A fundamental theorem in the theory of ruled surfaces.

By

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In previous papers*) I have shown that the general theory of non-developable ruled surfaces may be based upon the consideration of the system of differential equations

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\begin{aligned}
y'' + p_{11} y' + p_{12} z' + q_{11} y + q_{12} z &= 0, \\
z'' + p_{21} y' + p_{22} z' + q_{21} y + q_{22} z &= 0,
\end{aligned} \tag{1}
$$

where

$$y' = \frac{dy}{dx}, \qquad y'' = \frac{d^2y}{dx^2}, \text{ etc.}$$

For, if we denote by (y_k, z_k) for $k = 1, 2, 3, 4$, four systems of simultaneous solutions of (1), for which the determinant

$$D = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ y_1' & y_2' & y_3' & y_4' \\ z_1' & z_2' & z_3' & z_4' \end{vmatrix}$$

does not vanish, the general solutions of (1) will be

$$y = \sum_{k=1}^{4} c_k y_k, \qquad z = \sum_{k=1}^{4} c_k z_k.$$

We can interpret $(y_1, \cdots, y_4)$ and $(z_1, \cdots, z_4)$ as the homogeneous coordinates of two points P_y and P_z. As x changes these points describe two curves C_y and C_z. The ruled surface S is obtained by joining corresponding points P_y and P_z by straight lines.

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*) In volumes II to IV of the Transactions of the American Mathematical Society.

The transformation

(2a) $\quad y = \alpha(x)\bar{y} + \beta(x)\bar{z}, \quad z = \gamma(x)\bar{y} + \delta(x)\bar{z}, \quad \alpha\delta - \beta\gamma \neq 0,$

in which α, β, γ, δ are arbitrary functions of x, couverts the curves C_y and C_z into any other two curves $C_{\bar{y}}$ and $C_{\bar{z}}$ upon the surface S. The further transformation

(2b) $$x = f(\bar{x}),$$

where $f(\bar{x})$ is also an arbitrary function, merely transforms the parameter x.

If we denote by U_k and V_k the coordinates of the planes tangent to S at P_y and P_z respectively, we find that these functions constitute a simultaneous fundamental system of solutions for the system of equations adjoined to (1),

(3) $$\begin{aligned} &U'' + p_{11}U' + p_{12}V' + \left\{q_{11} + \tfrac{1}{4}(u_{11} - u_{22})\right\}U + \left(q_{12} + \tfrac{1}{2}u_{12}\right)V = 0, \\ &V'' + p_{21}U' + p_{22}V' + \left(q_{21} + \tfrac{1}{2}u_{21}\right)U + \left\{q_{22} + \tfrac{1}{4}(u_{22} - u_{11})\right\}V = 0^{*}), \end{aligned}$$

where the quantities u_{ik} are formed from the p_{ik} and q_{ik} in a way that will be indicated immediately. The relation between (1) and (3) is a reciprocal one. If U_k and V_k are interpreted as point coordinates, the ruled surface of (3) is dualistic to S.

When (1) is transformed by equations (2), a new system of differential equations is obtained of the same form as (1). The functions of p_{ik}, q_{ik}, p'_{ik}, q'_{ik}, etc. which have the same value for the transformed as for the original system are called invariants. There are four relative invariants θ_4, θ_6, θ_9, θ_{10}, which are defined as follows.

Let

(4) $$\begin{aligned} u_{11} &= 2p'_{11} - 4q_{11} + p_{11}^2 + p_{12}p_{21}, \\ u_{12} &= 2p'_{12} - 4q_{12} + p_{12}(p_{11} + p_{22}), \\ u_{21} &= 2p'_{21} - 4q_{21} + p_{21}(p_{11} + p_{22}), \\ u_{22} &= 2p'_{22} - 4q_{22} + p_{22}^2 + p_{12}p_{21}, \end{aligned}$$

and

(5) $$\begin{aligned} v_{11} &= 2u'_{11} + p_{12}u_{21} - p_{21}u_{12}, \\ v_{12} &= 2u'_{12} + (p_{11} - p_{22})u_{12} - p_{12}(u_{11} - u_{22}), \\ v_{21} &= 2u'_{21} - (p_{11} - p_{22})u_{21} + p_{21}(u_{11} - u_{22}), \\ v_{22} &= 2u'_{22} - p_{12}u_{21} + p_{21}u_{12}. \end{aligned}$$

Moreover let there be a third set of quantities w_{ik} formed from v_{ik}

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*) Reciprocal Systems of Linear Differential Equations. Trans. Am. Math. Soc. Vol. III, p. 64.

and p_{ik} in the same way as the quantities v_{ik} are formed from u_{ik} and p_{ik}. Then the relative invariants, which we shall consider, are*).

$$
\begin{aligned}
\theta_4 &= (u_{11}-u_{22})^2 + 4u_{12}u_{21},\\
\theta_6 &= 2(u_{11}+u_{22})\theta_4 - \frac{5}{4}[(v_{11}-v_{22})^2+4v_{12}v_{21}] + (u_{11}-u_{22})(w_{11}-w_{22})\\
&\qquad + 2(w_{12}u_{21}+w_{21}u_{12}),\\
\theta_{10} &= (u_{12}v_{21}-u_{21}v_{12})^2\\
&\qquad - [(u_{11}-u_{22})v_{12}-(v_{11}-v_{22})u_{12}]\,[(u_{11}-u_{22})v_{21}-(v_{11}-v_{22})u_{21}],\\
\theta_9 &= \Delta,
\end{aligned} \tag{6}
$$

where

$$
\Delta = \begin{vmatrix} u_{11}-u_{22}, & u_{12}, & u_{21}\\ v_{11}-v_{22}, & v_{12}, & v_{21}\\ w_{11}-w_{22}, & w_{12}, & w_{21} \end{vmatrix} \text{**)}. \tag{7}
$$

If these same invariants be formed for (3) we find that they have the same value, all except Δ, which has the opposite sign.

Now system (1) being given θ_4, θ_6, θ_9, θ_{10} are definite functions of x. We wish to show conversely, that if θ_4, θ_6, θ_9, θ_{10} are given as arbitrary functions of x there essentially exists just one system of differential equations of which they are the invariants. But all ruled surfaces belonging to the same system (1) are projectively equivalent. We shall therefore find the theorem:

If θ_4, θ_6, θ_9, θ_{10} are given as arbitrary functions of x, they determine a ruled surface uniquely, except for projective transformations. If, in the second place one determines a second surface by the invariants $\bar\theta_4$, $\bar\theta_6$, $\bar\theta_9$, $\bar\theta_{10}$, where

$$\bar\theta_4 = \theta_4, \quad \bar\theta_6 = \theta_6, \quad \bar\theta_9 = -\theta_9, \quad \bar\theta_{10} = \theta_{10},$$

this second surface can be obtained from the first by a dualistic transformation. The ruled surface is not determined uniquely if θ_4 or θ_{10} vanish identically.

We proceed now to prove the theorem. Let us assume in the first place that θ_4 is not equal to zero. Then, the flecnode curve on S has two distinct branches, and we may take these as fundamental curves C_y and C_z. Then $u_{12} = u_{21} = 0$***). Moreover, we may take the indepen-

*) Invariants of Systems of Linear Differential Equations. Trans. Am. Math. Soc. Vol. II, pp. 1—24.

**) Geometry of a simultaneous system of two linear homogeneous differential equations of the second order. Trans. Am. Soc. Vol. II, p. 351.

***) Covariants of Systems of linear differential equations and applications to the theory of ruled surfaces. Trans. Am. Math. Soc. Vol. III, p. 435.

dent variable so as to make $\theta_4 = 1$*). Then $u_{11} - u_{22} = \varepsilon$, where $\varepsilon = \pm 1$. Finally, we may assume $p_{11} = p_{22} = 0$. For, if the system (1) be transformed by putting

$$y = \alpha(x)\eta, \qquad z = \delta(x)\zeta,$$

a transformation, which clearly does not affect the previously assumed conditions, one may determine α and δ so as to have p_{11} and p_{22} in the new system equal to zero.

We have therefore

$$(8) \qquad u_{12} = u_{21} = 0, \quad u_{11} - u_{22} = \varepsilon = \pm 1, \quad p_{11} = p_{22} = 0,$$

whence

$$v_{12} = -\varepsilon p_{12}, \quad v_{21} = \varepsilon p_{21}, \quad p'_{12} - 2q_{12} = 0, \quad p'_{21} - 2q_{21} = 0,$$

and therefore

$$(9) \qquad p_{12} = -\varepsilon v_{12}, \quad p_{21} = \varepsilon v_{21}, \quad q_{12} = -\frac{1}{2}\varepsilon v'_{12}, \quad q_{21} = \frac{1}{2}\varepsilon v'_{21}.$$

We have further

$$(10) \qquad \begin{aligned} \theta_6 &= 2(u_{11} + u_{22}) - 9v_{12}v_{21}, \\ \theta_{10} &= -v_{12}v_{21}, \\ \theta_9 &= \varepsilon(v_{12}w_{21} - v_{21}w_{12}) = 2\varepsilon(v_{12}v'_{21} - v_{21}v'_{12}). \end{aligned}$$

Therefore we find

$$\frac{v'_{12}}{v_{12}} - \frac{v'_{21}}{v_{21}} = \frac{1}{2}\varepsilon\frac{\theta_9}{\theta_{10}},$$

or integrating

$$\frac{v_{12}}{v_{21}} = Ce^{\frac{1}{2}\varepsilon\int\frac{\theta_9}{\theta_{10}}dx}.$$

But

$$v_{12}v_{21} = -\theta_{10},$$

whence

$$v_{12} = \pm\sqrt{-C\theta_{10}}\,e^{\frac{1}{4}\varepsilon\int\frac{\theta_9}{\theta_{10}}dx},$$

$$v_{21} = \mp\sqrt{\frac{\theta_{10}}{-C}}\,e^{-\frac{1}{4}\varepsilon\int\frac{\theta_9}{\theta_{10}}dx}.$$

The constant C is quite immaterial. For, if we transform (1) by putting $y = k\eta$, $z = l\xi$, where k and l are constants, we merely multiply v_{12} by $\frac{l}{k}$ and v_{21} by $\frac{k}{l}$. If then we put

$$\frac{l}{k} = \frac{1}{\sqrt{-C}},$$

we shall have

$$(11) \qquad v_{12} = \pm\sqrt{\theta_{10}}\,e^{\frac{1}{4}\varepsilon\int\frac{\theta_9}{\theta_{10}}dx}, \qquad v_{21} = \mp\sqrt{\theta_{10}}\,e^{-\frac{1}{4}\varepsilon\int\frac{\theta_9}{\theta_{10}}dx}.$$

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*) ibid. p. 448.

We also find

$$u_{11} = \frac{1}{2}\varepsilon + \frac{1}{4}\theta_6 - \frac{9}{4}\theta_{10},$$

$$u_{22} = -\frac{1}{2}\varepsilon + \frac{1}{4}\theta_6 - \frac{9}{4}\theta_{10},$$

so that we have finally

$$(12)\quad \begin{aligned} &p_{11} = p_{22} = 0, \\ &p_{12} = \mp \varepsilon\sqrt{\theta_{10}}\, e^{\frac{1}{4}\varepsilon\int\frac{\theta_9}{\theta_{10}}dx}, \qquad p_{21} = \mp\varepsilon\sqrt{\theta_{10}}\, e^{-\frac{1}{4}\varepsilon\int\frac{\theta_9}{\theta_{10}}dx}, \\ &q_{11} = -\frac{1}{8}\varepsilon - \frac{1}{16}\theta_6 + \frac{13}{16}\theta_{10}, \quad q_{12} = \frac{1}{2}p_{12}', \\ &q_{22} = +\frac{1}{8}\varepsilon - \frac{1}{16}\theta_6 + \frac{13}{16}\theta_{10}, \quad q_{21} = \frac{1}{2}p_{21}'. \end{aligned}$$

If we distinguish the systems with $\varepsilon = \pm 1$ from each other by writing p_{ik} and q_{ik} for one, and $\bar{p}_{ik}$ and $\bar{q}_{ik}$ for the other, we shall have

$$\begin{aligned} &\bar{p}_{11} = p_{11} = 0, \quad \bar{p}_{22} = p_{22} = 0, \quad \bar{p}_{12} = -p_{21}, \quad \bar{p}_{21} = -p_{12}, \\ &\bar{q}_{11} = q_{22}, \qquad \bar{q}_{22} = q_{11} \qquad \bar{q}_{12} = -q_{21}, \quad \bar{q}_{21} = -q_{12}. \end{aligned}$$

But two such systems can be transformed into each other by putting

$$\bar{y} = -z, \quad \bar{z} = y.$$

We may therefore assume $\varepsilon = +1$. We still have two systems of coefficients according as we take the other ambiguous sign in (12) to be plus or minus. Again denoting one system by p_{ik}, q_{ik} and the other by $\bar{p}_{ik}$, $\bar{q}_{ik}$, we have

$$\begin{aligned} &\bar{p}_{11} = p_{11} = 0, \quad \bar{p}_{22} = p_{22} = 0, \quad \bar{p}_{12} = -p_{12}, \quad \bar{p}_{21} = -p_{21}, \\ &\bar{q}_{11} = q_{11}, \qquad \bar{q}_{22} = q_{22}, \qquad \bar{q}_{12} = -q_{12}, \quad \bar{q}_{21} = -q_{21}; \end{aligned}$$

but the two systems are again easily transformed into each other by putting

$$\bar{y} = -y, \quad \bar{z} = z.$$

If finally θ_9 be changed into $-\theta_9$, we obtain a system of equations which is easily seen to be equivalent to (3).

The equations (12) are valid only if θ_4 and θ_{10} are different from zero. Let us again assume $\theta_4 \neq 0$, but $\theta_{10} = 0$. We may again take $u_{12} = u_{21} = 0$, $u_{11} - u_{22} = 1$, $p_{11} = p_{22} = 0$. From $\theta_{10} = 0$ then follows either $v_{12} = 0$ or $v_{21} = 0$. Let us assume $v_{12} = 0$. We shall then find

$$(13)\quad \begin{aligned} &p_{11} = 0, \qquad p_{12} = 0, \quad q_{11} = -\frac{1}{8} - \frac{1}{8}\theta_6, \quad q_{12} = 0, \\ &p_{21} = f(x), \quad p_{22} = 0, \quad q_{21} = \frac{1}{2}p_{21}', \qquad q_{22} = +\frac{1}{8} - \frac{1}{8}\theta_6, \end{aligned}$$

where $f(x)$ remains arbitrary. To complete the determination of the surface, an additional equation is therefore necessary in this case. We notice incidentally that $\theta_4 \neq 0$, $\theta_{10} = 0$ are the conditions under which S has a straight line directrix.

If $\theta_4 = 0$, we may put $u_{12} = 0$, and therefore $u_{11} - u_{22} = 0$. The independent variable may be so chosen as to make $u_{11} + u_{22} = 0$*). Unless S is a quadric we shall have $u_{21} \neq 0$. We find that θ_{10} must vanish. Assuming further $p_{11} = p_{22} = 0$ as before, and also $p_{21} = 0$ which amounts to taking for C_z an asymptotic curve on S, we have.

$$(14)\qquad \begin{aligned} &p_{11} = 0,\ p_{12} = f(x), && q_{11} = 0, \quad q_{12} = \frac{1}{2} p'_{12}, \\ &p_{21} = 0,\ p_{22} = 0, && -\frac{1}{4} u_{21} = q_{21} = g(x),\ q_{22} = 0, \end{aligned} \qquad p_{12} u_{21} = -\frac{9}{4}\frac{\theta_9}{\theta_6},$$

where, of the two functions $f(x)$ and $g(x)$, one remains arbitrary. Again, therefore a further condition is necessary to determine the surface. If $\theta_6 = 0$, we have either $u_{21} = 0$, i. e. a quadric surface, or $p_{12} = q_{12} = 0$, i. e. a surface belonging to a linear congruence with coincident directrices. $g(x)$ remains arbitrary. We have seen that in general the invariants determine the surface, while in the exceptional cases a further condition is necessary. This is the theorem we wished to prove.

We wish to make a few further remarks. *We have seen that the conditions* $\theta_4 \neq 0$, $\theta_{10} = 0$ *characterize a ruled surface with a straight line directrix.* This is a much simpler criterion than that given in a former paper. Further, *if all of the invariants vanish, the surface is either a quadric or belongs to a linear congruence with coincident directrices, according as the simultaneous equations*

$$u_{11} - u_{22} = u_{12} = u_{21} = 0,$$

are satisfied or not.

Let us assume $\theta_9 = \theta_4 = 0$. Then we may put $u_{12} = u_{11} - u_{22} = 0$, and assume $u_{21} \neq 0$. One easily deduces this consequence: *If the two branches of the flecnode curve of a ruled surface belonging to a linear complex coincide, it is a straight line.* This theorem is due to Voss**).

Systems (1) and (3) are referred to the same independent variable. If they are equivalent, it must therefore possible be to transform (1) into (3) by a transformation of the dependent variables alone. But such a transformation leaves the invariants *absolutely* unchanged. Therefore (1) and (3) can be equivalent only if

$$\theta_9 = \Delta = 0,$$

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*) On a certain congruence associated with a given ruled surface. Trans. Am. Math. Soc. vol. IV, p. 197.

**) Voss, Math. Ann. Bd. VIII, p. 92.

i. e. if S belongs to a linear complex. Moreover if $\theta_9 = 0$, equations (12) show that (1) and (3) actually become equivalent, if θ_4 and θ_{10} do not vanish.

If $\theta_4 \neq 0$, $\theta_{10} = 0$, we have seen that our system may be written

$$(15) \qquad \begin{aligned} &y'' + q_{11} y = 0, \\ &z'' + p_{21} y' + \frac{1}{2} p'_{21} y + q_{22} z = 0, \end{aligned} \qquad p_{21} = f(x),$$

to which belongs the adjoined system

$$(16) \qquad \begin{aligned} &U'' + q_{22} U = 0, \\ &V'' + p_{21} U' + \frac{1}{2} p'_{21} U + q_{11} V = 0. \end{aligned}$$

The independent variable is the same for both systems. Moreover both systems are referred to their flecnode curves, so that the only transformations which could couvert (15) in to (16) are of the form either

$$y = \alpha U, \quad z = \delta V;$$

or

$$y = \alpha V, \quad z = \delta U.$$

Moreover α and δ must be constants, so as to preserve the conditions $p_{11} = p_{22} = 0$. Since we have $q_{22} \neq q_{11}$, the first transformation can never accomplish this. The second, however, can do this, if and only if $p_{21} = 0$, i. e. if there exists upon S a second straight line directrix.

Consider now the case $\theta_4 = \theta_{10} = 0$. In order that the system may be equivalent to its adjoined system we must of course have $\theta_9 = 0$. This gives either $p_{12} = 0$ or else $u_{21} = 0$. In either case θ_6 also would vanish, and S would be either a quadric or a surface contained in a linear congruence with coincident directrices. In the former case the adjoined system is identical with the original. In the latter case our system becomes

$$(17) \qquad y'' = 0, \qquad z'' + g(x) y = 0,$$

to which belongs the adjoined system

$$(18) \qquad U'' = 0, \qquad V'' - g(x) U = 0.$$

But if we put in (17)

$$y = U, \qquad z = -V$$

we obtain (18).

But whenever it is possible to transform a system (1) into its adjoined system by a transformation of the form

$$y = \alpha U + \beta V. \qquad z = \gamma U + \delta V,$$

the corresponding ruled surface remains invariant under a certain dualistic transformation.

We have therefore the following theorem.

If a ruled surface is self-dual, it must belong to a linear complex. If however, this complex is special, the surface must belong to still another linear complex, i. e. it must have two straight line directrices, which may or may not coincide. All such surfaces are self-dual.

Göttingen, July 29th, 1903.