face (ii.), so as to touch alternately the two ovals of this curve of intersection. Taking any portion of it, the tangent lines along it touch the surface (i.), and, when produced geodesically along it, form the corresponding arcs of the geodesic system of (i.), *i.e.*, those arcs which run in the same direction, in the sense that we can proceed by continuous change from the one to the other through the intermediate arcs.

If, therefore, we take any two points A and B on (i.), the corresponding geodesic arcs drawn from them, and carried geodesically on to (ii.), finally run into one another and are continuous.

Again, if we take any point on (ii.) and follow the two geodesic arcs of the system which cross each other there, they will run on to (i.) so as to be non-corresponding geodesic arcs on it.

If, therefore, we take any two points A and B on (i.), the non-corresponding geodesic arcs drawn from them and carried geodesically on to (ii.) will meet in a point P on (ii.), such that the sum or difference of AP and BP is constant, according to the relative position of A and B considered fixed, in a way that is most easily illustrated by the analogy of Roberts' theorem for a single quadric.

10. If the quadric (i.) is taken to be a focal conic, and the points A and B are taken on its boundary, all lines through A and B are tangential to (i.), so that the condition of geodesic contact with (i.) imposes no restriction.

Thus taking, for definiteness, (ii.) to be an ellipsoid, we see that a thread drawn tight over it between two points A and B on opposite quadrants of the focal hyperbola is free to slip all over it; and also that the locus of the space-foci, in the geodesic sense, of the lines of curvature is the focal hyperbola, which, of course, contains the umbilics.

On the Motion of Two Spheres in a Liquid, and Allied Problems.

By A. B. Basset, M.A.

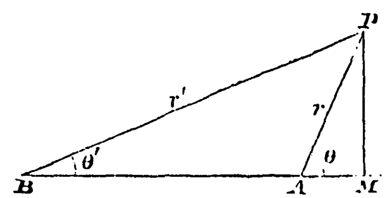
[*Read May 12th*, 1887.]

1. When two spheres are moving in an infinite liquid along the line joining their centres, the coefficients of the velocities in the expression for the kinetic energy, have been completely determined by the method of images, in the form of infinite series, by Mr. W. M.

Hicks*; but when the spheres are moving perpendicularly to the line joining their centres, the successive images become so very complicated that Mr. Hicks has been compelled to abandon the attempt of obtaining the complete solution, and to resort to an approximate one.

In the present paper I propose to explain a different method, by means of which approximate values of the coefficients may be obtained in a series of powers of the inverse distance of the centres of the two spheres, and the calculation is carried as far as the inverse twelfth power. I have also pointed out that the same method may be applied to obtain an approximate solution of the problem of two electrified spheres, which are placed in a field of force whose potential is given.

2. It will first be necessary to establish the following proposition.



In the figure, let $PM = \rho$, $AM = z$, $BM = z'$, $AB = c$, $\cos\theta = \mu$, $\cos\theta' = \mu'$; also let $P_n^m(\mu)$ be an associated function of degree n and order m, whose origin is A, and axis is AM; and let $P_n'^m(\mu')$ denote a similar function having the same axis and whose origin is B. Then we shall prove that, when $r < c$,

$$\frac{P_n'^m}{r'^{n+1}} = \frac{r^m}{(n-m)!\,c^{n+m+1}}\left[\frac{(n+m)!}{2m!}P_m^m - \frac{(n+m+1)!}{(2m+1)!}\frac{r}{c}P_{m+1}^m + \dots + \frac{(-)^s(n+m+s)!}{(2m+s)!}\left(\frac{r}{c}\right)^s P_{m+s}^m + \dots\right] \dots (1),$$

and when $r' < c$,

$$\frac{(-)^{n-m}P_n^m}{r^{n+1}} = \frac{r'^m}{(n-m)!\,c^{n+m+1}}\left[\frac{(n+m)!}{2m!}P_m'^m + \frac{(n+m+1)!}{(2m+1)!}\frac{r'}{c}P_{m+1}'^m + \frac{(n+m+s)!}{(2m+s)!}\left(\frac{r'}{c}\right)^s P_{m+s}'^m + \dots\right] \dots (2).$$

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* *Phil. Trans.*, 1880.

[A complete account of the history of this problem, together with a description of the papers by Bjerknes and others, will be found in Mr. W. M. Hicks' Report on Hydrodynamics to the British Association, 1882, Part II., pages 13 to 16. Since the publication of this report, the following papers have appeared:—

K. Pearson, "On the Motion of Spherical and Ellipsoidal Bodies in Fluid Media." *Quarterly Journal*, Vol. XX., p. 60.

A. H. Leahy, "On the Pulsations of Spheres in an Elastic Medium," *Camb. Trans.*, Vol. XIV., p. 45.

R. A. Herman, "On the Motion of Two Spheres in a Fluid, and Allied Problems," *Quarterly Journal*, Vol. XXII., p. 204 (June, 1887).

In the last paper the approximation is carried as far as the *fifteenth* power of the inverse distance between the centres of the spheres (see p. 229), and the results agree with those given in the present paper. (*June 10th*.)]

It is known that P_n^m can be expressed in either of the forms

$$M(1-\mu^2)^{\frac{1}{2}m}\int_0^\pi \{\mu+\sqrt{\mu^2-1}\cos\phi\}^{n-m}\sin^{2m}\phi\,d\phi,$$

or
$$M(1-\mu^2)^{\frac{1}{2}m}\int_0^\pi \frac{\sin^{2m}\phi\,d\phi}{\{\mu+\sqrt{\mu^2-1}\cos\phi\}^{n+m+1}},$$

where
$$M=\frac{(n+m)!}{(n-m)!\,1.3\ldots(2m-1)\,\pi}.$$

Therefore

$$\frac{P_n'^m(\mu')}{r'^{n+1}} = M\rho^m\int_0^\pi \frac{\sin^{2m}\phi\,d\phi}{(z'+\iota\rho\cos\phi)^{n+m+1}}$$
$$= M\rho^m\int_0^\pi \frac{\sin^{2m}\phi\,d\phi}{\left[c+r\{\mu+\sqrt{\mu^2-1}\cos\phi\}\right]^{n+m+1}};$$

whence, if $\lambda=\mu+\sqrt{\mu^2-1}\cos\phi$, and $r<c$,

$$\frac{P_n'^m(\mu')}{r'^{n+1}} = \frac{M\rho^m}{c^{n+m+1}}\int_0^\pi\Big\{1-(n+m+1)\frac{r\lambda}{c}+\frac{(n+m+1)(n+m+2)}{2!}\left(\frac{r\lambda}{c}\right)^2\ldots$$
$$\ldots+\frac{(-)^s(n+m+1)\ldots(n+m+s)}{s!}\left(\frac{r\lambda}{c}\right)^s+\ldots\Big\}\sin^{2m}\phi\,d\phi;$$

whence, by the first form of P_n^m, we obtain

$$\frac{P_n'^m}{r'^{n+1}} = \frac{r^m}{(n-m)!\,c^{n+m+1}}\left[\frac{(n+m)!}{2m!}P_m^m-\frac{(n+m+1)!}{(2m+1)!}\frac{r}{c}P_{m+1}^m+\ldots\right.$$
$$\left.\ldots+\frac{(-)^s(n+m+s)!}{(2m+s)!}\left(\frac{r}{c}\right)^s P_{m+s}^m+\ldots\right].$$

In order to obtain the second equation, let us change θ and θ' into their supplements; then, since

$$P_n^m\{\cos(\pi-\theta)\} = (-)^{n-m}P_n^m(\cos\theta),$$

we obtain

$$\frac{(-)^{n-m}P_n^m(\mu)}{r^{n+1}} = \frac{r'^m}{(n-m)!\,c^{n+m+1}}\left[\frac{(n+m)!}{2m!}P_m'^m+\frac{(n+m+1)!}{(2m+1)!}\frac{r'}{c}P_{m+1}'^m\right.$$
$$\left.\ldots+\frac{(n+m+s)!}{(2m+s)!}\left(\frac{r'}{c}\right)^s P_{m+s}'^m+\ldots\right].$$

The corresponding formulæ when $r>c$ could be easily obtained, but they are not required for the present investigation.

3. Let A and B be the centres of the two moving spheres, a and b their radii, v_1, v_2 their parallel component velocities perpendicular to

AB. Also let ϕ_1 be the velocity potential of the liquid when A is moving with velocity v_1, whilst B is kept at rest, and let ϕ_2 be the velocity potential when B is in motion and A is fixed. Then if ϕ be the velocity potential of the whole motion,

$$\phi = \phi_1 + \phi_2 \quad \dots\dots\dots\dots\dots\dots (3).$$

The problem is therefore reduced to the determination of ϕ_1, for when this is known, ϕ_2 can be written down by symmetry.

Let χ be the angle which a plane through AB and any point P makes with the plane through AB which contains the directions of motion of A and B; also let Q_n, Q'_n be written for P^1_n and P'^1_n. Then, in the neighbourhood of A, ϕ_1 must be expressible in the form of the series

$$\phi_1 = \left\{ -\frac{v_1 a^3 Q_1}{2r^3} + A_1\left(r + \frac{a^3}{2r^2}\right) Q_1 + A_2\left(r^2 + \frac{2a^5}{3r^3}\right) + \dots \right\} \cos\chi \quad \dots (4),$$

for this value of ϕ_1 satisfies the surface condition

$$\left(\frac{d\phi_1}{dr}\right)_a = v_1 \sin\theta \cos\chi.$$

In the neighbourhood of B, ϕ_1 must be expressible in the form

$$\phi_1 = \left\{ B_1\left(r' + \frac{b^3}{2r'^2}\right) Q'_1 + B_2\left(r'^2 + \frac{2b^5}{3r'^3}\right) Q'_2 + \dots \right\} \cos\chi \quad \dots\dots (5),$$

for the value of ϕ_1 satisfies the surface condition

$$\left(\frac{d\phi_1}{dr'}\right)_b = 0.$$

The series consisting of powers of r^{-1} and r'^{-1} are convergent at all points outside the two spheres, but the series consisting of powers of r and r' will be divergent if r and r' be sufficiently great; but we shall only require these latter series in the neighbourhood of the two spheres where they are convergent.

The kinetic energy consists of a series of terms of the form

$$\int_0^{2\pi} d\chi \int_0^{\pi} Q_n v_1 a^2 \cos^2\chi \sin^2\theta \, d\theta = \pi a^2 v_1 \int_{-1}^{1} (1-\mu^2)\frac{dP_n}{d\mu} d\mu$$

$$= 2\pi a^2 v_1 \int_{-1}^{1} \mu P_n \, d\mu$$

$$= \frac{4\pi a^2 v_1}{3} \quad (n = 1) \quad \dots\dots\dots\dots (6)$$

$$= 0 \quad (n \text{ any other value}).$$

Hence the terms involving Q_2, Q_3, &c. contribute nothing to the energy, and we may therefore, in writing down the final value of ϕ_1, reject all terms except those involving Q_1 or Q_1'.

Dropping the factor $\cos\chi$ for the present, we should have, if B were absent,

$$\phi_1 = -\frac{v_1 a^3 Q_1}{2r^2}.$$

Putting $m = 1$, $n = 1$ in (2), the value of this near B is

$$\phi_1 = -\frac{v_1 a^3}{2c^3}\left(r'Q_1' + \frac{r'^2 Q_2'}{c} + \frac{r'^3 Q_3'}{c^3} + \frac{r'^4 Q_4'}{c^4} + \dots\right).$$

From (5) it follows that, in order to make the velocity at B vanish, we must add the series

$$-\frac{v_1 a^3}{2c^3}\left(\frac{b^3 Q_1'}{2r'^2} + \frac{2b^5 Q_2'}{3cr'^3} + \frac{3b^7 Q_3'}{4c^2 r'^4} + \frac{4b^9 Q_4'}{5c^3 r'^5} + \dots\right).$$

Transforming this latter series back again to A by (1), and retaining the important terms only, the value of ϕ_1 near A becomes

$$\phi_1 = -\frac{v_1 a^3 Q_1}{2r^2} - \frac{v_1 a^3 b^3}{c^6}\left(\frac{1}{4} + \frac{b^2}{c^2} + \frac{9b^4}{4c^4} + \frac{4b^6}{c^6}\right) Qr + \frac{v_1 a^3 b^3}{4c^7} Q_2 r^2.$$

In order to satisfy the surface conditions at A, add the terms

$$-\frac{v_1 a^3 b^3}{c^6}\left(\frac{1}{4} + \frac{b^2}{c^2} + \frac{9b^4}{4c^4} + \frac{4b^6}{c^6}\right)\frac{Q_1 a^3}{2r^2} + \frac{v_1 a^8 b^3 Q_2}{6c^7 r^3}.$$

Neglecting powers of c^{-1} higher than the twelfth, the value of these added terms near B is

$$-\frac{v_1 a^6 b^3}{2c^9}\left(\frac{1}{4} + \frac{b^2}{c^2}\right) Q_1' r' - \frac{v_1 a^8 b^3}{2c^{11}} Q_1' r'.$$

Adding the terms $-\dfrac{v_1 a^6 b^3}{4c^9}\left(\dfrac{1}{4} + \dfrac{b^2}{c^2} + \dfrac{a^2}{c^2}\right)\dfrac{Q_1' b^3}{r'^2}$ ....(7),

omitting Q_2', &c., and restoring $\cos\chi$, the value of the velocity potential near B becomes

$$\phi_1 = -\frac{v_1 a^3}{2c^3}\left\{1 + \frac{a^3 b^3}{4c^6} + \frac{a^3 b^3 (a^2 + b^2)}{c^8}\right\}\left(r' + \frac{b^3}{2r'^2}\right) Q_1' \cos\chi \quad \dots(8).$$

The first term of (7) on transformation becomes

$$-v_1 a^6 b^6 Q_1 r / 16c^{12},$$

whence the value of ϕ_1 near A is

$$\phi_1 = -\frac{v_1 a^3 Q_1}{2r^2}\cos\chi - \frac{v_1 a^3 b^3}{c^6}\left\{\frac{1}{4}+\frac{b^2}{c^2}+\frac{9b^4}{4c^4}+\frac{(a^3+64b^3)\,b^3}{16c^6}\right\}$$
$$\times\left(r+\frac{a^3}{2r^2}\right)Q_1\cos\chi \quad\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

The values of ϕ_2 at A and B can be written down by symmetry; whence, if T be the kinetic energy of the system

$$2T = A'v_1^2+2B'v_1v_2+C'v_2^2,$$

where

$$\left.\begin{aligned} A' &= m_1 - \iint \phi_1\frac{d\phi_1}{dn}\,dS_1 \\ &= m_1+\tfrac{1}{2}M_1\left[1+\frac{3a^3b^3}{c^6}\left\{\frac{1}{4}+\frac{b^2}{c^2}+\frac{9b^4}{4c^4}+\frac{b^3(a^3+64b^3)}{16c^6}\right\}\right] \\ C' &= m_2+\tfrac{1}{2}M_2\left[1+\frac{3a^3b^3}{c^6}\left\{\frac{1}{4}+\frac{a^2}{c^2}+\frac{9a^4}{4c^4}+\frac{a^3(b^3+64a^3)}{16c^6}\right\}\right] \\ B' &= -\rho\iint \phi_1\frac{d\phi_2}{dn}\,dS_2 \\ &= \frac{\pi\rho a^3b^3}{c^3}\left\{1+\frac{a^3b^3}{4c^6}+\frac{a^3b^3(a^2+b^2)}{c^8}\right\} \end{aligned}\right\}\ldots(10),$$

where m_1, m_2 are the masses of the spheres A and B; M_1, M_2 those of the liquid displaced by them, and ρ is the density of the liquid.*

4. When the motion takes place along the line of centres, the energy could be calculated in a similar manner; but in this case it is better to employ the method of images. By means of the formulæ given on p. 463 of Mr. Hicks' paper, it can easily be shown that, if the kinetic energy be $(Au_1^2-2Bu_1u_2+Cu_2^2)/2$, then

$$\left.\begin{aligned} A &= m_1+\tfrac{1}{2}M_1\left\{1+\frac{3a^3b^3}{c^6}\left(1+\frac{3b^2}{c^2}+\frac{6b^4}{c^4}+\frac{11b^6}{c^6}\right)\right\} \\ B &= \frac{2\pi\rho a^3b^3}{c^3}\left\{1+\frac{a^3b^3}{c^6}+\frac{3a^3b^3(a^2+b^2)}{c^8}\right\} \\ C &= m_2+\tfrac{1}{2}M_2\left\{1+\frac{3a^3b^3}{c^6}\left(1+\frac{3a^2}{c^2}+\frac{6a^4}{c^4}+\frac{11a^6}{c^6}\right)\right\} \end{aligned}\right\}\ldots\ldots(11).$$

These formulæ give the values of the coefficients as far as c^{-12}.

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* There seems to be an error in equation (17) of Mr. Hicks' paper. I find

$$\frac{\mu_1}{\mu_0} = \tfrac{1}{2}\left\{\left(\frac{ab}{c^2-b^2}\right)^3 - \frac{a^3b^3}{2c^2(c^2-b^2)^2}\right\},$$

which is of the order c^{-6}. The next ratio μ_2/μ_0 is of the order c^{-12}, and the above expression for μ_1/μ_0 agrees with (10) as far as c^{-10}, as ought to be the case.

5. If a sphere is projected in a liquid which is bounded by a fixed plane, we must put $a = b$, $u_1 = -u_2 = u$, $v_1 = v_2 = v$; then

$$2T = (A+B)\,u^2 + (A'+B')\,v^2,$$

and, if higher powers than c^{-6} be neglected,

$$A+B = m + \tfrac{1}{2}M\left(1 + \frac{3a^3}{c^3} + \frac{3a^6}{c^6}\right),$$

$$A'+B' = m + \tfrac{1}{2}M\left(1 + \frac{3a^3}{2c^3} + \frac{3a^6}{4c^6}\right),$$

where $c/2$ is the distance of the sphere from the plane. Lagrange's equation

$$\frac{d}{dt}\,\frac{dT}{du} - 2\,\frac{dT}{dc} = 0$$

gives $$(A+B)\,\dot{u} = v^2\,\frac{d}{dc}\,(A'+B') - u^2\,\frac{d}{dc}\,(A+B).$$

Also, since the momentum parallel to the plane is constant,

$$(A'+B')\,v = \text{const.} = G.$$

Let V be the resultant velocity of the sphere, θ the angle which its direction makes with the normal to the plane, then

$$(A+B)\,\dot{u} = V^2\left\{\sin^2\theta\,\frac{d}{dc}\,(A'+B') - \cos^2\theta\,\frac{d}{dc}\,(A+B)\right\}$$

$$= \frac{9MV^2a^3}{2c^4}\left\{\left(1+\frac{2a^3}{c^3}\right)\cos^2\theta - \tfrac{1}{2}\left(1+\frac{a^3}{c^3}\right)\sin^2\theta\right\}.$$

If, therefore, $$\tan\alpha = \sqrt{\frac{2\,(c^3+2a^3)}{c^3+a^3}},$$

it follows that, whenever the direction of motion makes with the normal to the plane an angle which is $< \alpha$ or $> \pi - \alpha$, the sphere will be repelled from the plane; but, whenever this angle lies between α and $\pi - \alpha$, the sphere will be attracted. Also, since $A'+B'$ increases as c diminishes, the velocity parallel to the plane will be accelerated when the direction of motion lies between α and $\pi - \alpha$; and retarded when this direction makes with the normal an angle $< \alpha$ or $> \pi - \alpha$. If, therefore, the sphere be projected parallel to the plane, it will ultimately strike it.

Mr. Hicks has elsewhere shown,* that in the case of a cylinder

* *Quarterly Journal*, Vol. XVI., p. 195.

$\alpha = \pi/4$, and we have shown that in the case of a sphere $\alpha > \pi/4$. The discussion of the subsequent motion of a sphere projected in any given direction in a liquid bounded by a fixed plane, can be carried on in the same manner as in the corresponding case of a cylinder, but it must be recollected that the preceding values of the coefficients will not give correct results if the sphere gets too close to the plane.

6. In order to obtain, by the preceding method, the potential when two charged spheres are placed in a field of electric force, we must first expand the potential of the field into two convergent series of spherical harmonics, whose origins are the centres of the two spheres respectively, and then obtain the potential corresponding to each term separately and add the results.

Let us illustrate this by considering a field whose potential is $L + Mz$.

Let E_1, E_2 be the charges, and V the potential of the two spheres; A and B the constant values of the potential of the spheres and field at the surface of the two spheres; then, near A,

$$V = A - L\frac{a}{r} - \frac{Ma^3P_1}{r^2} + A_0\left(1 - \frac{a}{r}\right) + A_1\left(\frac{r}{a} - \frac{a^2}{r^2}\right)P_1$$

$$+ \dots A_n\left(\frac{r^n}{a^n} - \frac{a^{n+1}}{r^{n+1}}\right)P_n,$$

also

$$\frac{E_1}{a} + L = -A_0,$$

whence

$$V = A - L - \frac{Ma^3P_1}{r^2} - \frac{E_1}{a}\left(1 - \frac{a}{r}\right) + \Sigma_1^\infty A_n\left(\frac{r^n}{a^n} - \frac{a^{n+1}}{r^{n+1}}\right)P_n,$$

Similarly, near B,

$$V = B - L + Mc - \frac{Mb^3P_1'}{r'^2} - \frac{E_2}{b}\left(1 - \frac{b}{r'}\right) + \Sigma_1^\infty B_n\left(\frac{r'^n}{b^n} - \frac{b^{n+1}}{r'^{n+1}}\right)P_n'.$$

And the successive coefficients must be determined by putting $m = 0$ in (1) and (2), in the manner already explained.

I have elsewhere pointed out that, if ψ be the current function when a solid of revolution is moving parallel to its axis with velocity V, then $2\psi\rho^{-1}\sin\chi$ is the potential of the induced charge when the solid is placed in a field of electric force whose potential is

$$-V\rho\sin\chi,*$$

where z, ρ, χ are cylindrical coordinates.

The value of ψ for a doublet of strength m, whose axis coincides with that of z is $m\rho^2/r^3$, and therefore, when the spheres are moving along the line of centres, the complete value of ψ can be obtained by the method of images. Hence the potential can be obtained when the spheres are placed in a uniform field of force, which is perpendicular to a fixed plane which passes through their centres.

Second Note on Elliptic Transformation Annihilators.

By JOHN GRIFFITHS, M.A.

[*Read May 12th*, 1887.]

If P and Q be algebraic functions of x, involving the constant k, such that $y = \dfrac{P}{Q}$ leads to the transformation

$$\frac{dy}{\sqrt{1-y^2\,.\,1-\lambda^2y^2}} = \frac{M\,dx}{\sqrt{1-x^2\,.\,1-k^2x^2}},$$

where
$$M^2 = \frac{nkk'^2}{\lambda\lambda'^2}\,\frac{d\lambda}{dk},$$

then there are two operators Ω and O which reduce $P\div Q$ and $\lambda P\div Q$, respectively, to zero, *i.e.*, which give

$$\Omega\,\frac{P}{Q} = 0,\quad O\,\frac{\lambda P}{Q} = 0.$$

If the partial differentiation operators $\dfrac{d}{dk}$ and $\dfrac{d}{dx}$ be denoted by ∂_k and ∂_x, the general forms of Ω, O are

$$\Omega = nkk'^2\partial_k + \sqrt{1-x^2\,.\,1-k^2x^2}\;G\,(x,\,k)\,\partial_x,$$

$$O = nkk'^2\partial_k + \sqrt{1-x^2\,.\,1-k^2x^2}\;g\,(x,\,k)\,\partial^x,$$

where n is a number.

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* *Proc. Camb. Phil. Soc.*, Vol. VI., p. 7.