

NOTE ON THE PROBABLE ERROR OF URBAN'S FORMULA FOR THE METHOD OF JUST PERCEPTIBLE DIFFERENCES.

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THE object of this note is to correct an error in Urban's application of Bernoulli's Theorem for the calculation of the probable error of the Method of Just Perceptible Differences¹, which was followed by the writer in an article on "The Best Form of the Method of Serial Groups²." Fortunately the conclusions drawn in both articles are not seriously affected by the correction.

By the Method of Just Perceptible Differences Urban really means a process of calculation (the Limiting Process³), which can be applied to data collected by several psychologically different methods. The full calculations as carried out by Urban can, however, only be performed if a large number of experiments have been made with each stimulus value: he himself in the article quoted applied the Limiting Process to data collected by the Method of Right and Wrong Cases⁴ with lifted weights. The standard weight was 100 grams, and was lifted before each of the seven comparison weights, which were so chosen that the subject nearly always recognised the extreme weights as lighter or heavier respectively than the standard. The experimenter presented the comparison weights to the subject in an irregular sequence, and the whole series was repeated 450 times. The judgments given were *lighter*, *equal*, or *heavier* than the standard. The answers were entered in a

¹ F. M. Urban, "Die Psychophysischen Massmethoden als Grundlagen empirischer Messungen," *Arch. f. d. ges. Psychol.* 1903, xv. 261-415.

² This *Journal*, 1913, v. 398-416. The writer will be obliged if readers, in referring to this article, will note this correction.

³ G. H. Thomson, "Comparison of Psychophysical Methods," this *Journal*, 1912, v. 210.

⁴ See Thomson, *op. cit.* 204,
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table containing 450 horizontal rows and seven vertical columns corresponding to the seven comparison weights.

Provided with this table, the calculator applies the Limiting Process as follows. Beginning at the left-hand end of each row with the weight 84 grams, he passes along until a judgment *heavier* is met. The weight at which this occurs is noted as one reading of the just perceptible positive difference. For some reason which I cannot discover, Urban only uses 400 rows of the table obtaining that number of readings which in the case of his Subject I. were distributed as follows¹:

TABLE I.

Comparison weights r	84	88	92	96	100	104	108
Frequencies N	0	7	30	76	106	169	12

The mean of all these readings is 100·36 grams, which is taken as the mean just perceptible positive difference, and it is the probable error of this number which is required. Before examining Urban's processes for finding this, we may see approximately what it must be. The "quartiles" of the above distribution occur at 96 grams and 104 grams, the semi-interquartile range is 4 grams and the probable error of the mean is therefore about

$$\frac{4}{\sqrt{400}} = 0\cdot2 \text{ gram.}$$

Urban² indicates three processes for finding this quantity more accurately. One of these he does not recommend or use. It is the ordinary algorithm of Least Squares, and gives about 0·16 gram. The two formulae which Urban does use are based on an inverse use of Bernoulli's Theorem, and are both incorrect³. They give values 2·832 grams and 2·373 grams respectively, more than ten times too large.

In his first formula Urban assumes that with each of the comparison weights r_x there is associated a probability P_x that it will be recorded as a reading of the just perceptible positive difference. The frequencies N in Table I, when divided by 400, are experimental determinations of

¹ Urban, *op. cit.* 322, Table 18.

² Urban, *op. cit.* 313-317.

³ I am indebted to Professor Karl Pearson for confirmation of this statement.

these probabilities. The inverse use of Bernoulli's Theorem gives as the probable error of P_κ

$$\omega_\kappa = .6745 \sqrt{\frac{P_\kappa(1-P_\kappa)}{s}} \dots\dots\dots(1),$$

where s is the total number of experiments (400). The mean just perceptible difference T is

$$T = \frac{\sum_1^n N_\kappa r_\kappa}{s} = \sum_1^n P_\kappa r_\kappa \dots\dots\dots(2).$$

Therefore (and this is where the mistake occurs) the probable error F of T will be given by

$$F^2 = \sum_1^n \omega_\kappa^2 r_\kappa^2 \dots\dots\dots(3).$$

But this last step would only be correct if the P 's were independent of one another. By the nature of their formation, however, their sum is necessarily unity. The largest possible value for ΣPr is therefore 108 grams and the smallest possible value 84 grams, a range of 24 grams. Were the P 's independently measured, their sum would not necessarily be unity. Independent measurement would mean that in one set of experiments, using all the weights, we would ascertain how often the weight 84 grams was noted as a just perceptible positive difference and how often it was not so noted, and nothing else. Then in *another* set of experiments we would do the same for 88 grams, and so on for each weight. Now the chances might possibly be against each weight in turn just as we were doing the set of experiments which were focussed upon it; or on the other hand the chances might be in favour of each weight in turn just at the right time. In the first case each P might even be zero, which would give a value zero for the threshold ΣPr . In the second case each P might even be unity (except the P at 84 grams, which must in any case be nearly zero, for otherwise we would simply take a still lower weight as the beginning of our set of comparison weights). This would then give a value of 588 grams for the threshold, namely

$$\Sigma Pr = 88 + 92 + 96 + 100 + 104 + 108.$$

The possible range assumed by equation (3) for the threshold is therefore from zero to 588 grams. Of course no experimenter would accept such results, but Urban's formula assumes their possibility. Were experiments really made independently, they would be continued until ΣP approximated to unity and then the values would be adjusted

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as are the angles of a closed polygon in a survey. We may expect therefore that Urban's probable error 2.832 will be too large in something like the proportion of these two ranges and that the correct value will be approximately of the order

$$\frac{108 - 84}{588 - 0} \times 2.832 = 0.12 \text{ gram,}$$

a value much more in accordance with what we found by Least Squares¹.

Urban's other process is based upon the actual number of answers *heavier* recorded for each comparison weight. These were as follows in 450 trials²:

TABLE II.

Comparison weights <i>r</i>	84	88	92	96	100	104	108
Answers <i>heavier</i>	1	9	40	100	186	403	423

Urban now assumes that associated with each comparison weight r_k there is a probability p_k that the subject will answer *heavier*. The numbers in Table 2, when divided by 450, are experimental determinations of these probabilities. Each is therefore subject to a probable error

$$\omega_k = .6745 \sqrt{\frac{p_k q_k}{s}} \dots\dots\dots(4),$$

where

$$s = 450 \text{ and } q = 1 - p.$$

The former probabilities P can be calculated from the p 's and then the threshold T can be found from them. Since therefore T is ultimately compounded of the values p , it ought to be possible to calculate its probable error from the probable errors of the p 's; and since these latter probabilities are quite independently measured, the objection previously raised does not hold here. Unfortunately Urban performs the algebra in two steps, calculating first the probable errors of the P 's

¹ This is not suggested as an exact or practical way of finding the probable error. The alternative to Least Squares, if only Table I is known, is indicated towards the end of this Note, after Urban's second formula has been discussed.

² Multiply the numbers in Urban, *op. cit.* 287, Table 11, column *Vp. I grösser*, by 450; or read direct from Urban, *Application of Statistical Methods to the Problems of Psychophysics*, Philadelphia, 1908, 174, Table 3.

and from these that of T , thus reintroducing the same mistake. His second equation on page 316 *op. cit.* is incorrect. It assumes that

$$\frac{dT}{dP_\kappa} = r_\kappa \dots\dots\dots(5),$$

which is not true. The correct formula is

$$P^2 = 2\rho^2 \sum_1^n \left(\frac{dT}{d\rho_\kappa} \right)^2 \frac{p_\kappa q_\kappa}{s_\kappa} \dots\dots\dots(6),$$

where $\sqrt{2\rho^2} = \cdot 6745$.

After performing the differentiation, and remembering that q_n and p_1 must nearly equal zero, we get

$$F = \cdot 6745 \sqrt{\sum_{\kappa=1}^{\kappa=n} \frac{p_\kappa}{s_\kappa q_\kappa} \left(\sum_{\lambda=\kappa}^{\lambda=n} P_\lambda r_\lambda - r_\kappa \sum_{\lambda=\kappa}^{\lambda=n} P_\lambda \right)^2} \dots\dots\dots(7),$$

a much simpler formula to calculate than Urban's. It gives in the present case the value 0.133 gram approximately. This is the formula for just perceptible positive differences, that is for ascents. For descents interchange p and q , and the suffixes 1 and n . Similar formulae hold for the negative differences.

Urban's conclusion that the Limiting Process of calculation is, for what it attempts, more accurate than calculation by the $\Phi(\gamma)$ hypothesis is of course not altered by this correction: the accuracy is even greater than he supposed, and is hardly distinguishable from that of the Lagrange interpolation formula,—as might be expected, since both processes are alike in accepting the data as given and in finding the fifty per cent. point without any attempt at smoothing the curve.

The writer, in the article cited above on the Method of Serial Groups, fell into the same error. The Limiting Process is an extreme form of the Group Process, and the writer checked his equation IV¹ by seeing that for certain values it reduced to Urban's equation. The correct form of equation IV is

$$F = \cdot 6745 C \sqrt{\sum_{\kappa=1}^{\kappa=n} \frac{p_\kappa^{2t+1} q_\kappa^{2g-2t-1}}{s_\kappa l_\kappa^2} \left(\sum_{\lambda=1}^{\lambda=\kappa} \mathfrak{P}_\lambda' r_\lambda - r_\kappa \sum_{\lambda=1}^{\lambda=\kappa} \mathfrak{P}_\lambda' \right)^2} \dots\dots(8),$$

where the letters have the same meaning as in the article quoted. For the Limiting Process, $t = 0$, $g = 1$, $l = p$ and $C = 1$; and the equation reduces to equation (7) for descents.

The equation on page 409 of the same article is also incorrect. This is Urban's case (1a) where the P 's but not the p 's are known², to which

¹ This *Journal*, 1913, v. 415, Appendix II.

² Urban, *op. cit.* 313.

he applies his first formula discussed above. If it is desired to avoid making the assumption of a certain distribution which underlies the Least Squares process, then the only way of handling this case would be to calculate the p 's from the P 's and use the observed P 's and calculated p 's in equation (7) or (8) as the case might be.

The incorrect probable errors in the writer's article are, like Urban's, too large; but the difference is not so great, because an absolute not a difference threshold is being calculated, so that the values of r are small. The correct probable errors are from one-third to a quarter of those given; and fortunately this proportion is sufficiently constant to keep the various forms of the process in the same order of merit. The two general conclusions on page 412 are therefore still correct although the advantage of small groups is weakened.

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