



# XVII. Rotating elastic solid cylinders of elliptic section

C. Chree M.A.

To cite this article: C. Chree M.A. (1892) XVII. Rotating elastic solid cylinders of elliptic section , Philosophical Magazine Series 5, 34:207, 154-173, DOI: [10.1080/14786449208620302](https://doi.org/10.1080/14786449208620302)

To link to this article: <http://dx.doi.org/10.1080/14786449208620302>



Published online: 07 May 2010.



Submit your article to this journal [↗](#)



Article views: 4



View related articles [↗](#)

If gravity be supposed operative in aid of the restoration of equilibrium, we should have to include in the boundary condition relative to pressure a term  $g\rho\xi$  in addition to  $Tk^3\xi$ ; so that the more general result is obtainable by adding  $g\rho/k^2$  to  $T$ . Thus

$$in = -\frac{k}{2\mu} \left( T + \frac{g\rho}{k^2} \right), \quad . \quad . \quad . \quad (38)$$

giving the rate of subsidence of waves upon the surface of a highly viscous material. It could of course be more readily obtained directly.

When gravity operates alone,

$$in = -\frac{g\rho}{2\mu k} = -\frac{g}{2\nu k}, \dots \dots \dots (39)$$

which agrees with a conclusion of Prof. Darwin\*. A like result may be obtained from equations given by Mr. Basset†.

**XVII. *Rotating Elastic Solid Cylinders of Elliptic Section.***  
By C. CHREE, M.A., Fellow of King's College, Cambridge‡.

**PART II.—*The Long Elliptic Cylinder.***

§ 35. **BY** a long cylinder is here meant one whose length  $2l$  bears to its greatest diameter  $2a$  a ratio such as is required for the legitimate application of Saint-Venant's solution for beams. What this ratio may be depends on the degree of accuracy aimed at, but the best authorities seem satisfied with values of  $l/a$  which are not markedly less than 10. The cylinder is supposed to be rotating uniformly, and to be free from all but "centrifugal" forces. In the paper in the Quarterly Journal, already referred to, I obtained a solution for a rotating elliptic cylinder, but its length was supposed to be maintained constant by the application of suitable forces over the ends. This is a totally different case from the present, in which the cylinder is supposed free from all surface forces and capable of altering alike in length and diameter. The present solution is thus completely new, except for the case of a circular section which I have already treated elsewhere §, and for the limiting value 0 of  $\eta$  when the alteration

\* Phil. Trans. 1879, p. 10. In equation (12) write  $i/a = k$ , and make  $i = \infty$ .

† Hydrodynamics, vol. ii. § 520, equations (21), (27). See also Tait, Edinb. Proc. 1890, p. 110.

‡ Communicated by the Author.

§ Cambridge Philosophical Society's Proceedings, vol. vii. part vi.

in the length of the cylinder is zero\*. It seems, however, unnecessary to reproduce the algebraical work of determining the arbitrary constants occurring in my general solution of the elastic solid equations. The work is of no interest in itself, and the accuracy of the solution may be easily tested by reference to the internal equations and surface conditions it has to satisfy. It satisfies exactly the internal equations (3), the three conditions (7), (8), and (9) over the curved surface, and the conditions (4) and (5) over the flat ends. There remains only the last surface condition over the ends,  $z = \pm l$ , viz.

$$\widehat{zz} = 0.$$

If this were satisfied the solution would be absolutely exact, and applicable however great or small  $l/a$  might be. This condition is satisfied when  $\eta = 0$ , but otherwise we have to avail ourselves of the principle of statically equivalent load systems, replacing the above condition by

$$\int_{-a}^a \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} \widehat{zz} \, dx \, dy = 0. \quad . \quad . \quad . \quad (75)$$

This equation the following solution will be found to satisfy for every cross section, and not merely for the ends. The notation is the same as in Part I.

$$\begin{aligned} E\Delta(1-\eta)(3a^4+2a^2b^2+3b^4)/\omega^2\rho = \\ (1-2\eta)[(a^2+b^2)\{a^4+a^2b^2+b^4-\frac{1}{4}\eta(3a^4-2a^2b^2+3b^4)-\frac{1}{4}\eta^2(a^2-b^2)^2\} \\ -(1+\eta)x^2\{a^4+a^2b^2+2b^4-\eta(a^4-b^4)\}-(1+\eta)y^2\{2a^4+a^2b^2+b^4+\eta(a^4-b^4)\}], \quad (76) \end{aligned}$$

$$\begin{aligned} E\alpha(1-\eta)(3a^4+2a^2b^2+3b^4)/\omega^2\rho = \\ x\{a^2(a^4+a^2b^2+b^4)-\eta(a^6+a^4b^2+2a^2b^4+b^6)-\frac{1}{4}\eta^2(a^2-b^2)(a^4+3b^4) \\ +\frac{1}{4}\eta^3(a^2-b^2)^2(a^2+b^2)\} \\ -\frac{1}{3}(1+\eta)x^3\{a^4+a^2b^2+b^4-\eta(2a^4+a^2b^2+3b^4)+\eta^2(a^4-b^4)\} \\ -(1+\eta)xy^2\{a^4-\eta b^2(a^2+b^2)-\eta^2(a^4-b^4)\}, \quad . \quad . \quad . \quad (77) \end{aligned}$$

\* In this case the solution is the same as for the thin disk, and also as equations (131)-(133) on pp. 31-32 Quarterly Journal, vol. xxiii., when  $m$  is put  $=n$ , and an obvious printer's error interchanging  $x$  and  $x^3$  in (131) is corrected.

$$E\beta(1-\eta)(3a^4 + 2a^2b^2 + 3b^4)/\omega^2\rho =$$

$$\begin{aligned} & y\{b^2(a^4 + a^2b^2 + b^4) - \eta(a^6 + 2a^4b^2 + a^2b^4 + b^6) + \frac{1}{4}\eta^2(a^2 - b^2)(3a^4 + b^4) \\ & \quad + \frac{1}{4}\eta^3(a^2 - b^2)^2(a^2 + b^2)\} \\ & - \frac{1}{3}(1+\eta)y^3\{a^4 + a^2b^2 + b^4 - \eta(3a^4 + a^2b^2 + 2b^4) - \eta^2(a^4 - b^4)\} \\ & - (1+\eta)yx^2\{b^4 - \eta a^2(a^2 + b^2) + \eta^2(a^4 - b^4)\}, \quad \dots \dots \dots (78) \end{aligned}$$

$$\gamma = -\omega^2\rho\eta(a^2 + b^2)z \div (4E), \quad \dots \dots \dots (79)$$

$$\begin{aligned} \widehat{xx}(1-\eta)(3a^4 + 2a^2b^2 + 3b^4)/\omega^2\rho &= (a^2 - x^2)\{a^4 + a^2b^2 + b^4 - \eta(a^4 + b^4)\} \\ & - y^2a^4(1+2\eta), \quad \dots \dots (80) \end{aligned}$$

$$\begin{aligned} \widehat{yy}(1-\eta)(3a^4 + 2a^2b^2 + 3b^4)/\omega^2\rho &= (b^2 - y^2)\{a^4 + a^2b^2 + b^4 - \eta(a^4 + b^4)\} \\ & - x^2b^4(1+2\eta), \quad \dots \dots (81) \end{aligned}$$

$$\begin{aligned} \widehat{zz}(1-\eta)(3a^4 + 2a^2b^2 + 3b^4)/\omega^2\rho &= \frac{1}{4}\eta(a^2 + b^2)\{(a^2 + b^2)^2 - \eta(a^2 - b^2)^2\} \\ & - \eta x^2\{a^4 + a^2b^2 + 2b^4 - \eta(a^4 - b^4)\} - \eta y^2\{2a^4 + a^2b^2 + b^4 + \eta(a^4 - b^4)\}, (82) \end{aligned}$$

$$\widehat{xy}(1-\eta)(3a^4 + 2a^2b^2 + 3b^4)/\omega^2\rho = -xy\{a^4 + b^4 - \eta(a^2 + b^2)^2\}, \quad \dots (83)$$

$$\widehat{yz} = \widehat{zx} = 0. \quad \dots \dots \dots (84)$$

§ 36. As already stated,  $\widehat{zz}$  vanishes when  $\eta=0$ . For other values of  $\eta$  the solution applies only under the same restrictions as Saint-Venant's solution for beams, and portions of the cylinder immediately adjacent to the ends should be excluded from its domain.

At every point  $\widehat{zz}$  is one of the three principal stresses, but the other two vary in direction from point to point of the cross section, being parallel to the axes of the ellipse only at points which lie on these axes. The stress system other than  $\widehat{zz}$  may be conveniently analysed into a series of simple systems. For shortness let

$$\{a^4 + a^2b^2 + b^4 - \eta(a^4 + b^4)\} \div \{(1-\eta)(3a^4 + 2a^2b^2 + 3b^4)\} = K', \quad (85)$$

then the simple systems are as follows:—

- (i.) A uniform normal tension  $\omega^2\rho a^2K'$  parallel to the major axis.
- (ii.) A uniform normal tension  $\omega^2\rho b^2K'$  parallel to the minor axis.

- (iii.) A normal pressure  $-\omega^2 \rho r^2 K'$  directed along the radius  $r$  from the axis of rotation.
- (iv.) A normal pressure directed along the tangent at the point considered to the ellipse which passes through the point and is similar and similarly situated to the cross section. If  $a', b'$  denote the semi-axes of this ellipse,  $p'$  the perpendicular from its centre on the tangent, this pressure is

$$-\omega^2 \rho \frac{1+2\eta}{1-\eta} \frac{a^2 b^2}{3a^4 + 2a^2 b^2 + 3b^4} \left( \frac{a'b'}{p'} \right)^2. \quad (86)$$

On the curved surface of the cylinder, of the two principal stresses other than  $zz$  one, directed along the normal, is zero, and the other, whose line of action is the tangent to the cross section, is given by the equation

$$\hat{t} = \omega^2 \rho \frac{a^4 + b^4 - \eta(a^2 + b^2)^2}{(1-\eta)(3a^4 + 2a^2 b^2 + 3b^4)} \left( \frac{ab}{p} \right)^2, \quad (87)$$

where  $p$  is the perpendicular from the centre on the tangent to the elliptic section. This stress vanishes when the section is circular for the limiting value  $\cdot 5$  of  $\eta$ . Under all other conditions it is a tension. This analysis of the stress is very similar to that given for the thin disk in § 5. When  $\eta$  is so small that its square is negligible the two analyses are absolutely identical.

§ 37. The stress  $\widehat{zz}$  vanishes over the cylindrical surface

$$x^2/a_{11}^2 + y^2/b_{11}^2 = 1, \quad (88)$$

where

$$a_{11}^2 = \frac{1}{4}(a^2 + b^2) \{ (a^2 + b^2)^2 - \eta(a^2 - b^2)^2 \} \div \{ a^4 + a^2 b^2 + 2b^4 - \eta(a^4 - b^4) \}, \quad (89)$$

$$b_{11}^2 = \frac{1}{4}(a^2 + b^2) \{ (a^2 + b^2)^2 - \eta(a^2 - b^2)^2 \} \div \{ 2a^4 + a^2 b^2 + b^4 + \eta(a^4 - b^4) \}. \quad (90)$$

It is a tension at points within, a pressure at points outside this surface. If  $e_{11}$  denote the eccentricity of (88) and  $e$  that of the rotating cylinder,

$$e_{11}^2/e^2 = (1+2\eta)a^2(a^2+b^2) \div \{ 2a^4 + a^2 b^2 + b^4 + \eta(a^4 - b^4) \}. \quad (91)$$

When  $b=a$  and  $\eta=\cdot 5$  then  $e_{11}=e$ . But under all other conditions

$$e_{11} < e,$$

or the surface of no longitudinal stress and the surfaces of equal longitudinal stress, which are similar to it, are of

smaller eccentricity than the surface of the rotating cylinder. It is also easy to prove

$$a_{11} < a.$$

Thus the surface of no longitudinal stress lies wholly within the cylinder when

$$b_{11} < b,$$

$$i. e. \quad 2b^2(a^2 + b^2)^2 - (a^2 - b^2)^3 + \eta(a^4 - b^4)(a^2 + 3b^2) > 0.$$

When this inequality holds the portion of the cross section wherein the longitudinal stress is a pressure forms a complete annulus limited externally by the surface of the cylinder and internally by (88). When, however, the above inequality does not hold—and by taking  $b/a$  small enough it can always be reversed even when  $\eta = \cdot 5$ —the portion of the cross section wherein  $\widehat{zz}$  is a pressure consists of two detached areas surrounding the ends of the major axis.

§ 38. When  $\eta = 0$  the maximum stress-difference is always correctly given by the axial value of  $\widehat{xx}$ . But for other values of  $\eta$  it is given by the axial value of  $\widehat{xx-yy}$  or by the axial value of  $\widehat{xx-zz}$  according as  $b/a$  is less or greater than a certain value. This value of  $b/a$  increases with  $\eta$ , being *approximately*  $\cdot 277$  when  $\eta = \cdot 25$ , and  $\cdot 511$  when  $\eta = \cdot 5$ . The greatest strain is always correctly given by the axial value of  $\frac{d\alpha}{dx}$ . The expressions for the maximum stress-difference  $\bar{S}$

and greatest strain  $\bar{s}$  may easily be found from equations (77)–(82). It seems unnecessary to write them down.

The limiting safe speed cannot be determined solely by reference to a limiting elastic stress or strain on account of a species of instability which may arise. This question of instability will be discussed presently, but in the meantime it is convenient to record results from which the limiting speed, according to the stress-difference and greatest-strain theories, might be simply derived when the circumstances are such that these theories apply. In Table XIII. the angular velocity is termed  $\omega_1$ , and in Table XIV. it is for the sake of distinction termed  $\omega_2$ . Assigning to  $\bar{S}$  and  $\bar{s}$  in these tables their limiting values for the material under consideration, we obtain the limiting speeds according to the stress-difference and greatest-strain theories, while by assigning a given value to  $\omega_1 a$  and  $\omega_2 a$  we obtain the corresponding maximum stress-difference and greatest strain. The tables should be compared with Tables I. and II.

TABLE XIII.—Value of  $\omega_1 a \div \sqrt{S/\rho}$ .

$\eta$ .	$b/a=$	0.	.2.	.4.	.6.	.8.	1.0.
0		1.732	1.721	1.693	1.661	1.639	1.633
.25		1.732	1.745	1.737	1.706	1.703	1.732
.5		1.732	1.724	1.734	1.784	1.848	2.0

TABLE XIV.—Value of  $\omega_2 a \div \sqrt{E s/\rho}$ .

$\eta$ .	$b/a=$	0.	.2.	.4.	.6.	.8.	1.0.
0		1.732	1.721	1.693	1.661	1.639	1.633
.25		1.746	1.734	1.710	1.704	1.745	1.852
.5		1.789	1.775	1.762	1.822	2.066	2.828

The last result in Table XIII. is *exact*, the rest are *approximate*. When  $\eta=0$  the results are the same on the two theories, and apply, as already stated, to cylinders of all lengths. For large values of  $\eta$  the greatest-strain theory would allow a considerably more rapid rotation than the other theory for all values of  $b/a$ . But for values of  $\eta$  such as .25 the difference between the two theories is remarkably small. For a given value of  $\eta$ , other than 0, the limiting speed has on both theories at least one minimum as  $b/a$  increases from 0 to 1; but for ordinary values of  $\eta$  the limiting speed depends wonderfully little on the value of  $b/a$ .

§ 39. We shall next consider the principal displacements in the cylinder. The longitudinal displacement vanishes when  $\eta=0$ , and for any other value of  $\eta$  the cylinder shortens under rotation. Each cross section remains plane, which is perhaps the most striking difference between the phenomena and those in thin disks. The reduction in length per unit length ( $-\delta l/l$ ) varies directly as  $\eta$ , so it will suffice to give its value when  $\eta=.25$ .

TABLE XV.—Value of  $(-\delta l/l)$ ,  $\eta=.25$ .

$b/a=$	0.	.2.	.4.	.6.	.8.	1.0.
$(-\delta l/l) \div (\omega^2 \rho a^2/E) =$	.0625	.065	.0725	.085	.1025	.125

The results are all *exact*. This table should be compared with Table III.

§ 40. In the cross section the most important displacements are the alterations in the lengths of the principal axes. The major axis always lengthens under rotation. The increase  $\delta a$  in the semi-axis is got by putting  $x=a, y=0$  in (77). The alteration  $\delta b$  in the minor semi-axis is got by putting  $x=0, y=b$  in (78).

When  $\eta=0$  it is easily seen that the minor axis lengthens under rotation for all finite values of  $b/a$ . For other values of  $\eta$ , however, the minor axis shortens when  $b/a$  is less than a critical value  $b_1/a$ . The value of  $b_1/a$  increases with  $\eta$ ; thus, answering to  $\eta=.25, .3, .5$  we find *approximately*

$$b_1/a = .584, .663, .783 \text{ respectively.}$$

These do not differ very much from the corresponding results in the case of a thin disk (see Table VI.). In the following table of values of  $\delta a/a$  and  $\delta b/b$  no sign is attached to the former quantity as being always positive.

TABLE XVI.

Values of  $(\delta a/a) \div (\omega^2 \rho a^2/E)$  and  $(\delta b/b) \div (\omega^2 \rho a^2/E)$ .

$b/a =$	0.	.2.	.4.	.6.	.8.	1.0.
$(\delta a/a) \div (\omega^2 \rho a^2/E) = \begin{cases} \eta=0 \\ \eta=.25 \\ \eta=.5 \end{cases}$	.2 .224 .229	.225 .226 .230	.233 .229 .231	.242 .229 .220	.248 .216 .187	.25 .1875 .125
$(\delta b/b) \div (\omega^2 \rho a^2/E) = \begin{cases} \eta=0 \\ \eta=.25 \\ \eta=.5 \end{cases}$	0 -.0885 -.1875	+0.0090 -.0794 -.1780	+0.0372 -.0496 -.1458	+0.0870 +.0052 -.0840	+0.1587 +.0858 +.0087	+.25 +.1875 +.125

The results for  $b/a=1$  are *exact*, the others are nearly all only *approximate*. This table should be compared with Table IV.

§ 41. The expressions for the displacements and strains in the long cylinder are more complicated than in the thin disk, and their full consideration would require more analysis than the interest of the results seems likely to warrant. I shall thus merely call attention to the more striking features of the radial strain along the principal axes of the cross section.

Along the major axis the radial strain is the value of  $\frac{d\alpha}{dx}$  with  $y=0$ . In a circular section when  $\eta=.5$  the radial strain



has a constant value along the radius. But for all other possible combinations in the values of  $b/a$  and  $\eta$  the radial strain along the major axis continually diminishes algebraically as the distance from the centre increases. At the centre the radial strain along the major axis is always positive, but under certain conditions it may be negative, *i.e.* a compression throughout a small length at the ends of the axis. These conditions are most easily investigated by determining

the points where  $\frac{d\alpha}{dx}$  vanishes in the major axis. From the

symmetry we need only consider the point on the positive side of the axis of  $y$ , and we shall denote its abscissa by  $x_0$ . When  $x_0 > a$  the radial strain is an extension along the whole semi-axis; but when  $x_0 < a$ , this strain is a compression throughout a length  $a - x_0$  at the end of the axis. The expression for  $x_0/a$  given by (77) may be thrown into the form

$$\begin{aligned} (x_0^2 - a^2)(1 + \eta) \{ a^4 + a^2b^2 + b^4 - \eta(2a^4 + a^2b^2 + 3b^4) + \eta^2(a^4 - b^4) \} \\ = \frac{1}{12} \eta [(a^2 - b^2) \{ 3a^4 - 4a^2b^2 + 9b^4 - 3\eta^2(3a^4 + 4a^2b^2 + b^4) \} \\ + (3\eta - 1)(3a^6 + 5a^4b^2 + 13a^2b^4 + 3b^6)]. \quad . \quad . \quad (92) \end{aligned}$$

Remembering that  $\eta$  cannot exceed  $\cdot 5$ , we see that the coefficient of  $x_0^2 - a^2$  in (92) is essentially positive.

When  $\eta = 0$  we have obviously  $x_0 = a$  for all values of  $b/a$ , *i.e.* the radial strain vanishes at the end of the major semi-axis, and at every other point of it is an extension.

When  $\eta$  is very small, and  $b/a$  is not very small, we find from (92), neglecting terms in  $\eta^2$ ,

$$a - x_0 = \frac{1}{2} \eta \frac{b^2}{a} \frac{a^4 + b^4}{a^4 + a^2b^2 + b^4}. \quad . \quad . \quad . \quad (93)$$

The radial strain is a compression throughout this very small length  $a - x_0$  of the major semi-axis and elsewhere is an extension. When  $\eta$  and  $a$  are constant,  $a - x_0$  has a maximum value  $\eta a/3$  when  $b/a = 1$ .

When  $b/a$  is very small as well as  $\eta$ , we find that the radial strain is an extension over the whole major axis when

$$b/a < \frac{1}{2} \sqrt{3\eta} \text{ approximately.}$$

When  $\eta$ , though no longer very small, remains less than  $\cdot 3$ , the radial strain is an extension throughout the whole major semi-axis when  $b/a$  is small, but is a compression over a small portion  $a - x_0$  at the end when  $b/a$  exceeds a certain value

increasing with  $\eta$ . Thus for  $\eta = \cdot 25$  this critical value of  $b/a$  is *approximately*  $\cdot 4034$ , and the *approximate* values of  $a - x_0$  answering to the values  $\cdot 6$ ,  $\cdot 8$ , and  $1$  of  $b/a$  are respectively

$$\cdot 0045 a, \cdot 0150 a, \text{ and } \cdot 0339 a.$$

When  $\eta = \cdot 3$  it is obvious from (92) that  $x_0 = a$  when  $b/a = 1$ . Also the coefficient of  $a^2 - b^2$  in the right-hand side of (92) is easily proved to be positive for this value of  $\eta$ ; thus  $x_0 > a$  for all other values of  $b/a$ , or the radial strain is never a compression at any point of the major axis in any form of elliptic disk, though in a circular disk it just vanishes at the rim.

As  $\eta$  approaches close to  $\cdot 5$  the coefficient of  $a^2 - b^2$  in (92) may become negative, but it is easily shown that the term containing  $3\eta - 1$  is then always sufficiently great to keep the right-hand side of (92) positive. Thus when  $\eta > \cdot 3$  we have  $x_0 > a$  for all values of  $b/a$ , or the radial strain is an extension at every point of the major axis.

§ 42. Along the minor axis the radial strain is the value of  $\frac{d\beta}{dy}$  with  $x = 0$ . When  $\eta$  is less than

$$\frac{1}{2}(\sqrt{13} - 3), \text{ or } \cdot 3028 \text{ approximately,}$$

the radial strain along the minor axis diminishes algebraically as  $y$  increases for all values of  $b/a$ . For larger values of  $\eta$  the radial strain diminishes or increases algebraically as  $y$  increases according as  $b/a$  is greater or less than a certain value depending on  $\eta$ . This critical value of  $b/a$  increases from  $0$  when  $\eta = \cdot 3028$ , and approaches  $1$  as  $\eta$  approaches  $\cdot 5$ . For  $\eta = \cdot 3$  the critical value is  $\frac{1}{2}\sqrt{\sqrt{13} - 3}$ , or  $\cdot 3891$  *approximately*. For  $b/a = 1$ , with  $\eta = \cdot 5$ , the radial strain has everywhere a constant value. When  $\eta = 0$  the radial strain is for all values of  $b/a$  an extension at every point of the minor axis except the ends, where it vanishes. When  $b/a$  is very small the radial strain is a compression throughout the whole minor axis unless  $\eta$  be very small. When both  $b/a$  and  $\eta$  are very small, we find for points in the minor axis the approximate formula

$$\frac{d\beta}{dy} = \frac{\omega^2 \rho}{3E} (b^2 - y^2 - \eta a^2). \quad \dots \quad (94)$$

So in this case the radial strain is a compression along the whole minor axis when  $b/a < \sqrt{\eta}$ , but for greater values of  $b/a$  it is an extension between  $y = 0$  and  $y = \sqrt{b^2 - \eta a^2}$ .

When  $\eta$  is not very small it may be proved that  $\frac{d\beta}{dy}$  is negative along the whole minor axis when  $b/a$  is less than a certain critical value  $b_3/a$  depending on  $\eta$ . The *approximate* values of  $b_3/a$  answering to the values  $\cdot 25$ ,  $\cdot 3$  and  $\cdot 5$  of  $\eta$  are respectively  $\cdot 523$ ,  $\cdot 628$ , and  $\cdot 577$ .

As  $b/a$  increases from  $b_3/a$  the radial strain becomes an extension throughout a portion of the minor axis, and this portion expands until eventually when  $\eta > \cdot 3$  it includes the whole axis. The *approximate* values of  $y_0$ , the distance from the centre of the points where the radial strain vanishes in the minor axis, are given in the following table:—

TABLE XVII.—Value of  $y_0/b$ .

$b/a$ .	$\eta =$	$\cdot 25$ .	$\cdot 3$ .	$\cdot 5$ .
$\cdot 6$		$\cdot 633$	—	$\cdot 963$
$\cdot 8$		$\cdot 909$	$\cdot 926$	$\cdot 507$
$1\cdot 0$		$\cdot 966$	$1\cdot 0$	—

In the case of the first blank the radial strain is a compression, in the case of the second an extension at every point of the minor axis. When  $\eta$  has the values  $\cdot 25$  and  $\cdot 3$  the radial strain is a compression in the length  $b - y_0$ , an extension in the length  $y_0$ ; but when  $\eta = \cdot 5$  it is an extension in the length  $b - y_0$ , a compression in the length  $y_0$ .

§ 43. As already indicated in § 38 an elastic theory of rupture, such as the stress-difference or greatest-strain theory, even if satisfactorily established, would not necessarily suffice in the present case to fix the limiting speed. This failure arises from the possible occurrence of instability, consisting in a tendency in the axis of the cylinder to bend, it being driven as it were by "centrifugal" force from coincidence with the straight line joining its ends. This question has been investigated by Prof. Greenhill\*, who has deduced formulæ for the limiting speed in isotropic cylinders. Before applying his formulæ we shall briefly consider what is the actual problem he has solved. He takes the cylinder, with its axis displaced by rotation into a curved line, and supposes

\* Institution of Mechanical Engineers, Proceedings 1883, pp. 182 *et seq.*

the action of the "centrifugal" forces the same as if the mass were collected in the axis. This axial distribution of force is then supposed in equilibrium with the elastic forces, the distribution of stress over each cross section being assumed to give a couple as in the ordinary Bernoulli-Eulerian treatment of beams under flexure. This leads to a differential equation of the form

$$\frac{d^4 y}{dx^4} - \mu^4 y = 0, \quad . \quad . \quad . \quad . \quad . \quad (95)$$

where  $x$  is measured along the line joining the ends of the cylinder's axis,  $y$  is the distance from this line of a point in the axis in its displaced position, while  $\mu$  is a constant depending on the velocity, material, and dimensions of the cylinder. Supposing the origin at an end of the axis, we may represent the solution of (95) by

$$y = A \sinh \mu x + B \cosh \mu x + C \sin \mu x + D \cos \mu x, \quad . \quad (96)$$

where  $A, B, C, D$  are constants depending on the terminal conditions. Prof. Greenhill takes two alternative sets of conditions:—

$$(1) \quad y=0 = \frac{dy}{dx}, \text{ at both ends, } \quad . \quad . \quad . \quad . \quad (97)$$

$$(2) \quad y=0 = \frac{d^2 y}{dx^2}, \quad ,, \quad ,, \quad . \quad . \quad . \quad . \quad (98)$$

These lead to different results for the limiting speed.

§ 44. The condition  $y=0$  merely fixes the origin of co-ordinates, assuming the line joining the ends of the axis to be fixed in space. The condition  $\frac{dy}{dx} = 0$  at an end signifies that

the direction of the axis is there fixed, while  $\frac{d^2 y}{dx^2} = 0$  signifies

the vanishing of the elastic couple given by the Bernoulli-Eulerian method of treatment\*. This latter condition is thus required by Professor Greenhill's theory when no applied couple acts over a terminal section. Now, supposing Professor Greenhill's method of reducing the physical problem to a mathematical form sufficiently exact, so that (95) is satisfactory, it is clear that the reliance to be placed on his results

\* Prof. Greenhill introduces the condition  $\frac{d^2 y}{dx^2} = 0$  on his p. 200, without explicit reference to his elastic theory, but the above is, I believe, the explanation he had in view.

must largely depend on how closely actual conditions are represented by the assumed terminal conditions. This point is not one which a mathematician is competent to decide, but I do not see that either set given by Professor Greenhill seems more likely *à priori* than the combined set

$$y = \frac{dy}{dx} = \frac{d^2y}{dx^2} = 0. \quad . \quad . \quad . \quad . \quad . \quad (99)$$

Thus, suppose the bearings on which the axle rests each of breadth  $c$ , where  $c$  is, as must happen in practice, finite, though small compared to the length of the cylinder. Further, suppose the bearing to consist of a hollow circular cylinder, inside which the circular axle can move freely without appreciable friction. If the axle fitted the bearing exactly, then  $y$  and  $\frac{dy}{dx}$  would each require to vanish both when  $x=0$  and when  $x=c$ , with similar conditions of course at the other end. Thus, without introducing any elastic principle, we should deduce that  $y$ ,  $\frac{dy}{dx}$ ,  $\frac{d^2y}{dx^2}$ , and, perhaps, some higher differential coefficients must vanish at each end. We cannot, however, satisfy all these conditions with a solution such as (96) without having each of the 4 constants identically zero, in which case we reach no conclusion whatever as to instability.

If we accept Professor Greenhill's elastic theory, the condition  $\frac{d^2y}{dx^2} = 0$  must hold over the terminal sections, unless their faces are in contact with some stops or held in some way. Since the cylinder naturally shortens under rotation, it is difficult to conceive how any system of support which did not introduce very large frictional forces when the rotation was slow could leave the terminal sections anything but *free* when the rotation was rapid. Thus, if it be possible for the supports to keep  $\frac{dy}{dx} = 0$  at both ends, as Professor Greenhill supposes in his first set of terminal conditions, it seems doubtful whether his theory leads to any conclusions whatsoever as to instability.

§ 45. It is only fair to recognize that the hypothesis that the axle exactly fits the bearings—though apparently implied in Professor Greenhill's first set of terminal conditions—is hardly possible in the strict mathematical sense, and some of

the above difficulties might be avoided by a judicious use of this fact. Thus, if we suppose the radius of the bearing to exceed that of the axle by  $\delta r^*$ , we could satisfy the set (98) of terminal conditions by putting in (96)

$$A=B=D=0, \quad 2\mu l=\pi, \quad . \quad . \quad . \quad . \quad (100)$$

whence

$$y=C \sin (\pi x/2l). \quad . \quad . \quad . \quad . \quad (101)$$

The constant  $C$  is not, however, altogether arbitrary. The axle must press against the bearing at the section  $x=c$ , and it seems most reasonable to suppose it would also press against it at the section  $x=0$ , though this is not apparently absolutely necessary. On the first view we must have

$$C=4l\delta r/(\pi c);$$

on the second view  $C$  is only limited to being less than this value. This solution, it will be noticed, answers to one definite value of  $\mu$ , *i. e.* to one given angular velocity. With either a smaller or a greater angular velocity—with definite exceptions in the latter case to be noticed presently—the solution (96) can satisfy the terminal conditions only when  $C$  vanishes as well as the other three arbitrary constants. It would thus appear that the meaning of our solution is as follows:—Supposing the angular velocity gradually to increase from zero, the axis of the cylinder *must* remain straight until the velocity is reached for which  $2\mu l=\pi$ . The axis *may* remain straight when the velocity passes through this value, or it may not. If it remains straight for this critical value it *must* continue straight while the velocity continues to increase until the value is reached for which  $2\mu l=2\pi$ , when again it is possible for it to bend. It may, however, happen that as the velocity is attained for which  $2\mu l=\pi$  the axis bends. The bending will take place suddenly, the consequence being an impulse between the cylinder and the sections  $x=c$ ,  $x=2l-c$  of the bearings. This may suffice of course to smash the bearings or the cylinder, in which case the instability theory may perhaps be considered satisfactory. If, however, this impact does not smash either the bearings or the cylinder, the danger seems to be passed unless the rotation be kept exactly at the critical velocity. Now it is clear that if  $l\delta r/c$  be very small, the resultant “centrifugal” force answering to the displacement (101) is small, and the impulse on the bearings may be but trifling. Thus the danger which the instability theory

\* This implies, however, that when the axis of the cylinder bends, the line joining its ends ceases to be fixed in space.

recognizes may conceivably be very small with many forms of support. There are obviously a series of other critical velocities answering to

$$2\mu l/\pi = 2, 3, \dots, i, \dots$$

where  $i$  is any positive integer. The larger the value of  $i$  the greater would be the danger attending the bending of the axis. Since  $\mu \propto \omega^{\frac{1}{2}}$ , the corresponding angular velocities are as the squares of successive integers.

The previous considerations, it must be clearly understood, are intended not to throw doubt on the existence of a species of instability, such as Professor Greenhill imagines, but merely to give a general idea of the uncertainties attaching to any numerical details to which his theory leads on account of possible divergences between the terminal conditions he assumes and those existing in practice. The only positive conclusion we have come to is that a tendency to instability may be expected to show itself by a want of smoothness in the motion and an undue wearing away of the inner edges of the bearings. This tendency to instability might be seriously increased by a slight departure of the centre of gravity of the cylinder when at rest from its axis.

§ 46. We have next to consider the nature of the hypotheses by which the equation (95) is obtained. The assumptions that the action of the "centrifugal" forces may be calculated by collecting the mass of the cylinder into its axis, and that the elastic stresses over a cross section give origin to a couple proportional to the curvature of the axis, are certainly not more exact, even near the centre of a long cylinder, than the Bernoulli-Eulerian treatment of the flexure of a rod under its own weight. Near the ends of the cylinder the strain and stress must differ widely from that assumed by Professor Greenhill, as he takes no account of the displacements which exist in the absence of instability in a rotating cylinder. These considerations show that while it is quite possible Professor Greenhill's formulæ may lead to correct results for short cylinders, there is no apparent reason from an elastic-solid point of view why they should. My own view is that the application of these formulæ to cylinders whose length is less than 8 or 10 times their diameter is certainly not justifiable—an opinion in which I hope Professor Greenhill will concur—and that in longer cylinders the application of either formula can hardly be considered satisfactory unless some definite evidence exists that the terminal conditions it supposes are approximately satisfied.

§ 47. Considering the uncertainty which prevails, it will

suffice to indicate how these instability formulæ may be applied to check the application of the elastic theories of rupture without entering on any elaborate calculations.

In applying his first set of terminal conditions\* Professor Greenhill makes a numerical slip. His amended formula for the limiting speed in a cylinder of length  $2l$  is

$$\omega^2 \rho / E \kappa^2 = (2.36502/l)^4, \quad . \quad . \quad . \quad (102)$$

where  $\kappa$  is the radius of gyration of the cross section about an axis through its centroid perpendicular to the plane of bending. The corresponding formula for the second set of terminal conditions\* is

$$\omega^2 \rho / E \kappa^2 = (\pi/2l)^4. \quad . \quad . \quad . \quad (103)$$

As we require the least velocity for which instability may arise,  $\kappa$  is the least radius of gyration obtainable, *i. e.* is  $b/2$  in an elliptic section. The corresponding plane of bending con-

tains the minor axis. The terminal condition  $\frac{d^2 y}{dx^2} = 0$  of the

second set of surface-conditions depends on Professor Greenhill's elastic theory, which does not seem a close representation of matters near the ends. Thus, in selecting one of the formulæ for illustration I have preferred the first, as based only on geometrical considerations. The results it leads to in a cylinder of length  $L$  are, however, the same as the second formula would give for a cylinder of length  $\pi L/(4.73)$ .

Taking then (102), suppose we determine by means of our previous formulæ the value of the maximum stress-difference or greatest strain answering to the velocity which this formula allows in a given cylinder. Then if this value of the stress-difference or greatest strain be within the limits allowed by the elastic theory of rupture, the greatest safe velocity is that assigned by the instability formula, assuming of course that the theory it is based on is satisfactory. Take, for instance, the greatest-strain theory, and suppose our formula gives for the greatest strain  $\bar{s}$

$$E \bar{s} / \omega^2 \rho a^2 = N,$$

where  $N$  is a certain function of  $\eta$  and  $b/a$ . Then ascribing to  $\omega$  the value given by (102), and putting  $\kappa = b/2$ , we find

$$\bar{s} = (1.18251)^4 (2b/a)^2 (a/l)^4 N. \quad . \quad . \quad (104)$$

This shows how very rapidly the greatest strain answering to the limiting velocity of the instability theory diminishes as  $l/a$

\* *L. c.* pp. 198-200.



increases. Also the value of  $\bar{s}$  in (104), while varying with  $\eta$ , is quite independent of the density or of Young's modulus. Taking  $\eta = \cdot 25$ , we deduce from (104) the following values for  $\bar{s}$  :—

TABLE XVIII.

Greatest Strain answering to Instability Velocity.

$b/a =$	$\cdot 2.$	$\cdot 4.$	$\cdot 6.$	$\cdot 8.$	$1\cdot 0.$
$(l/a)^4 \bar{s} =$	$\cdot 104$	$\cdot 428$	$\cdot 970$	$1\cdot 644$	$2\cdot 281$

In a material such as steel or good wrought-iron the strain given by this table for a circular cylinder in which  $l=10a$  would answer to a longitudinal load of some 3 tons per square inch, and a slight ellipticity in the section reduces this but slightly. Thus in circular, or nearly circular, cylinders whose length is not decidedly greater than 10 times their diameter, it would certainly be only prudent to consider the magnitude of the stress-difference and greatest strain before applying so rapid a rotation as the instability theory allows. In cylinders in which  $b/a$  is as small as  $\cdot 2$ , instability may be expected to arise under quite a slow rotation, and to attempt to rotate such cylinders with the velocity allowed by the elastic theories would be extremely rash.

§ 48. As regards cylinders whose length is less than 8 or 10 times their greatest diameter, the results of the instability theory are hardly likely to prove satisfactory ; but there can be little doubt of the general truth of the conclusion the theory leads to, viz. that the tendency to instability diminishes rapidly as the ratio of length to diameter is reduced. On the other hand, while the application of the results deduced from our elastic equations is not legitimate in short cylinders unless  $\eta$  be zero, or at least very small, there is no reason to suppose that the maximum stress-difference, or the greatest strain, will be either very much greater or very much less than in long cylinders under similar conditions. One of the strongest reasons for this statement is derived from the comparison of Tables I. and II. with Tables XIII. and XIV. According to these tables the limiting speeds allowed by either elastic theory are fairly similar for long cylinders and for thin disks, and it seems most unlikely that any disproportionately large difference will exist in cylinders of intermediate length. The greatest difference between disks and long cylinders occurs

in a circular section with  $\eta = \cdot 5$ , when the velocity allowed by the stress-difference theory in a long cylinder is approximately 1·323 times that which it allows in a thin disk of the same radius. For ordinary values of  $\eta$  the differences are always much less than this. Suppose for instance  $\eta = \cdot 25$ , then, employing the suffixes 1 and 2 as before, and using  $\Omega$  for the limiting angular velocity in the thin disk,  $\omega$  for that in the long cylinder of the same cross section and material, we deduce the following results :—

TABLE XIX.

Ratio of Limiting Velocities in Cylinder and Disk.

$b/a =$	·0.	·2.	·4.	·6.	·8.	1·0
$\omega_1/\Omega_1 =$	1	1·019	1·043	1·058	1·078	1·104
$\omega_2/\Omega_2 =$	1·008	1·007	1·007	1·008	1·013	1·022

§ 49. The solutions we have obtained both for thin disks and long cylinders are, unless  $\eta = 0$ , only approximate, and the principle of statically equivalent surface-forces on which they are based is one as to whose degree of accuracy opinions may differ. It is thus very desirable to subject our results to some independent test.

If we take the value of  $(-\gamma)$  in (16), integrate it over a cross section of the disk and divide the integral by  $\pi ab$ , we obtain the mean approach to the central plane of points originally in a plane section at distance  $z$ . Representing this mean value by  $(-\bar{\delta}z)$  we easily find

$$(-\bar{\delta}z/z) = \frac{\eta\omega^2\rho}{4E} \left\{ a^2 + b^2 + \frac{4\eta(1+\eta)}{3(1-\eta)} (l^2 - z^2) \right\}, \quad (105)$$

$$(-\bar{\delta}l/l) = \frac{\eta\omega^2\rho}{4E} (a^2 + b^2). \quad \dots \dots \dots (106)$$

Now for the change in length of a long elliptic cylinder of the same section as the disk we find by (79)

$$(-\delta l/l) = \frac{\eta\omega^2\rho}{4E} (a^2 + b^2). \quad \dots \dots \dots (107)$$

But the right-hand sides of (106) and (107) are identical, or we have

$$(-\bar{\delta}l/l) = (-\delta l/l) = \eta\omega^2\rho\kappa^2E, \quad \dots \dots \dots (108)$$

where  $\kappa$  is the radius of gyration of the elliptic cross section about a perpendicular to its plane through its centre.

Again, let  $A \equiv \pi ab$  denote the area of the section of an elliptic disk, and let  $\delta A_z$  be the increase produced by rotation in a section at distance  $z$  from the central plane. Then denoting the direction cosines of the outwardly directed normal to the rim by  $\lambda$  and  $\mu$ , we have

$$\delta A_z = \int (\lambda \alpha + \mu \beta) ds = \iint \left( \frac{d\alpha}{dx} + \frac{d\beta}{dy} \right) dx dy, \quad (109)$$

where  $\alpha$  and  $\beta$  are given by (14) and (15). The line-integral is to be taken round the entire rim, the double integral over the whole cross section. Employing the double integral we easily deduce

$$\delta A_z = \frac{A \omega^2 \rho}{4E} \{ (1-\eta)(a^2 + b^2) + \frac{4}{3}\eta(1+\eta)(l^2 - 3z^2) \}; \quad (110)$$

whence, if  $\overline{\delta A}$  be the mean value of  $\delta A_z$  between  $\pm l$ ,

$$\overline{\delta A}/A = (1-\eta)\omega^2 \rho \kappa^2/E, \quad (111)$$

where  $\kappa$  has the same meaning as in (108).

Employing the values of  $\alpha$  and  $\beta$  given by (77) and (78), we obtain precisely the same result as (111) for the change of cross section in the long cylinder.

Finally, let  $v = 2\pi abl$  denote the volume of the disk, and  $\delta v$  its increase under rotation, then

$$\delta v = \iiint \Delta dx dy dz, \quad (112)$$

the integral extending throughout the entire disk.

From this we easily deduce by means of (13)

$$\delta v/v = (1-2\eta)\omega^2 \rho \kappa^2/E, \quad (113)$$

or

$$\delta v = \omega^2 I/3k, \quad (114)$$

where  $k \equiv E \div \{3(1-2\eta)\}$  is the bulk modulus, and  $I$  the moment of inertia of the disk about the axis of rotation. Taking the value of  $\Delta$  given by (76), we arrive in the case of the long cylinder at precisely the same formula (114).

§ 50. Now these coincidences it must be admitted are of a striking character, and if it can be shown that they are not the outcome of mere accident, but the exact results to which the complete theory of rotating elastic solids is bound to lead, this would seem strong evidence of the trustworthiness of the present solution as regards both its soundness of basis and its

accuracy of detail. That the results are exactly true is easily shown as follows :—

Consider a right cylinder of any length or form of cross section, rotating round the axis formed by the centroids of the sections, taking as before the centre of the cylinder for origin and the axis for axis of  $z$ . We may write the internal equations in the form :—

$$\frac{\widehat{dxx}}{dx} + \frac{\widehat{dxy}}{dy} + \frac{\widehat{d zx}}{dz} + \omega^2 \rho x = 0, \quad . \quad . \quad . \quad (115)$$

$$\frac{\widehat{dxy}}{dx} + \frac{\widehat{dyy}}{dy} + \frac{\widehat{d yz}}{dz} + \omega^2 \rho y = 0, \quad . \quad . \quad . \quad (116)$$

$$\frac{\widehat{d zx}}{dx} + \frac{\widehat{d yz}}{dy} + \frac{\widehat{d zz}}{dz} = 0, \quad . \quad . \quad . \quad (117)$$

the notation being the same as before. Now multiply (115) by  $\eta x$  and (116) by  $\eta y$ ; then from the sum of these two equations subtract (117) multiplied by  $z$ , and integrate throughout the volume of the cylinder. Integrating the terms involving the stresses by parts, we find the surface-integrals vanish in virtue of the surface-conditions at the free surface of an elastic solid, and thus obtain

$$\iiint \{ z\widehat{z} - \eta(\widehat{xx} + \widehat{yy}) \} dx dy dz + \eta \omega^2 \rho \iiint (x^2 + y^2) dx dy dz = 0.$$

Now by the ordinary stress-strain relations,

$$\widehat{zz} - \eta(\widehat{xx} + \widehat{yy}) = E \frac{d\gamma}{dz};$$

thus

$$\iiint \frac{d\gamma}{dz} dx dy dz = -\eta \omega^2 \rho \kappa^2 \cdot 2A l / E,$$

where  $A$  is the area of the cross section, and  $\kappa$  its radius of gyration as in (108).

But

$$\int \frac{d\gamma}{dz} dz = \gamma_l - \gamma_{-l} = 2\gamma_l,$$

where  $\gamma_l$  is the value of  $\gamma$  at an end of the cylinder, and

$$\iiint \gamma_l dx dy = \bar{\delta} l \cdot A.$$

Thus the formula (108) is exactly true for all forms of cross section in all right cylinders long or short.

Again, multiply (115) by  $x$ , (116) by  $y$ , (117) by  $z$ , and add. Then integrating throughout the entire volume, we find

$$-\iiint(\widehat{xx} + \widehat{yy} + \widehat{zz})dx dy dz + \omega^2\iiint\rho(x^2 + y^2)dx dy dz = 0,$$

the surface-integrals vanishing as before.

But by the ordinary stress-strain relations,

$$\widehat{xx} + \widehat{yy} + \widehat{zz} = 3k\Delta,$$

where  $k$  is the bulk modulus, and thus we get

$$\delta v = \iiint \Delta dx dy dz = \omega^2 I / 3k,$$

where  $I$  has the same meaning as in (114).

Since

$$v = 2Al,$$

we have

$$\begin{aligned} \overline{\delta A} / A &= \delta v / v - \overline{\delta l} / l, \\ &= (1 - \eta) \omega^2 \rho \kappa^2 / E. \end{aligned}$$

Thus (111) and (114) are also proved to be absolutely true in all right cylinders rotating about their axis of figure.

The preceding formulæ by which the solution has been tested are particular cases of certain much more general results\*, to whose discovery the author was led by the recognition of the coincidences pointed out in § 49.

### XVIII. *Note on the Measurement of the Internal Resistance of Cells.* By E. WYTHE SMITH†.

IN order to determine the actions which take place in an accumulator during charge and discharge, it is necessary to know the working electromotive force at the different stages. This might be observed by breaking the circuit; but immediately on doing this the electromotive force varies at a very rapid rate, so that if only four or five seconds be occupied in taking the measurement an error of 25 per cent. may be made in the difference between the electromotive force and the terminal potential difference. If time-readings be taken after breaking the circuit and a curve drawn connecting E.M.F. and time, this curve may be produced back in the way described by Prof. Ayrton and others in a paper

\* Cambridge Philosophical Society's Transactions, vol. xv. part iii.

† Communicated by the Physical Society: read June 24th, 1892.