then the (inner) content of E must also be zero; so that E is measurable.* Let G be any set of (inner) content a and (outer) content b, having no point common with E. Then $G+E$ can be enclosed in a set of intervals of content as near as we please to b, but not in a set of content less than b; thus b is the (outer) content of the sum. Again, $E+G$ contains closed sets of content as near as we please to a. Suppose it contains a closed set K of content a' greater than a. Let E' be an ordinary inner limiting set containing E and having zero content. Then, since K and E' are both additive sets, their common part K' is additive and has content zero. Therefore $(K-K')$ is additive and has content a'. But $(K-K')$ is a component of G, and G contains no components of content higher than a; so that this is impossible; therefore $E+G$ does not contain any components of content higher than a; so that a is the content of $E+G$. Thus E is additive. Q. E. D.

Now, if F be a perfect set of content zero, any component E of F has (outer) content zero, and belongs therefore to the additive class; but the potency of the components of F is evidently the same as that of all possible sets. This proves the whole of (6).

It is unnecessary to say more to show the importance of this class of sets; it includes all the familiar sets and has all the advantages of Lebesgue's class of measurable sets, while, if there be other than

* Cp. Lebesgue, § 6.

measurable sets, it possesses distinct advantages over the class of measurable sets *in toto*. The fundamental property of additive sets embodied in the definition enables us to extend the theory of content to all sets of the additive class without any scruple. The extent to which that theory can be still further extended, on the one hand to the (inner), and on the other to the (outer), additive class, and a step further to all measurable sets, has been now fully discussed. The only point which remains uncertain is whether or no sets other than these exist.

20.

It will be noticed that the additive class includes all countable sets, and that, with the definition of the content of an additive set which I have adopted, we have the theorem that *the (inner) content of every countable set is zero.*

Again, *the content of the set of irrational numbers in any segment of a straight line is that of the segment itself.*

By making use of the theorems of the present paper, we prove not only this theorem, but the more general one for space of any number of dimensions. For the sake of variety, and also because it throws fresh light on the subject, I give an independent proof of the theorem for one dimension.

Take the following construction :—

Divide the segment (0, 1) into m parts, where m is any odd number except unity. Blacken the central part.

Divide each of the $(m-1)$ remaining parts into m^2 parts, and blacken each central part.

Then divide each of the $(m-1)(m^2-1)$ remaining parts into m^3 parts, and blacken each central one; and so on.

The set consisting of the end-points and external points of the set of intervals constructed thus is easily seen to be a perfect set, nowhere dense, whose content is the same as the corresponding H. J. S. Smith's set of the second kind, viz.,

$$1-\frac{1}{m}-\frac{1}{m^2}\left(1-\frac{1}{m}\right)-\frac{1}{m^3}\left(1-\frac{1}{m}\right)\left(1-\frac{1}{m^2}\right)-\dots,$$

which lies between 1 and $1-1/(m-1)$.*

Thus, by suitably choosing m, we can get a perfect set, nowhere

* For $m = 3$, the content, expressed in the ternary scale, is 1 02 212 2000 01001 01 … .

dense, in the segment (0, 1), whose content is as near as we please to unity. The points of this perfect set are not all irrational, but I will now show how to obtain from it a similar set in which every point is irrational.

A theorem of Scheeffer* asserts that, *given two sets, one closed and nowhere dense, and the other countable, and any two quantities a and b, we can find a quantity c, $a < c < b$, such that, if one of the sets of points be pushed a distance c along the straight line, all the points of the countable set lie inside the black intervals of the closed set.*

Choose as the countable set all the rational numbers between 0 and 1, and as the closed set the perfect set just constructed, so that its content is greater than $1-\frac{1}{2}e$, where e is as small as we please. Then we can find a positive quantity $c < \frac{1}{2}e$, such that, shifting the perfect set to the left a distance c, all its points which remain in the segment (0, 1) become irrational. Since these points form a perfect set nowhere dense of content greater than $1-e$, *we have in this way constructed a perfect set of irrational numbers in the segment* (0, 1) *of content as near as we please to unity.* Q. E. F.

* *Acta Math.*, 5.