

# 10.

## Additamentum ad functionis $\Gamma(a) = \int_0^\infty e^{-x} \cdot x^{a-1} \partial x$ theoriā.

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**F**unctionem  $\Gamma(a) = \int_0^\infty e^{-x} \cdot x^{a-1} \partial x$  (sive  $\Pi(a-1)$  secundum *Gaussii* significationem) inde ab *Eulero* saepius pertractatam a Geometris celeberrimis anno modo praeterito felicissime perscrutatus est *Bern. Jos. Féaux*, Philosoph. Dr. et superioris magisterii candidatus illustris in dissertatione inaugurali mathematica, cui titulus: „De functione, quae littera  $\Gamma$ , obsignatur, sive de integrali *Euleriano* secundae speciei; Monasterii typis Copenrathianis.” Invenit inter alia formulam sive seriem, quae infinite multas continet series infinitas, et quidem hanc

$$(1.) \quad \Gamma(a) = \frac{1}{2} \log 2\pi - a + (a - \frac{1}{2}) \log a + \frac{1}{2} \cdot \frac{1}{2 \cdot 3} \left( \frac{1}{a^2} + \frac{1}{(a+1)^2} + \frac{1}{(a+2)^2} + \dots \right) \\ - \frac{1}{2} \cdot \frac{2}{3 \cdot 4} \left( \frac{1}{a^3} + \frac{1}{(a+1)^3} + \frac{1}{(a+2)^3} + \dots \right) \\ + \frac{1}{2} \cdot \frac{3}{4 \cdot 5} \left( \frac{1}{a^4} + \frac{1}{(a+1)^4} + \frac{1}{(a+2)^4} + \dots \right) \\ - \frac{1}{2} \cdot \frac{4}{5 \cdot 6} \left( \frac{1}{a^5} + \frac{1}{(a+1)^5} + \frac{1}{(a+2)^5} + \dots \right) \\ + \text{etc.},$$

cui calculum functionis numericum superstruendum esse voluit, quod vero negotium, quia series tarde convergunt, non sine molestiis peragetur. Quam ego seriem alio disposui ordine, ut singulae series infinitae, quibus illa constat, summari queant, quod contigit adiumento formulae

$$\frac{1}{2} \left( \frac{1}{2 \cdot 3} \cdot \frac{1}{a^2} - \frac{2}{3 \cdot 4} \cdot \frac{1}{a^3} + \frac{3}{4 \cdot 5} \cdot \frac{1}{a^4} - \frac{4}{5 \cdot 6} \cdot \frac{1}{a^5} + \frac{5}{6 \cdot 7} \cdot \frac{1}{a^6} - \frac{6}{7 \cdot 8} \cdot \frac{1}{a^7} + \dots \right) \\ = (a + \frac{1}{2}) \log \left( 1 + \frac{1}{a} \right) - 1.$$

Hoc modo oritur series simpliciter infinita convergens, quemcunque valorem argumento  $a$  tribuas, ideoque ad calculum propositum absolvendum expeditissima

$$(2.) \quad \log \Gamma(a) = \frac{1}{2} \log 2\pi - a + (a - \frac{1}{2}) \log a + (a + \frac{1}{2}) \log \left(1 + \frac{1}{a}\right) - 1 \\ + (a + \frac{3}{2}) \log \left(1 + \frac{1}{a+1}\right) - 1 \\ + (a + \frac{5}{2}) \log \left(1 + \frac{1}{a+2}\right) - 1 \\ + (a + \frac{7}{2}) \log \left(1 + \frac{1}{a+3}\right) - 1 \\ + \text{etc.}$$

in qua logarithmi obvii sunt naturales. Si vis concedere locum nonnisi logarithmis vulgaribus, sive Briggicis, adhibeas formulam

$$(3.) \quad \log \Gamma(a) = \frac{1}{2} \log 2\pi - \mu a + (a - \frac{1}{2}) \log a + (a + \frac{1}{2}) \log \left(1 + \frac{1}{a}\right) - \mu \\ + (a + \frac{3}{2}) \log \left(1 + \frac{1}{a+1}\right) - \mu \\ + (a + \frac{5}{2}) \log \left(1 + \frac{1}{a+2}\right) - \mu \\ + (a + \frac{7}{2}) \log \left(1 + \frac{1}{a+3}\right) - \mu \\ + \text{etc.}$$

in qua est  $\mu = 0,43429448190$ . Si functionis argumentum  $a$  in formula (2.) unitate augetur, series abit in similem

$$\log \Gamma(a+1) = \frac{1}{2} \log 2\pi - (a+1) + (a + \frac{1}{2}) \log(a+1) + (a + \frac{3}{2}) \log \left(1 + \frac{1}{a+1}\right) - 1 \\ + (a + \frac{5}{2}) \log \left(1 + \frac{1}{a+2}\right) - 1 \\ + \text{etc.,}$$

quae vero, quia  $\log(a+1) = \log a + \log \left(1 + \frac{1}{a}\right)$  est, ita disponi potest:

$$(4.) \quad \log \Pi(a) = \\ \log \Gamma(a+1) = \frac{1}{2} \log 2\pi - a + (a + \frac{1}{2}) \log a + (a + \frac{1}{2}) \log \left(1 + \frac{1}{a}\right) - 1 \\ + (a + \frac{3}{2}) \log \left(1 + \frac{1}{a+1}\right) - 1 \\ + (a + \frac{5}{2}) \log \left(1 + \frac{1}{a+2}\right) - 1 \\ + (a + \frac{7}{2}) \log \left(1 + \frac{1}{a+3}\right) - 1 \\ + \text{etc.}$$

Si aequationem (2.) subtrahis ab hac, remanet aequatio simplex notissima  $\log \Gamma(a+1) - \log \Gamma(a) = \log a$ , sive

$$\Gamma(a+1) = a \cdot \Gamma(a),$$

unde perspicis, formulas (1.) et (2.) esse revera eas, ut functionis naturae, quam exprimit formula  $\Gamma(a+1) = a \cdot \Gamma(a)$ , egregie satisfaciant.

Formula (4.) in primis saltem terminis congruit cum formula notissima *Gaussiana*, vel si mavis, *Euleriana*

$$\log \Pi(a) = \frac{1}{2} \log 2\pi - a + (a + \frac{1}{2}) \log a + \frac{1}{1 \cdot 2 \cdot a} - \frac{1}{3 \cdot 4 \cdot a^3} + \frac{1}{5 \cdot 6 \cdot a^5} - \frac{1}{7 \cdot 8 \cdot a^7} + \dots,$$

at in eo longe ab ipsa discrepat, quod in reliquis terminis deest potentia  $\frac{1}{a}$ , quam in hac continet terminus  $\frac{1}{1 \cdot 2 \cdot a}$ ; qua e re gravissimi momenti, ut alia

silentio praeteream, coniciendum esse videtur, alterutram formulam esse falsam. Si formulae (4.) vis inesse logarithmos vulgares loco naturalium, ipsam mutabis in

$$(5.) \quad \log \Pi(a) =$$

$$\begin{aligned} \log \Gamma(a+1) = & \frac{1}{2} \log 2\pi - \mu a + (a + \frac{1}{2}) \log a + (a + \frac{1}{2}) \log \left(1 + \frac{1}{a}\right) - \mu \\ & + (a + \frac{3}{2}) \log \left(1 + \frac{1}{a+1}\right) - \mu \\ & + (a + \frac{5}{2}) \log \left(1 + \frac{1}{a+2}\right) - \mu \\ & + (a + \frac{7}{2}) \log \left(1 + \frac{1}{a+3}\right) - \mu \\ & + \text{etc.} \end{aligned}$$

Formulae prolatae novae eo magis convergunt, quo maius sit argumentum  $a$ , at semper convergunt, quicumque argumento  $a$  tribuatur valor, quam eximiam proprietatem formulae modo dictae *Gaussianae* deesse notissimum est. Si obsignationem a celeberrimo *Gaussio* nuncupatam ulterius prosequimur, et  $\frac{\partial \log \Pi a}{\partial a} = \Psi(a)$  ponimus, aequatio (4.) illico praebet formulam novam

$$\begin{aligned} (6.) \quad \Psi(a) = & \log a + \frac{1}{2a} + \log \left(1 + \frac{1}{a}\right) - \frac{2a+1}{2a(a+1)} \\ & + \log \left(1 + \frac{1}{a+1}\right) - \frac{2a+3}{2(a+1)(a+2)} \\ & + \log \left(1 + \frac{1}{a+2}\right) - \frac{2a+5}{2(a+2)(a+3)} \\ & + \log \left(1 + \frac{1}{a+3}\right) - \frac{2a+7}{2(a+3)(a+4)} \\ & + \text{etc.} \end{aligned}$$

in qua est

$$\log\left(1 + \frac{1}{a}\right) - \frac{2a+1}{2a(a+1)} = -\frac{1}{2} \left( \frac{1}{3} \cdot \frac{1}{a^3} - \frac{2}{4} \cdot \frac{1}{a^4} + \frac{3}{5} \cdot \frac{1}{a^5} - \frac{4}{6} \cdot \frac{1}{a^6} + \frac{5}{7} \cdot \frac{1}{a^7} - + \dots \right),$$

et quae satisfacit aequationi notae

$$\Psi(a+1) = \Psi(a) + \frac{1}{a+1}.$$

Quia  $\frac{2a+1}{2a(a+1)} = \frac{1}{2} \cdot \frac{1}{a} + \frac{1}{2} \cdot \frac{1}{a+1}$  et pari modo omnes fractiones subtractivae decomponi possunt, series etiam hoc modo se habet

$$\begin{aligned} \Psi(a) = \log(a+k) - \frac{1}{a+1} - \frac{1}{a+2} - \frac{1}{a+3} - \dots - \frac{1}{a+k-1} - \frac{1}{2} \cdot \frac{1}{a+k} \\ + \log\left(1 + \frac{1}{a+k}\right) - \frac{1}{2} \cdot \frac{2a+2k+1}{(a+k)(a+k+1)} \\ + \log\left(1 + \frac{1}{a+k+1}\right) - \frac{1}{2} \cdot \frac{2a+2k+3}{(a+k+1)(a+k+2)} \\ + \log\left(1 + \frac{1}{a+k+2}\right) - \frac{1}{2} \cdot \frac{2a+2k+5}{(a+k+2)(a+k+3)} \\ + \text{etc.}; \end{aligned}$$

quare vides, functionem  $\Psi(a)$  esse limitem expressionis

$$\log(a+k) - \frac{1}{a+1} - \frac{1}{a+2} - \frac{1}{a+3} - \frac{1}{a+k-1} - \frac{1}{2} \cdot \frac{1}{a+k}$$

crescente numero positivo  $k$  in infinitum.

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