was engaged in revising the first edition of the *Principia*. The reference to prop. 7 cor. 3 looks at first sight as if the manuscript were subsequent to the second edition; but I think that this reference relates merely to the manuscript of his proposed addition to that proposition, of which the three rules mentioned above are, I believe, a rough draft. On the other hand, the numerical calculation relating to book III. prop. 4 gives a number as occurring on page 406, line 25, and can refer only to the first edition; moreover many of the remarks alluded to above in art. 3 would be meaningless if written subsequent to the publication of the second edition in 1713. Altogether I feel no doubt that the manuscript was written before the issue of the second edition.

The Harmonic Functions for the Elliptic Cone. By E. W. HOBSON.

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The harmonic functions for the circular cone were introduced by Mehler; an account of his theory is given in Heine's *Kugelfunctionen*. In the present communication, I give some indications of a theory of the more general harmonic functions which are required for the corresponding potential problems connected with the elliptic cone; I propose to call these harmonics *elliptic conal harmonics*.

It is first shown that the normal functions are of the form

$$\frac{1}{\sqrt{r}}\,{\sin \atop \cos}\,(p\log r)\,A_p(\mu)\,B_p(\nu),$$

where r is the radius vector, and μ, ν are elliptic coordinates, the latter referring to the elliptic cones, and p is a constant; the functions $A_p(\mu)$, $B_p(\nu)$ satisfy differential equations which are the same as Lamé's, except that the degree n is no longer a positive integer, but a complex quantity $-\tfrac{1}{2}+pc$, so that the functions are really Lamé's functions of complex degree. I have next considered the forms of the solutions of the differential equations satisfied by

$A_p(\mu)$, $B_p(\nu)$ which must be taken for the potential problem. There are no solutions, as in Lamé's case, which can be found in finite terms. The constant parameter which occurs in the differential equations is determined from the condition that $A_p(\mu)$ must be periodic with respect to a certain angle ϕ, of which μ is a function; it appears that $A_p(\mu)$ falls into four distinct classes corresponding to Lamé's four classes, and that the constant is determined as a root of one of four transcendental equations. I have next considered the nature of the roots of these equations, and shown that these roots are all real and are contained between certain intervals. It is next shown that two solutions $B_p(\nu)$, $B_p(-\nu)$ of the equation satisfied by B, exist, one of which is finite everywhere within the cone, but infinite over a certain space lying in a principal plane of the cone produced beyond the vertex.

So far as I am aware, Lamé's equation has been hitherto studied only in the case in which n is a positive integer (by Lamé and Hermite), and in the case in which n is half an odd positive integer.* It appeared to me that the indications I have been able to give as to the nature of the solutions in the case of these more complicated functions, might not be without interest.

It may be remarked that the problem of electrical distribution on an infinite plate bounded by two straight edges meeting at a point, is solvable in terms of these functions. Mehler's functions are, of course, a particular case of the functions I have considered.

1. If for the rectangular coordinates x, y, z we put

$$x = r\frac{\mu\nu}{bc}, \quad y = r\frac{\sqrt{\mu^2-b^2}\,\sqrt{b^2-\nu^2}}{b\sqrt{c^2-b^2}}, \quad z = r\frac{\sqrt{c^2-\mu^2}\,\sqrt{c^2-\nu^2}}{c\sqrt{c^2-b^2}},$$

where $c > b$, we have $\qquad x^2+y^2+z^2 = r^2,$

$$\frac{x^2}{\mu^2}+\frac{y^2}{\mu^2-b^2}-\frac{z^2}{c^2-\mu^2} = 0,$$

$$\frac{x^2}{\nu^2}-\frac{y^2}{b^2-\nu^2}-\frac{z^2}{c^2-\nu^2} = 0;$$

thus the equations $r =$ constant, $\mu =$ constant, $\nu =$ constant, represent three systems of surfaces cutting each other orthogonally, the

* See Halphen's *Fonctions elliptiques*, Vol. II., p. 482; also Lindemann, *British Association Report*, 1883.

first being concentric spheres, and the second and third two systems of confocal cones having their vertices at the centre of the spheres. We are about to consider the potential problem for spaces in which a boundary is the half-cone for which ν is constant and positive; for the other half of the cone ν will be negative. We observe that ν may have values of 0 to b^2, and μ^2 may have values from b^2 to c^2.

It is known* that Laplace's equation $\nabla^2 V = 0$, when the quantities r, μ, ν are considered as the coordinates of a point, takes the form

$$(\mu^2-\nu^2)\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{\partial^2 V}{\partial \epsilon^2}+\frac{\partial^2 V}{\partial \zeta^2}=0,$$

where $$\mu = c\,\mathrm{dn}\,(K-c\epsilon,\, k), \quad \nu = b\,\mathrm{sn}\,(\zeta,\, k'),$$

k being $\frac{1}{c}\sqrt{c^2-b^2}$, and k' the complementary modulus; this equation is satisfied by $(Cr^n+Dr^{-n-1})\,A(\mu)\,B(\nu)$, where $A(\mu)$, $B(\nu)$ satisfy the equations

$$\frac{d^2 A(\mu)}{d\epsilon^2}+\{n(n+1)\mu^2-(b^2+c^2)\,a\}\,A(\mu)=0,$$

$$\frac{d^2 B(\nu)}{d\zeta^2}-\{n(n+1)\,\nu^2-(b^2+c^2)\,a\}\,B(\nu)=0,$$

a being any constant quantity.

If n is determined so that the solution $(Cr^n+Dr^{-n-1})\,A(\mu)\,B(\nu)$ vanishes for two given values r_1 and r_2 of r, we must have

$$Cr_1^n+Dr_1^{-n-1}=0, \quad Cr_2^n+Dr_2^{-n-1}=0;$$

hence $$r_1^{2n+1}=r_2^{2n+1},$$

or $$(2n+1)\log\frac{r_1}{r_2}=2k\iota\pi,$$

where $\iota=\sqrt{-1}$, and k is an integer; hence

$$n=-\tfrac{1}{2}+\iota\frac{k\pi}{\log\dfrac{r_1}{r_2}}.$$

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* See Heine's *Kugelfunctionen*, p. 354.

Thus, as in the case of Mehler's functions, we take n of the form $-\frac{1}{2}+\iota p$, for harmonics adapted to the case of the cone; the normal functions are therefore of the form

$$\frac{1}{\sqrt{r}}\frac{\sin}{\cos}p\rho \,.\, A_p(\mu)\, B_p(\nu),$$

when $$\rho = \log r,$$

and where $A_p(\mu)$, $B_p(\nu)$ satisfy the equations

$$\frac{d^2A_p(\mu)}{de^2}-\left[(p^2+\tfrac{1}{4})\mu^2+(b^2+c^2)a\right]A_p(\mu)=0 \quad \text{.........(1)},$$

$$\frac{d^2B_p(\nu)}{d\zeta^2}+\left[(p^2+\tfrac{1}{4})\nu^2+(b^2+c^2)a\right]B_p(\nu)=0 \quad \text{.........(2)},$$

which differ from Lamé's equation only in having $-\frac{1}{2}+p\sqrt{-1}$ for n; thus $A_p(\mu)$, $B_p(\nu)$ may be regarded as Lamé's functions of complex degree.

These equations are reducible to

$$(\mu^2-b^2)(\mu^2-c^2)\frac{d^2A(\mu)}{d\mu^2}+\mu(2\mu^2-b^2-c^2)\frac{dA(\mu)}{d\mu}$$
$$+\left[(b^2+c^2)a+(p^2+\tfrac{1}{4})\mu^2\right]A(\mu)=0,$$

and a precisely similar equation in ν.

2. In order to determine the forms of $A(\mu)$, we shall put

$$\cos\phi=\sqrt{\frac{\mu^2-b^2}{c^2-b^2}},$$

and we observe that, in order that the normal function may be a potential function either for the space inside the half-cone $\nu =$ constant, or for the space outside this half-cone, it is necessary that $A(\mu)$ should be a periodic function of ϕ, the period being 2π; the values of the constant a will be determined for given values of p by means of a certain convergency condition.

The equation $$\frac{d^2A}{de^2}-\left[(p^2+\tfrac{1}{4})\mu^2+(b^2+c^2)a\right]A=0 \quad \text{...........(1)}$$

is equivalent, when μ is the independent variable, to the equation

$$(\mu^2-b^2)(\mu^2-c^2)\frac{d^2A}{d\mu^2}+\mu(2\mu^2-b^2-c^2)\frac{dA}{d\mu}$$
$$+\left\{(b^2+c^2)\,a+(p^2+\tfrac{1}{4})\,\mu^2\right\}A=0.$$

Putting $\qquad \mu=(b^2\sin^2\phi+c^2\cos^2\phi)^{\frac{1}{2}},$

and making ϕ the independent variable, this equation becomes

$$(b^2\sin^2\phi+c^2\cos^2\phi)\frac{d^2A}{d\phi^2}-(c^2-b^2)\sin\phi\cos\phi\frac{dA}{d\phi}$$
$$-\left\{(b^2+c^2)a+(p^2+\tfrac{1}{4})(b^2\sin^2\phi+c^2\cos^2\phi)\right\}A=0,$$

or, writing $\qquad \dfrac{c^2-b^2}{c^2+b^2}=\kappa,$

$$(1+\kappa\cos2\phi)\frac{d^2A}{d\phi^2}-\kappa\sin2\phi\frac{dA}{d\phi}-\left\{2a+(p^2+\tfrac{1}{4})(1+\kappa\cos2\phi)\right\}A=0.$$

To solve this equation, we may take A of one of the four forms which correspond to Lamé's four classes,

$$\tfrac{1}{2}a_0+a_2\cos2\phi+a_4\cos4\phi+\ldots,$$
$$a_2\sin2\phi+a_4\sin4\phi+a_6\sin6\phi+\ldots,$$
$$a_1\cos\phi+a_3\cos3\phi+a_5\cos5\phi+\ldots,$$
$$a_1\sin\phi+a_3\sin3\phi+a_5\sin5\phi+\ldots.$$

These series are infinite instead of finite as in Lamé's case.

Substituting the first of these series in the differential equation, arranging the result of the substitution in a series of cosines of multiples of ϕ, and equating the coefficient of the general term to zero, we find

$$-4n^2a_{2n}-2(n-1)^2\kappa a_{2n-2}-2(n+1)^2\kappa a_{2n+2}+(n+1)\kappa a_{2n+2}-(n-1)\kappa a_{2n-2}$$
$$-(2a+p^2+\tfrac{1}{4})\,a_{2n}-(p^2+\tfrac{1}{4})\,\tfrac{1}{2}\kappa a_{2n-2}-(p^2+\tfrac{1}{4})\,\tfrac{1}{2}\kappa a_{2n+2}=0,$$

and in particular $\;a_0(2a+p^2+\tfrac{1}{4})+\kappa a_2(p^2+\tfrac{1}{4})=0.$

Arranging the above result, we have for the equation connecting a_{2n+2}, a_{2n}, a_{2n-2},

$$a_{2n+2}\left\{(p^2+\tfrac{1}{4})\,\tfrac{1}{2}\kappa+(n+1)(2n+1)\kappa\right\}+a_{2n}\left\{4n^2+(2a+p^2+\tfrac{1}{4})\right\}$$
$$+a_{2n-2}\left\{(n-1)(2n-1)\kappa+(p^2+\tfrac{1}{4})\right\}\tfrac{1}{2}\kappa=0.$$

Write $$-\beta = 2a+p^2+\tfrac{1}{4},\quad \tfrac{1}{2}\kappa\,(p^2+\tfrac{1}{4}) = \gamma\,;$$

then, changing n into $n-1$, we have

$$a_{2n}\left\{\gamma+n\,(2n-1)\,\kappa\right\}+a_{2n-2}\left\{4\,(n-1)^2-\beta\right\}$$
$$+a_{2n-4}\left\{(n-2)(2n-3)\,\kappa+\gamma\right\}=0 \quad \text{.........(3)}$$

as the equation connecting a_{2n}, a_{2n-2}, a_{2n-4}.

By means of the n equations obtained by giving n the values 1, 2, 3 ... n, we can express a_{2n} in terms of a_0, the value of a_{2n}/a_0 being a rational integral expression in β or in a of degree n. The condition of convergency of the series is that the limit of a_{2n}, when n is indefinitely increased, may be zero; this condition gives the values of β or a as the roots of a transcendental equation which it is necessary to examine.

If any one of the other three series is substituted in the differential equation, we shall obtain in a similar manner a relation connecting the coefficients, and thence a transcendental equation for the determination of β or a, as the convergency condition of the series. It will, however, be sufficient for us to examine the first class of solutions, as the treatment of the other three classes is precisely similar.

3. It will now be shown that all the n roots of the equation $a_{2n}=0$, in the quantity β, are real and unequal.

Writing down the equations (3) for the values 1, 2, 3, ... n of n, we see that, when $\beta=-\infty$, the functions a_0, a_2, a_4, ... a_{2n} have only changes of sign, and when $\beta=+\infty$, they have only continuations; thus, as β changes from ∞ to $-\infty$, n continuations of sign are lost. When $a_{2r}=0$, where $r<n$, we see that a_{2r+2} and a_{2r-2} have opposite signs; hence, as in the case of Sturm's functions, a continuation of signs is lost only when a_{2n} goes through the value zero; it follows that the number of roots of the equation $a_{2n}=0$ in β, between any two given values of β, is equal to the number of continuations of sign in the series a_0, a_2, a_4, ... a_{2n} which are lost as β goes from the larger of the two values to the smaller one, and that all the roots are real and unequal.

Again, if $\beta=4\,(n-1)^2$, we see that a_{2n} and a_{2n-4} have opposite signs; hence $a_{2n}=0$ has one or more roots lying between ∞ and $4\,(n-1)^2$.

Next reduce $\frac{a_{2n}}{a_{2n-2}}$ to a continued fraction of the form

$$\beta-4(n-1)^2-\frac{1}{u_n\{\beta-4(n-2)^2\}-}\ \frac{1}{v_{n-1}\{\beta-4(n-3)^2\}-}\ \&c.,$$

where $$u_n=\frac{n(2n-1)\kappa+\gamma}{(n-2)(2n-3)\kappa+\gamma},\quad v_{n-1}=\frac{u_{n-1}}{u_n},\ \&c.$$

If we put $\beta=4n^2$, it can be seen that each denominator in the continued fraction is greater than 2, and thus that the whole expression is positive. The same is the case if β has any value greater than $4n^2$; hence for the value $\beta=4n^2$ the signs of $a_{2n}, a_{2n-2}, a_{2n-4}, \ldots$ are all the same, and thus there is no root of the equation $a_{2n}=0$ greater than $4n^2$. When $\beta=4(n-1)^2$, the series $a_{2n-2}, a_{2n-4}\ldots$ are all of the same sign, and thus there is only one change in the series $a_{2n}, a_{2n-2}, a_{2n-4}, \ldots$; and therefore the equation $a_{2n}=0$ has one and only one root lying between the values $4n^2$ and $4(n-1)^2$ of β; this root is the greatest one which the equation has.

We shall next show that the equation $a_{2n}=0$ has r roots less than $4r^2$. In the series $a_0, a_2, \ldots a_{2r}$, there are only continuations when $\beta=4r^2$; hence in the series $a_0, a_2, \ldots a_{2n}$ there are $n-r$ changes; thus there are $n-r$ roots of $a_{2n}=0$ greater than $4r^2$, and therefore r roots less than $4r^2$. It has thus been shown that, of the roots of the equation $a_{2n}=0$, one lies between each of the numbers $0, 2^2, 4^2, \ldots (2n)^2$. The roots of $a_\infty=0$ are therefore comprised between $0, 2^2, 4^2, \ldots$, one root being in each interval.

In the special case $\kappa=0$, we have $b=c$, and the differential equation reduces to

$$\frac{d^2A}{d\phi^2}+\beta A=0;$$

in this case the values of β are $0, 2^2, 4^2, \ldots$, and the corresponding values of the functions of this class are 0, $a_2\cos 2\phi$, $a_4\cos 4\phi$, $a_6\cos 6\phi, \ldots$; these are the functions for the circular cone.

In the case of the functions A of the second class, it is shown in a similar manner that there are an infinite number of values of β, corresponding to any one of which a function is determined, and that these values of β lie between the same intervals as in the case of the first class.

In the functions of the third and fourth classes, the values of β lie in the intervals between $0, 1^2, 3^2, 5^2, \ldots$.

In the case of the circular cone ($\kappa = 0$), the four classes of functions A are $\cos 2n\phi$, $\sin 2n\phi$, $\cos(2n+1)\phi$, $\sin(2n+1)\phi$.

4. We have next to consider the form of the function $B_p(\nu)$, a having a value corresponding to a particular function $A_p(\mu)$ as above determined.

We have to integrate the equation

$$(\nu^2-b^2)(\nu^2-c^2)\frac{d^2B(\nu)}{d\nu^2}+\nu(2\nu^2-b^2-c^2)\frac{dB(\nu)}{d\nu}$$
$$+\left[(b^2+c^2)a+(p^2+\tfrac{1}{4})\nu^2\right]B(\nu)=0,$$

in a form suitable for values of ν numerically less than b. The singular points being $\nu = \pm b$, $\nu = \pm c$, the equation has an integral of the form

$$A_0\left\{1+a_2\left(\frac{\nu}{b}\right)^2+a_4\left(\frac{\nu}{b}\right)^4+\dots\right\}$$
$$+B_0\frac{\nu}{b}\left\{1+b_2\left(\frac{\nu}{b}\right)^2+b_4\left(\frac{\nu}{b}\right)^4+\dots\right\},$$

which is convergent when $\nu < b$, A_0, B_0 denoting arbitrary constants, and $a_2, a_4, \dots, b_2, b_4, \dots$ quantities which can be determined. When $\nu = \pm b$, each of the series becomes divergent, but it is possible to choose the value of A_0/B_0 so that the whole expression will be convergent for $\nu = b$; for this value of A_0/B_0, the expression will be infinite when $\nu = -b$. Taking for A_0/B_0 the same value as before, but with opposite sign, we obtain a solution which is convergent for $\nu = -b$ but divergent for $\nu = b$. Denoting this value of A_0/B_0 by $1/\lambda$, we see that the equation has two solutions

$$A_0\left\{\left(1+a_2\frac{\nu^2}{b^2}+a_4\frac{\nu^4}{b^4}+\dots\right)+\lambda\frac{\nu}{b}\left(1+b_2\frac{\nu^2}{b^2}+b_4\frac{\nu^4}{b^4}+\dots\right)\right\},$$

$$A_0\left\{\left(1+a_2\frac{\nu^2}{b^2}+a_4\frac{\nu^4}{b^4}+\dots\right)-\lambda\frac{\nu}{b}\left(1+b_2\frac{\nu^2}{b^2}+b_4\frac{\nu^4}{b^4}+\dots\right)\right\},$$

the first of which is finite when $\nu = b$, and is infinite when $\nu = -b$, and the second of which is finite when $\nu = -b$, and infinite when $\nu = b$. If we denote the expression

$$\left(1+a_2\frac{\nu^2}{b^2}+a_4\frac{\nu^4}{b^4}+\dots\right)+\lambda\left(1+b_2\frac{\nu^2}{b^2}+b_4\frac{\nu^4}{b^4}+\dots\right)$$

by $B_p(\nu)$, the complete solution of the equation is

$$B(\nu) = CB_p(\nu) + C'B_p(-\nu).$$

The function $B_p(\nu)$ is finite throughout the space which contains the infinite triangle $\nu = b$, and is infinite in the external space which contains the infinite triangle $\nu = -b$. There exists one such function $B_p(\nu)$ corresponding to each function $A_p(\mu)$. The constant λ is a transcendental quantity to be determined; in the case $(b = c)$ of the circular cone, its value can be determined.*

In this last case the functions $B_p(\nu)$, $B_p(-\nu)$ become Mehler's functions $K(\cos\theta)$, $K(-\cos\theta)$, the first of which is finite for $\theta = 0$, and the second for $\theta = \pi$.

5. As in the case of Lamé's functions, it can be shown that, if $A_{p_1}(\mu)$, $A_{p_2}(\mu)$ be two functions corresponding to a given value of p, and both belonging to the same one of the four classes, then

$$\int_0^{\omega} A_{p_1} A_{p_2} d\epsilon = 0.$$

To expand a function of μ in a series of the functions A, we divide the function into four parts

$$f_1(\mu) + \sqrt{\mu^2-b^2}\sqrt{c^2-\mu^2}\, f_2(\mu) + \sqrt{\mu^2-b^2}\, f_3(\mu) + \sqrt{c^2-\mu^2}\, f_4(\mu),$$

corresponding to the four classes of A functions; where $f_1(\mu)$, $f_2(\mu)$, $f_3(\mu)$, $f_4(\mu)$ are rational functions of μ; then, assuming the possibility of the expansions, the coefficients in the expansion may be determined as in the case of Lamé's functions.

6. Suppose it is required to find a potential function, inside the space bounded by the semi-cone $\nu = \nu_0$, which shall have a given value over the surface of this boundary. Let

$$V = \frac{1}{\sqrt{r}} f(r, \mu)$$

be the given value of V when $\nu = \nu_0$; then

$$\frac{1}{\sqrt{r}} f(r, \mu) = \frac{1}{2\pi\sqrt{r}} \int_{-\infty}^{\infty} dp \int_0^{\infty} f(\sigma, \mu) \cos p(r-\sigma)\, d\sigma$$

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* See *Camb. Phil. Trans.*, Vol. XIV., p. 218.

for all values of r between 0 and ∞. Next, suppose $f(\sigma, \mu)$ expanded in a series of A functions corresponding to the value p, say $\sum_{r=1}^{\infty} a_r A_{pr}$; we have then

$$\frac{1}{\sqrt{r}} f(r, \mu) = \frac{1}{2\sqrt{\pi r}} \int_{-\infty}^{\infty} dp \int_0^{\infty} \Sigma a_r A_{pr} \cos p\,(r-\sigma)\, d\sigma;$$

the required value of the potential function within $\nu = \nu_0$ is

$$\frac{1}{2\pi\sqrt{r}} \int_{-\infty}^{\infty} dp \int_0^{\infty} \Sigma a_r \frac{B_{pr}(\nu)}{B_{pr}(\nu_0)} A_{pr} \cos p\,(r-\sigma)\, d\sigma.$$

The potential function for the space outside the boundary $\nu = \nu_0$, which has the same value as before over the boundary, is

$$\frac{1}{2\pi\sqrt{r}} \int_{-\infty}^{\infty} dp \int_0^{\infty} \Sigma a_r \frac{B_{pr}(-\nu)}{B_{pr}(-\nu_0)} A_{pr} \cos p\,(r-\sigma)\, d\sigma.$$

Next consider the space bounded by two spheres $r = a$, $r = b$, and by the conal surface $\nu = \nu_0$; suppose the potential function V is to have the value zero over each of the spherical boundaries and to have prescribed values over the conal boundaries. In this case the values of p are $\dfrac{k\pi}{\log \frac{a}{b}}$, where k is an integer.

The value of V must be of the form

$$\sum_{k=1}^{k=\infty} \frac{1}{\sqrt{r}} \sin\left\{\frac{k\pi \log \frac{r}{b}}{\log \frac{a}{b}}\right\} \sum_{s=1}^{s=\infty} a A_{k\pi/(\log a/b), s}(\mu) \frac{B(\nu)}{B(\nu_0)},$$

the constants a being determined from the assigned values of V over the conal surface $\nu = \nu_0$.