Note on Bessel Functions. By H. M. Macdonald.

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The solutions of the differential equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0$$

can be represented by convergent series in ascending powers of x for all values of n, and by semiconvergent series in descending powers of x for values of n, which are such that the real part of n is greater than $-\frac{1}{2}$. If y_1 and y_2 are two solutions of the first kind, y_1' and y_2' of the second, then they are connected by relations of the form

$$y_1'=ay_1+by_2,\quad y_2'=cy_1+dy_2.$$

The constants in the above relations are usually obtained by calculating the numerical value of the two sides of the equation for certain values of x (Stokes, *Camb. Phil. Trans.*, Vol. IX., X.; Weber, *Crelle*, Vol. LXXV.; *Math. Ann.*, Vol. XXXVII.). The object of the following note is to show how the one form of solution can be directly obtained from the other.

1. *Solution of the Differential Equation.*

The solutions of the differential equation have been given by Sonine (*Math. Ann.*, Vol. XVI.) in the form

$$\frac{1}{2\pi\iota}\int e^{\frac{1}{2}x(t-1/t)}\frac{dt}{t^{n+1}},$$

the paths of integration being such that they begin and end at places where $e^{\frac{1}{2}x(t-1/t)}$ vanishes, n being arbitrary. These solutions are there obtained from the difference equation; they can be obtained from the differential equation in the following manner.

Assume $$y=\int_{s_0}^{s_1}e^{xs}S\,ds,$$

where S is a function of s not involving x. Then, substituting in the differential equation, after having multiplied by x^2,

$$\int_{s_0}^{s_1} e^{xs}(x^2s^2+xs+x^2-n^2)\,S\,ds = 0;$$

integrating by parts

$$e^{xs}\left[x(s^2+1)\,S-\frac{d}{ds}\{(s^2+1)\,S\}+sS\right]\Big|_{s_0}^{s_1}$$
$$+\int_{s_0}^{s_1} e^{xs}\left[\frac{d^2}{ds^2}\{(s^2+1)\,S\}-\frac{d}{ds}(sS)-n^2S\right]ds = 0.$$

Hence $$e^{xs}\left[x(s^2+1)\,S-\frac{d}{ds}\{(s^2+1)\,S\}+sS\right]\Big|_{s_0}^{s_1} = 0, \tag{1}$$

$$\frac{d^2}{ds^2}\{(s^2+1)\,S\}-\frac{d}{ds}(sS)-n^2S = 0. \tag{2}$$

Putting $s = \sinh\eta$, (2) becomes

$$\frac{1}{\cosh\eta}\frac{d}{d\eta}\left\{\frac{1}{\cosh\eta}\frac{d}{d\eta}(S\cosh^2\eta)\right\}-\frac{1}{\cosh\eta}\frac{d}{d\eta}(S\sinh\eta)-n^2S = 0,$$

that is, $$\frac{d^2}{d\eta^2}(S\cosh\eta)-n^2S\cosh\eta = 0.$$

A solution of this is $$S\cosh\eta = Ae^{-n\eta},$$

whence $$y = A\int_{\eta_0}^{\eta_1} e^{x\sinh\eta}\,e^{-n\eta}\,d\eta$$

is a solution of the original differential equation, η_0 and η_1 being given by

$$e^{x\sinh\eta}\left[xS\cosh^2\eta-\frac{d}{d\eta}(S\cosh\eta)\right]\Big|_{\eta_0}^{\eta_1} = 0,$$

that is, by $$e^{x\sinh\eta}(x\cosh\eta+n)\,e^{-n\eta}\Big|_{\eta_0}^{\eta_1} = 0.$$

Writing $e^\eta = t$, y takes the form

$$A\int_{t_0}^{t_1} e^{\frac{1}{2}x(t-1/t)}\frac{dt}{t^{n+1}},$$

where t_0, t_1 satisfy $$e^{\frac{1}{2}x(t-1/t)}\left[\frac{x}{2}(t+t^{-1})+n\right]t^{-n}\Big|_{t_0}^{t_1} = 0.$$

The following values of t_0 and t_1 satisfy this condition:—

$$t_0 = \infty e^{\alpha\iota}, \quad t_1 = \infty e^{\beta\iota},$$
$$t_0 = 0 e^{(\pi-\alpha)\iota}, \quad t_1 = 0 e^{(\pi-\beta)\iota},$$
$$t_0 = 0 e^{(\pi-\alpha)\iota}, \quad t_1 = \infty e^{\beta\iota},$$

where the real parts of $xe^{\alpha\iota}$, $xe^{\beta\iota}$ are less than zero. The solutions corresponding to these sets of values for the limits may be denoted by y_1, y_2, y_3; they are the same as those given by Sonine. The solution y_1 is immediately seen to be equivalent to the series which is usually taken to define $J_n(x)$, the constant A being put equal to $\frac{1}{2\pi\iota}$. The solution y_2 is equivalent to $e^{-n\pi\iota} J_{-n}(x)$ and y_3 to

$$\frac{J_{-n}(x) - e^{-n\pi\iota} J_n(x)}{2\iota \sin n\pi}.$$

These relations are established in the paper quoted.

2. *The Semiconvergent Forms of the Solutions.*

When a function can be represented, for certain values of the variables involved, by two series one of which is convergent and the other semiconvergent, it seems natural to expect that it can be represented by a double series of the form $\Sigma_p \Sigma_q u_{pq}$, where summation with respect to q first gives one of the series, and with respect to p first the other. In the present case it is required to find a double integral which shall be equal to the integral above given, change in the order of integration giving the semiconvergent form.

Taking the solution

$$y_3 = \frac{1}{2\pi\iota} \int_{0 e^{(\pi-\alpha)\iota}}^{\infty e^{\beta\iota}} e^{\frac{1}{2}x(t-1/t)} \frac{dt}{t^{n+1}},$$

and putting $xt = 2\tau$,

$$y_3 = \frac{1}{2\pi\iota} \left(\frac{x}{2}\right)^n \int_{0 x e^{(\pi-\alpha)\iota}}^{\infty x e^{\beta\iota}} e^{\tau - (x^2/4\tau)} \frac{d\tau}{\tau^{n+1}}.$$

If the real part of n is greater than -1, the upper limit may be taken to be $c + \infty\iota$, where c is any real positive quantity, for the part of the integral from $\infty x e^{\beta\iota}$ to $c + \infty\iota$ vanishes. Further, if the real part of x is taken to be greater than zero, the lower limit can be chosen so that the real part of τ is positive all along the path of integration. Now

$$\Pi(n-\tfrac{1}{2}) = \tau^{n+\frac{1}{2}}\int_0^\infty e^{-\tau s}s^{n-\frac{1}{2}}\,ds,$$

when the real part of τ is positive and the real part of n is greater than $-\frac{1}{2}$; therefore

$$y_8 = \frac{1}{2\pi\iota}\,\frac{x^n}{2^n\Pi(n-\frac{1}{2})}\int_0^{c+\infty\iota}\int_0^\infty e^{\tau-\tau s-(x^2/4\tau)}s^{n-\frac{1}{2}}\tau^{-\frac{1}{2}}\,d\tau\,ds;$$

where the real part of n is greater than $-\frac{1}{2}$, and the real part of x is greater than zero. This may be written

$$y_8 = \frac{1}{2\pi\iota}\,\frac{x^n}{2^n\Pi(n-\frac{1}{2})}\left[\int_0^{c+\infty\iota}\int_0^1 e^{\tau-\tau s-(x^2/4\tau)}s^{n-\frac{1}{2}}\tau^{-\frac{1}{2}}\,d\tau\,ds + \int_0^{c+\infty\iota}\int_1^\infty e^{\tau-\tau s-(x^2/4\tau)}s^{n-\frac{1}{2}}\tau^{-\frac{1}{2}}\,d\tau\,ds\right].$$

Writing in the first integral

$$\tau(1-s) = t,$$

and in the second $\tau(s-1) = t,$

it becomes

$$y_8 = \frac{1}{2\pi\iota}\,\frac{x^n}{2^n\Pi(n-\frac{1}{2})}\left[\int_0^{c+\infty\iota}\int_0^1 e^{t-[x^2(1-s)/4t]}s^{n-\frac{1}{2}}(1-s)^{-\frac{1}{2}}t^{-\frac{1}{2}}\,dt\,ds + \int_0^{c+\infty\iota}\int_1^\infty e^{-t-[x^2(s-1)/4t]}s^{n-\frac{1}{2}}(s-1)^{-\frac{1}{2}}t^{-\frac{1}{2}}\,dt\,ds\right].$$

Now, by the above,

$$\int_0^{c+\infty\iota} e^{t-[x^2(1-s)/4t]}\frac{dt}{t^{\frac{1}{2}}} = \iota\sqrt{\pi}\,e^{\iota x\sqrt{(1-s)}},$$

and

$$\int_0^{c+\infty\iota} e^{-t-[x^2(s-1)/4t]}\frac{dt}{t^{\frac{1}{2}}} = \sqrt{\pi}\,e^{-x\sqrt{(s-1)}};$$

hence, changing the order of integration in the expression for y_8,

$$y_8 = \frac{1}{2\sqrt{\pi}}\,\frac{x^n}{2^n\Pi(n-\frac{1}{2})}\left[\int_0^1 e^{\iota x\sqrt{(1-s)}}s^{n-\frac{1}{2}}(1-s)^{-\frac{1}{2}}\,ds + \frac{1}{\iota}\int_1^\infty e^{-x\sqrt{(s-1)}}s^{n-\frac{1}{2}}(s-1)^{-\frac{1}{2}}\,ds\right].$$

Writing D for $\dfrac{d}{dx}$, and observing that the expressions under the integral sign are uniform between the limits of integration,

$$y_3 = \frac{x^n}{2^{n+1}\sqrt{\pi}\,\Pi\left(n-\frac{1}{2}\right)}(1+D^2)^{n-\frac{1}{2}}\left[\int_0^1 e^{\iota x\sqrt{(1-s)}}(1-s)^{-\frac{1}{2}}\,ds + \frac{1}{\iota}\int_1^\infty e^{-x\sqrt{(s-1)}}(s-1)^{-\frac{1}{2}}\,ds\right],$$

that is,
$$y_3 = \frac{x^n}{2^n\sqrt{\pi}\,\Pi\left(n-\frac{1}{2}\right)}(1+D^2)^{n-\frac{1}{2}}\frac{e^{\iota x}}{\iota x}. \tag{3}$$

Substituting for y_3 the equivalent expression

$$\frac{J_{-n}(x)-e^{-n\pi\iota}J_n(x)}{2\iota\sin n\pi} = \frac{x^n}{2^n\sqrt{\pi}\,\Pi\left(n-\frac{1}{2}\right)}(1+D^2)^{n-\frac{1}{2}}\frac{e^{\iota x}}{\iota x}. \tag{3}$$

Similarly, it may be shown that

$$\frac{J_{-n}(x)-e^{n\pi\iota}J_n(x)}{2\iota\sin n\pi} = \frac{x^n}{2^n\sqrt{\pi}\,\Pi\left(n-\frac{1}{2}\right)}(1+D^2)^{n-\frac{1}{2}}\frac{e^{-\iota x}}{\iota x}. \tag{4}$$

These expressions can be extended to the case where the real part of x is negative after the manner of Weber, *Math. Ann.*, Vol. XXXVII. The symbolical forms for the Bessel functions (3) and (4) have been given by Hargreave, *Phil. Trans.*, 1848, for $J_n(x)$ and $Y_n(x)$. From the above, subtracting (4) from (3),

$$J_n(x) = \frac{2x^n}{2^n\sqrt{\pi}\,\Pi\left(n-\frac{1}{2}\right)}(1+D^2)^{n-\frac{1}{2}}\frac{\sin x}{x};$$

adding, $$\frac{J_{-n}(x)-J_n(x)\cos n\pi}{\sin n\pi} = \frac{2x^n}{2^n\sqrt{\pi}\,\Pi\left(n-\frac{1}{2}\right)}(1+D^2)^{n-\frac{1}{2}}\frac{\cos x}{x};$$

when n is an integer, this latter becomes

$$\frac{1}{\pi}Y_n = \frac{2x^n}{2^n\sqrt{\pi}\,\Pi\left(n-\frac{1}{2}\right)}(1+D^2)^{n-\frac{1}{2}}\frac{\cos x}{x},$$

where Y_n is Hankel's second solution of the equation.

The integral forms can be obtained from the symbolical as follows:—

$$\frac{x^n}{2^n\sqrt{\pi}\,\Pi\left(n-\frac{1}{2}\right)}(1+D^2)^{n-\frac{1}{2}}\frac{e^{\iota x}}{\iota x}$$
$$= \frac{x^n}{2^n\sqrt{\pi}\,\Pi\left(n-\frac{1}{2}\right)}(1+D^2)^{n-\frac{1}{2}}\frac{1}{\iota}\int_0^\infty e^{-x(t-\iota)}\,dt,$$

that is, $$y_3 = \frac{x^n}{2^n \sqrt{\pi}\, \Pi(n-\frac{1}{2})} \frac{1}{\iota} \int_0^\infty e^{-x(t-\iota)}(-2\iota t + t^2)^{n-\frac{1}{2}}\, dt;$$

hence $$y_3 = \frac{x^n 2^{n-\frac{1}{2}}}{2^n \sqrt{\pi}\, \Pi(n-\frac{1}{2})} e^{\iota x - \frac{1}{2}(2n+1)\pi\iota} \int_0^\infty e^{-xt} t^{n-\frac{1}{2}} \left(1 - \frac{t}{2\iota}\right)^{n-\frac{1}{2}} dt.$$

Putting $xt = s$,

$$y_3 = \frac{e^{\iota x - \frac{1}{2}(2n+1)\pi\iota}}{\sqrt{2\pi x}\, \Pi(n-\frac{1}{2})} \int_0^\infty e^{-s} s^{n-\frac{1}{2}} \left(1 - \frac{s}{2\iota x}\right)^{n-\frac{1}{2}} ds,$$

that is,

$$\frac{J_{-n}(x) - e^{-n\pi\iota} J_n(x)}{2\iota \sin n\pi} = \frac{e^{\iota x - \frac{1}{2}(2n+1)\pi\iota}}{\sqrt{2\pi x}\, \Pi(n-\frac{1}{2})} \int_0^\infty e^{-s} s^{n-\frac{1}{2}} \left(1 - \frac{s}{2\iota x}\right)^{n-\frac{1}{2}} ds,$$

$$\frac{J_{-n}(x) - e^{n\pi\iota} J_n(x)}{2\iota \sin n\pi} = -\frac{e^{-\iota x + \frac{1}{2}(2n+1)\pi\iota}}{\sqrt{2\pi x}\, \Pi(n-\frac{1}{2})} \int_0^\infty e^{-s} s^{n-\frac{1}{2}} \left(1 + \frac{s}{2\iota x}\right)^{n-\frac{1}{2}} ds.$$

These are equivalent to the relations given by Weber, *Math. Ann.*, XXXVII., and can be extended to the case where the real part of x is negative as before.

The Integral $\int P_n^2 dx$, and Allied Forms in Legendre's Functions, between Arbitrary Limits. By R. Hargreaves, M.A.

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For different positive integers the indefinite integral $\int P_m P_n\, dx$ has been expressed in simple form, and when $m = n$ the value of the definite integral between the special limits 0 and 1, or -1 and $+1$, is due to Legendre. The fundamental theorem of the present paper expresses the difference $(2n+3)\, P_{n+1}^2 - (2n+1)\, P_n^2$ as the differential coefficient of a simple expression involving P_n and P_{n+1}. From this follows the difference of two consecutive integrals between arbitrary limits, and a direct summation gives the value of the single integral. As the argument turns on the use of sequence equations, it is at once applicable to the forms $P_n Q_n$ and Q_n^2, and moreover the index n may