ON THE QUESTION OF THE EXISTENCE OF TRANSFINITE NUMBERS

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IN a recent paper in these *Proceedings** Dr. Hobson has initiated a discussion of the existence of certain transfinite numbers. His arguments may conveniently be divided into two classes. Firstly, while agreeing with me that the series (W) of all ordinal numbers, arranged in order of magnitude, has no type and no associated cardinal number,† he puts forward the suggestion that some *segment* of W may be "inconsistent," in the sense in which I used this word.‡

Secondly, there is his requirement of a "norm" for the definition of an aggregate,§ and the objections connected therewith to an infinite series of acts of arbitrary selection.‖

With regard to the first class of arguments, I give (§ 1) an exact statement of what I meant by the term "inconsistent," which seems to have been misunderstood by many people. In fact, an "inconsistent" aggregate is an aggregate (which is itself defined in a manner free from self-contradiction) of which the cardinal number (and type, if it is ordered) is contradictory; thus, I see no reason for denying the existence of W,¶ but I do see reason for denying that W has a type or associated cardinal number. Further, Hobson's remark** on the possible introduction of contradiction by Cantor's second principle of generation ignores the character of this principle; for the essence of it is, as shown in §§ 3–5, the constant avoidance of contradiction, while the difficulty as to the type of W is simply that the second principle is not applicable. And this requires the discussion of § 4 to show that there are no ordinal numbers other than those to which Cantor's third principle applies. Lastly,

* "On the General Theory of Transfinite Numbers and Order Types," *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 3 (1905), pp. 170–188.

† *Loc. cit.*, pp. 170–171, 180, 185.

‡ *Loc. cit.*, pp. 171–172.

§ *Loc. cit.*, pp. 172–175.

‖ *Loc. cit.*, pp. 182–188.

¶ Indeed, I admit a well-ordered series such that W is ordinally similar to merely a *segment* of it (§ 1).

** *Loc. cit.*, pp. 171–172, 178, 180.

Hobson's own criteria of existence,* and his objections to Russell,† are easily shown to fail (§ 6).

With regard to the second class of arguments, Hobson's requirement of a "norm" (§§ 8–9) can be proved rigorously to be too narrow (§ 10), but his objections to an infinite series of arbitrary selections, although intermixed with a psychology which, I think, is irrelevant to mathematics, lead to a discussion of what is known as the "multiplicative axiom" (§§ 11–13), which is necessary and sufficient to justify the process of arbitrary selection. Finally (§ 14), I have tried to state the grounds there are for admitting that the axiom in question is both evident and true.

1.

Hobson's statement‡ of the contradiction arising from supposing the series W to have a type does not seem to me to be the best possible. If, he says, we suppose that every well-ordered series has an ordinal number, W has one, which must be the maximum ordinal β. But, if we place any element after W, we form a series of the greater type $\beta+1$.

My own form was: If W has a type (β), we have, by this very supposition, indicated a series (W, β) of type (if it has one) $\beta+1$. This form is advantageous, because one emphasises, by using it, the fact that the supposition that W has a type implies that W is a segment of a well-ordered series; and this fact is necessary to refute a certain explanation of Burali-Forti's contradiction.§

But there is no reason for making the contradiction depend on the additional assumption that a series, like (W, β), which transcends W, has a type. For,‖ if W has an ordinal number β, β must occur in W, the series of *all* ordinals, and consequently

$$\beta > \beta,$$

a manifest contradiction.

Now, I see no reason for denying, with Hobson,¶ the existence of the series W (and the "field of the relation"** W, the class w of all ordinal numbers); for not only does there appear to be nothing objectionable in the definition of a series (W, γ) transcending W, so long as γ is not said to be the type of its preceding series W, but also, if there were no w, it is not evident what meaning could be given to the phrase "γ is an ordinal

* *Loc. cit.*, pp. 177–179, 180–181.

† *Loc. cit.*, pp. 179–180.

‡ *Loc. cit.*, p. 170.

§ Due to Bernstein ("Uber die Reihe der transfiniten Ordnungszahlen," *Math. Ann.*, Bd. LX., 1905, pp. 187–193; cf. *ibid.*, pp. 469–470).

‖ See *Math. Ann.*, Bd. LX., 1905, p. 466.

¶ *Loc. cit.*, p. 180.

** See Russell, *The Principles of Mathematics*, Vol. I., Cambridge, 1903, p. 97.

number," or $\gamma\epsilon\iota\upsilon$, in the symbolic logic of Peano and Russell, which is a necessary hypothesis to any theorem on ordinal numbers.*

There seemed, indeed, to be a ground for the assertion of the non-existence of w and W when it was assumed that every propositional function defines a class; † for, if u is a class, the propositional function—"the z's such that u is similar ‡ to z"—would define a class, the cardinal number of u.§ But when the hypothesis is abandoned, as it is now ‖ by Russell, from the non-existence of a cardinal number (or type, as the case may be) of u does not follow the non-existence of u.

As regards the above definition of series transcending W, W must, it appears, be distinguished from any of the series such that every well-ordered series is ordinally similar either to it or to a segment of it.¶ There *is* a contradiction involved in supposing any one of these series** ($\mathfrak{W}$) to be followed by a term (for, if it could, the new series would not be ordinally similar to $\mathfrak{W}$ or to any segment of it),†† and hence W is ordinally similar to a *segment* merely of $\mathfrak{W}$.

Thus, what I denoted as "inconsistent" aggregates or classes are not non-entities (which would make it absurd to speak of them as a kind of class), but classes such that cardinal numbers of them (and types, if the classes are ordered) are non-entities. The terms "non-entity" and "non-existent" must be kept distinct: Russell, following Peano, applied the term "non-existent" to classes which are null,‡‡ and considered that all the "existence-theorems" of mathematics were proofs that the various classes defined are not null. It was only later that "non-entities" were discovered; these were the results of giving propositional functions certain values,§§ looked like classes, and, if *every* propositional function were to

* [However, this difficulty is, I hear, avoided by Russell in his "No-Classes Theory." Here "γ is an ordinal number" is only a short way of stating something which does not imply a conception of *class*.—April 30th, 1906.]

† Russell, *op. cit.*, p. 20.

‡ *Op. cit.*, p. 113. This word replaces Cantor's *äquivalent*.

§ *Op. cit.*, p. 115.

‖ The first intimation of this I had in a letter of May 15th, 1904.

¶ Cf. *Phil. Mag.*, Jan. 1904, p. 67, and Jan. 1905, pp. 51, 53.

** Mr. Russell drew my attention to the fact that such a series is not unique (in fact, given one such series, we can get another such by interchanging two of its terms), and hence that my use of the words "*the* series," in the passages quoted in the preceding note, is incorrect.

†† Thus, we cannot remove the first term and put it after $\mathfrak{W}$; for there is no "after $\mathfrak{W}$," though there *is* an "after W."

‡‡ *Op. cit.*, pp. 32, 73-76.

§§ For example, such non-entities are "all the x's such that x is not a member of x" (Russell's contradictory "class"), and "all the x's such that x is similar to the class of ordinal numbers," which is, according to Russell, "the cardinal number of w" (a part of Burali-Forti's contradiction).

define a class, would be classes, and, indeed, *existent* (not null) classes. But "non-entities" are not classes, not even the null class, but are nothing at all, because self-contradictory, and have the characteristic property that they can be proved formally to be members of the null class.*

2.

Hobson finds the following difficulty:—The series of ordinal numbers and Alephs arise, says he, from the fundamental principle that every well-ordered aggregate has both a type and a cardinal number, but this principle fails with the series W of all ordinal numbers, owing to Burali-Forti's contradiction, and yet it is by means of this very principle that the existence of the successive ordinal numbers which make up W is regarded as having been established. And, further, it is not clear that *every segment* of W must have a type and cardinal number, and a criterion is needed which will enable us to distinguish (at least, theoretically) aggregates which have numbers from those which have not, at some less advanced stage than that at which we define W.

I have departed somewhat from Hobson's phraseology,† but I think I have reproduced accurately the substance of his difficulties. As regards the last, it is evident that we cannot, in a systematic exposition of the theory, use W itself as such a criterion. Historically, W was the first series without a type to be discovered, but we cannot say that we are to judge whether a well-ordered series has a type or not by seeing whether it is similar to a segment of W or not, without a palpable vicious circle. I pointed this out in my first paper‡ on the series in question, and suggested another criterion which will be discussed below (§ 7).

But the other difficulties seem to arise from a misconception of the relation of Burali-Forti's contradiction to Cantor's principles of generation of the ordinal numbers, which is fully discussed in the next section (§ 3), while this is illustrated, in § 4, by the incompleteness of Burali-Forti's argument, and the completion which I gave it.§

* See an article by myself entitled "De Infinito in Mathematica" in Peano's *Rivista di Matematica* (t. VIII., 1906).

† In particular, I avoid the use which Hobson makes, and I formerly made, of the term "inconsistent aggregate." In fact the term is misleading; for I see no reason to think that the aggregate is inconsistent (does not exist), but only that it has no type and no cardinal number (see § 1).

‡ *Phil. Mag.*, Jan., 1904, p. 67.

§ *Ibid.*, Jan., 1905, pp. 51–53.

3.

Cantor's first and second principles of generation of ordinal numbers may be combined and stated in the form: Whenever, starting with the ordinal number 1, we have a finite or infinite series, we posit (or create) a *new* number which is the next greater number to all the numbers of the series, provided always* that the new entity which we postulate forms, together with the old ones, a logically consistent scheme.† In fact, Cantor‡ has stated his belief that "mathematics is completely free in its development, and has only to pay attention to the self-evident condition that its conceptions are both free from self-contradiction and, in determinate relations, fixed by definitions, to the conception already present and verified. In particular, with the introduction of new numbers, it has only to give definitions of them by which such a definiteness, and, under circumstances, such a relation to the older numbers, is afforded that they can be distinguished from one another in given cases."§ Thus we form, successively, the numbers

$$1,\ 2,\ \ldots,\ \nu,\ \ldots,\ \omega,\ \omega+1,\ \ldots,\ \omega+\nu,$$
$$\ldots,\ \omega.2,\ \ldots,\ \omega.\nu,\ \ldots,\ \omega^2,\ \ldots,\ \omega^\omega,\ \ldots,\ \omega^{\omega^\omega},\ \ldots,\ \epsilon,\ \ldots,\ \alpha,\ \ldots,$$

of the second number-class, and then

$$\omega_1,\ \omega_2,\ \ldots,\ \omega_\omega,\ \ldots,\ \omega_{\omega_\omega},\ \ldots,\ \omega_\gamma,\ \ldots,$$

where ω_γ is the first number of the γ-th number-class [or the $(\gamma+2)$-th class, if γ is less than ω].

In particular, ω is not the greatest finite number (which is a self-contradictory conception, since we can easily prove that there is no greatest finite number), but is the first number which follows (is greater than) all the finite numbers and, consequently, is *transfinite*. Similarly, ω_1 is a number, transfinite indeed like the numbers of the second class, but of the *third class*. It is highly important to dwell on these distinctions; for, as I will show, Burali-Forti's contradiction (in its completed form) is analogous to the contradiction arising from the statement that there is a greatest finite ordinal number, in that both contradictions arise from supposing a next greater to all the numbers of a certain class to exist which is itself a member of that class.

* This addition shows that Cantor did not use Hobson's "principle" that *every* well-ordered series has a type.

† Cantor, *Grundlagen einer allgemeinen Mannichfaltigkeitslehre*, Leipzig, 1883, p. 33.

‡ *Ibid.*, pp. 19, 45–46.

§ It cannot, I think, be maintained as an historical fact that any advance in mathematics has been brought about by an arbitrary creation of the mind, of which the fruitfulness has only been discovered afterwards, but one cannot object, on logical grounds, to such a creation if non-contradictory.

(*a*) Suppose that there is a greatest finite ordinal number n. Let w be the class of finite integer ordinal numbers; then the proposition "m is a member of w" implies that $n > m$. But n is a member of w; consequently $n > n$, a palpable contradiction.

(*b*) Suppose that there is a greatest ordinal number β. Then, as in § 1, we get, if w is the class of ordinal numbers and β is a member of w, $\beta > \beta$.

Now (*b*) is a form of Burali-Forti's contradiction, and it is evident that both contradictions arise from the hypothesis that n [in (*a*)] or β [in (*b*)] is a member of w. If this is denied, the contradiction is avoided; thus there is no contradiction in the inequality $n > m$, for every finite m, if, for example, $n = \omega$; and the generalisation of this argument to the case of w being some or all of the numbers of, or preceding, a certain number-class, while n is the first number of the next number-class, is immediate.

Now, if we attempt a like alteration in (*b*), we fail, unless all ordinal numbers γ, which were defined or indicated by Cantor, show themselves as merely particular cases of a more general class of ordinal numbers. That this possibility was not to be thrown aside at once, the following considerations show.*

Suppose that the series W of all the numbers defined or indicated by Cantor has a type, which is an ordinal number, β; then the cardinal number corresponding to W is easily proved to be $\aleph_\beta$. On the other hand, we can show that β, if it exists, is the first number of a number-class, and thus of the form $\beta = \omega_\gamma$; while, if γ is a Cantor's ordinal number, the cardinal number of the ordinal numbers less than ω_γ is $\aleph_\gamma$.

Thus, then, unless

$$\aleph_{\omega_\gamma} = \aleph_\gamma, \tag{1}$$

* Cf. *Phil. Mag.*, Jan., 1905, p. 52. We see without difficulty that, if u be a class of ordinal numbers (and, if γ is a u, all less than γ are to be members of u), then, in order that there may be a type of the series of numbers of u, arranged in order of magnitude, it is necessary and sufficient that we should be able to define without contradiction a class of numbers containing u as a *proper* part. Now Cantor's advance lies in the perception that it is useful and indeed necessary to introduce what he called "number-classes" other than that of the finite integers; and I think it is clearly shown in the place referred to that Cantor's creation of "number-classes" is not possible *beyond* the series W. Hence it is not the case that Cantor's method of postulation leads to contradiction; such an idea arises from a neglect of the essential in Cantor's method, namely, that, if γ is an ordinal number, it is of the ζ-th class (ζ being some ordinal number), and hence there are numbers greater than γ [of the $(\zeta+1)$-th class, for example].

The first number of the $(\zeta+2)$-th class (or ζ-th, if $\zeta \geqslant \omega$) is ω_ζ, and ω_γ never $< \gamma$, it may be $> \gamma$, but I see no reason why there should not be γ's such that $\omega_\gamma = \gamma$ (in other words, why the limit of $\omega, \omega_\omega, \omega_{\omega_\omega}, \ldots$ should not exist). Hence my construction in the *Messenger of Maths.*, 1905, pp. 56–58, appears false.

we have no grounds for asserting that the series of Cantor's ordinal numbers has not a definite type (β); β would be, not the greatest ordinal number [a conception which leads to the contradiction (b)], but the least ordinal number which is greater than all of Cantor's ordinal numbers.

4.

The contradiction of Burali-Forti was, then, incompletely stated. Either it is a statement about the series of Cantor's numbers, in which case it is necessary for its validity to prove the equation (1); or else it is a statement about the *whole* series of ordinal numbers, in which case it is valid, but appears to use implicitly the conception of certain numbers whose existence has never been contemplated by Cantor or (if I except myself, for a short time) any one else.

However, we *can* prove (1), and hence that every ordinal number is a Cantor's ordinal number. In fact, if there is such a type as β or ω_γ, γ cannot be a Cantor's number, and hence, since γ can never be greater than ω_γ,

$$\gamma = \beta \qquad \text{or} \qquad \omega_\beta = \beta,$$

whence (1) follows.

5.

It will now, I think, be clear that Cantor's second principle of generation creates new numbers* by the constant avoidance of such contradictions as (a). Hence, instead of saying that the unrestricted application of the second principle leads to contradiction, it seems to me to be more correct to say that in the case (b) the second principle cannot be applied.

Further, I think that the above considerations show that doubts as to the existence (that is to say, the non-contradictory nature) of the type of any segment of W can hardly be maintained seriously.†

6.

Hobson proposes to adopt the "less ambitious procedure of postulating the existence of definite ordinal numbers of a limited number of classes

* I may be permitted to protest again here (cf. *Phil. Mag.*, Jan., 1905, p. 51) against the arbitrary restriction of Cantor's second principle to the *second* class, and the pretended necessity of a *third* principle to create ω_1, and so on. The second principle (together with the first), as stated by Cantor, suffices to create all the ordinal numbers.

† By this I mean that "the type of a segment of W" is not contradictory in the same way as "the type of W," and the latter is the only type of a series of ordinals which has been shown to lead to contradictions. It cannot, I think, be serious mathematics to doubt the existence of a thing without any reason for doing so, but, if anything, with reasons against doing so.

in accordance with Cantor's earlier method.* So long as the postulation of the existence of ordinal numbers does not go beyond some definite point, no contradiction will arise, and the utility of the scheme, for purposes of representation, will suffice to justify the postulations which have been made." † In this way, Burali-Forti's contradiction is certainly avoided, but Hobson's view does not, on this account, appear to have "an advantage over that of Russell," ‡ unless, indeed, discretion is the better part of valour.

Further, we must guard against confusing the utility which a mathematical conception may have—and but for which it would hardly have been conceived—with the purely logical question of whether the conception is possible (exists, or is non-contradictory).§

But, apart from these considerations, Hobson's criterion for the existence of a number || does not appear to be above criticism. This criterion (for ordinal numbers) seems to be capable of statement in the form: "The existence of an ordinal number cannot be inferred from the existence of that *single* series of the preceding ordinal numbers, but it can if, and only if, other series other than the above number-series (and similarly ordered to it), such as series of points on a straight line, can be exhibited." The motive for this requirement of *other* series was that, if only one series is considered, we must leave out of account that conception of a number as the common characteristic of *many* similarly ordered series.

But, in the first place, if we have one series A, we can obtain other series similarly ordered to A by interchanging two terms of A or by replacing a term by something else.¶ In the second place, since a point-series is merely a picturesque way of describing a series of real numbers, I am at a loss to understand why a series of integers should not be allowed a type until a series of real numbers can be found to support its claim. Moreover, it seems inconsequent to accept the number continuum (as a sort of substratum), if the existence of $\aleph_1$ is doubted because

* That is to say, the method of 1882 (see Cantor's *Grundlagen*, pp. 32–35, and § 3 above). In Cantor's memoir of 1897 (*Math. Ann.*, Bd. XLIX., pp. 207–246) the first order of things (which now seems preferable, because of Burali-Forti's contradiction) is inverted, and the principles of generation given a secondary place (p. 226).

† *Loc. cit.*, p. 176.

‡ *Loc. cit.*, p. 180.

§ See § 3 of this paper.

|| *Loc. cit.*, pp. 176–181.

¶ This is sufficient to show that those classes which are called "numbers" by Russell all have (except 0) more than one element; whereas the contrary was stated by Hobson (*loc. cit.*, p. 179).

Hobson doubts Hardy's construction of an aggregate of points of this cardinal number.

The criticism* of Russell's† objection to the assumption that a class of similar classes has "a common characteristic" contained in the words: "The mind does, however, in point of fact, in the case of finite aggregates at least, recognize the existence of such single entity, the number of the aggregates," must be mentioned. This is an appeal to common sense, and is quite irrelevant here. Russell, by defining numbers as classes, avoided the introduction of new indefinables, while such numbers satisfy all the formal laws.‡ Hobson would, apparently, introduce a new indefinable, "the mind," into mathematics, and make psychology a foundation of mathematics. It is, I think, true that in the history (and teaching) of mathematics one should endeavour to present the development of the science as a succession of human documents (of mathematicians), but such psychological information is irrelevant and intolerable in mathematics regarded as a body of logical doctrine.

7.

It is important to define a series ordinally similar to W, but in which numbers are not mentioned. My two attempts both failed: the first§ because $\mathfrak{W}$ is not a substitute for W, the second‖ (as already mentioned in a note to § 3) because it seems that we can find genuine ordinal numbers (γ) for which

$$\omega_\gamma = \gamma.$$

I see no alternative but to define such a series as any well-ordered series such that its "type" is a non-entity, while that of any segment of it is not. If we suppose the elements of this series to form a class, we can formally prove that class to be a member of the null-class, as indicated at the end of § 1.

8.

For the definition of an aggregate,¶ a "norm" is, according to Hobson,

* *Loc. cit.*, pp. 179-180.

† *Op. cit.*, pp. 114-115.

‡ I only mention this one advantage of Russell's definition; the other "common sense" objections were already sufficiently dealt with in *The Principles*, pp. 114-115, 304-307. [See also the Note at the end of this paper.]

§ *Phil. Mag.*, Jan. 1904, p. 67.

‖ "The Definition of a Series similarly ordered to the Series of all Ordinal Numbers," *Mess. of Math.*, 1905, pp. 56-58.

¶ Hobson also deals with *series*, and says (p. 174): "In order that a transfinite aggregate may be capable of being ordered, a principle of order must be explicitly or implicitly contained in the norm by which the aggregate is defined." I am unable to see in this more than the truism: "In necessary. A "norm" is the word he uses for "a law or set of laws by which the aggregate is defined," and he proceeds: "It is, however, convenient to admit the case of two or more alternative sets of conditions: thus an aggregate may contain all objects each of which satisfies either the conditions A or else one of the sets of conditions $B, C, \ldots, K$. The conditions forming the norm by which the aggregate is defined must be of a sufficiently precise character to make it logically determinate as regards any particular object whatever, whether such object does, or does not, belong to the aggregate."* And,† "In the case of a finite aggregate, the norm may take the form of individual specification of the objects which form the aggregate."

This, it appears to me, may be expressed more simply by the words: "A class (or aggregate) is all the entities x such that a certain propositional function $\phi(x)$ is true of each." ‡ It should be noted that a definable class need not be an existent class. Thus, if $\phi(x)$ is not true for any x (say: "x is not identical with x"), the class is the *null-class.*§

Hobson's definition seems to be (i.) redundant, since with *every* class it is logically determined whether a particular thing is or is not a member of it, and (ii.) incomplete, since it neglects the fact, which is vital in the explanation of Burali-Forti's (and Russell's ||) contradiction, that there are propositional functions which do not define classes.

As regards (i.), it amounts to an assertion that *every* propositional function is as "sufficiently precise" as Hobson requires it to be. If I may venture to interpret Hobson, it seems to me that he was thinking of propositional functions in which an indefinable occurs which is not one of logic and mathematics (is not a logical constant).¶ Thus "x is a poet" is a function which can hardly be said to divide humanity into two classes —poets and non-poets. For the term "poet" is probably incapable of

order that a transfinite aggregate may be capable of being ordered, it must be capable of being ordered." For I see no reason why an aggregate may not be ordered in accordance with "some law extrinsically imposed upon the aggregate," and, indeed, Zermelo's proof shows this to be possible, provided that his axiom be granted. In fact, Hobson seems to have been led to his condition by such observations as: "It is difficult, if not impossible, to see how order could be imposed upon" the aggregate of all functions of a real variable. But it is well known that it is possible to arrange all *continuous* functions of a real variable (which appears, *a priori*, just as little capable of order) in a series of type θ (for example).

* *Loc. cit.*, pp. 172, 173. Cf. Cantor, *Math. Ann.*, Bd. xx. (1882), p. 114.

† *Loc. cit.*, p. 173.

‡ Russell, *op. cit.*, p. 20.

§ *Ibid.*, pp. 22, 23.

|| *Ibid.*, pp. 79, 80, 101-107, 366-368.

¶ *Ibid.*, p. 3.

definition: if "poetry" is defined as "metrical composition," the present Poet Laureate would be a poet and Walt Whitman would not. And both statements might provoke discussion. Thus, also, the function by which du Bois-Reymond's decimal* is defined is "x is either 0 or 1, as determined by the chance throw of dice," and is not expressed in logical constants (and "chance" is, perhaps, as obscure in meaning as "poetry"). The "series of arbitrary selections," to which Hobson compares du Bois-Reymond's decimal, is, on the contrary, to be justified by an axiom (whose truth can hardly be denied seriously) which is expressible in terms of the logical constants (§§ 11–14).

9.

But the real meaning of Hobson's requirement of a "norm" appears, I think, later. Thus, in order to show that the totality of the numbers of the second number-class, taken in order, has a type or a cardinal number, "it would be necessary to show that a finite set of rules can be set up which will suffice to define a definite object corresponding to each ordinal number of the second class." And Hobson maintains† that an aggregate the elements of which are regarded as being successively defined by an endless series of separate acts of choice cannot be contemplated as existing, but that some "norm" must be assigned by which the successive elements are defined. On these grounds, he objects to Cantor's ‡ proof that every transfinite aggregate has an enumerable component; to Hardy's § proof that every cardinal number is either an Aleph or is greater than all Alephs, and, in particular, that

$$2^{\aleph_0} \geqslant \aleph_1;$$

to my || proof, that every cardinal number must be an Aleph; and to Zermelo's ¶ proof, that every aggregate can be well-ordered.

10.

This restriction of the "norm" to a *finite* set of rules seems to be demonstrably too narrow. In fact, I will show that the class of all entities which are definable by a finite set of rules is of cardinal number

* Hobson, *loc. cit.*, p. 182.

† *Loc cit.*, pp. 182–185.

‡ *Math. Ann.*, Bd. XLVI. (1895), p. 493.

§ *Quart. Journ. of Math.*, Vol. XXXV. (1903), pp. 87–94. Cf. Hobson, *loc. cit.*, pp. 183, 178, 185–188 (where also a method of Young's is noticed; cf. *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 1, p. 243).

|| *Phil. Mag.*, Jan. 1904, pp. 63, 64, 70 (in particular).

¶ *Math. Ann.*, Bd. LIX. (1904), pp. 514–516.

$\aleph_0$,* and we can always define perfectly definitely an enitity not in this class.

The predicate "definable by a finite set of rules" is expressed, more clearly, I think, by the words: "definable by what is, symbolically, a finite series of variables and logical constants,† in which any constant may be repeated a finite number of times," and the theorem in question can be proved in a manner quite analogous to my earlier theorem referred to in the note, as follows.

If $\mathfrak{m}$ and $\mathfrak{n}$ are finite cardinal numbers, and μ is the type of a finite series to which the cardinal number $\mathfrak{m}$ belongs, then the cardinal number of entities definable by a series of type μ of indefinables and variables with $\mathfrak{n}$ different indefinables and variables is at most‡

$$\mathfrak{n}^{\mathfrak{m}}.$$

We get certainly all such definable entities by letting μ take all values less than ω, in succession, and add the results; thus, if $\mathfrak{a}$ is the cardinal number of all such definable entities,

$$\mathfrak{a} \leqslant \sum_{\mu<\omega} \mathfrak{n}^{\mathfrak{m}} \leqslant \sum_{\mu<\omega} \aleph_0^{\mathfrak{m}} = \aleph_0, \tag{2}$$

and we see that, for our purpose, we may take $\mathfrak{n}$ as variable, but finite. Again, since an aggregate of cardinal number $\aleph_0$ (such as the aggregate of finite cardinal numbers) is so definable, we have also

$$\mathfrak{a} \geqslant \aleph_0. \tag{3}$$

From (2) and (3), now, $\mathfrak{a} = \aleph_0.$

Thus, only $\aleph_0$ real numbers are definable by "norms," and yet, by an obvious modification of Cantor's§ process for proving that

$$2^{\aleph_0} > \aleph_0,$$

we can construct a definite real number not defined by a "norm." Thus Hobson's "norms" cannot define all the real numbers which we have seen to be capable of definition.

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* Cf. my proof, in *Phil. Mag.*, Jan., 1905, that the cardinal number of all "actually" (better, *practically*) representable real numbers is $\aleph_0$. Soon after this (April 28th, 1905) Russell communicated to me what is, in effect the above generalisation, and which appears to me to be identical with König's somewhat obscurely expressed theorem (*Math. Ann.*, Bd. LXI., 1905, pp. 156–160) that the cardinal number of all "finitely defined" numbers is $\aleph_0$. [Cf. also A. C. Dixon, *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 4, 1906, pp. 18–20.]

† See Russell, *op. cit.*, pp. 3, 5, 6.

‡ This is an (unattained) upper limit; for some of the "definitions" thus formed are meaningless, just as some "words" of μ letters are.

§ *Jahresber. der Deutsch. Math.-Ver.*, Bd. I., 1892, p. 77.

11.

With regard to Cantor's argument* that it is possible to pick an enumerable aggregate out of *any* given transfinite aggregate (M), from which we may conclude, by the use of the Schröder-Bernstein theorem,† that

$$\mathfrak{m} \geqslant \aleph_0,$$

it rests on an endless series of acts of arbitrary selection of elements of M. But this can be avoided if only we admit the possibility of fixing on a single (arbitrary) element m_1 of M, and on a definite one-one correspondence between M and that part of M which arises by taking m_1 from M.‡ Let, then, $\phi(m_1) = m_2$ be the correlate of m_1, $\phi(m_2) = m_3$ that of m_2, and so on; it is easily seen that the sequence

$$m_1, m_2, m_3, \ldots, m_\nu, \ldots \tag{4}$$

consists of different elements of M, and is enumerable. Cantor's theorem is then proved.§

It is easily seen that, in order to continue the sequence (4) to obtain other elements

$$m_\omega, m_{\omega+1}, m_{\omega+2}, \ldots,$$

it is necessary to establish a *new* correspondence ϕ_1, if this is possible, between M and the part obtained by taking all the elements of (4) from M. Then

$$m_\omega = \phi_1(m_1), \quad m_{\omega+1} = \phi_1(m_\omega), \quad \ldots.$$

And when we try to take from M a sequence of elements of type ω_1 it evidently becomes necessary to perform an endless series of acts of arbitrary selection, in forming the series

$$\phi_1, \phi_2, \phi_3, \ldots;$$

for each of these ϕ's is one out of an infinity of equivalent ones—for there are an infinity of ways of establishing a one-one correspondence between two similar infinite aggregates.

* *Math. Ann.*, Bd. XLVI., 1895, p. 493.

† It has often been thought that this theorem is: If $\mathfrak{a} \geqslant \mathfrak{b}$ and $\mathfrak{b} \geqslant \mathfrak{a}$, then $\mathfrak{a} = \mathfrak{b}$. But this is a simple logical conclusion; while the Schröder-Bernstein theorem may be stated: If a part of the aggregate A is similar (equivalent) to B, then $\mathfrak{a} \geqslant \mathfrak{b}$, the signs $>$ and $=$ having been defined by Cantor (*Math. Ann.*, Bd. XLVI., 1895, pp. 482–484).

‡ This property may, of course, be taken for the *definition* of the "infinite" character of M.

§ The above method, as I subsequently recognised, is that in which Dedekind ("Was sind und was sollen die Zahlen?", 1887 and 1893, § 6) derives a "simply infinite system" (an aggregate arranged in type ω) from any infinite system, and is, in essentials, the method of Schröder and Bernstein, in the proof of the theorem known by their names (cf. Schoenflies, *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, Leipzig, 1900, pp. 16–18; Zermelo, *Gött. Nachr.*, 1901, pp. 34–38; A. E. Harward, *Phil. Mag.*, Oct., 1905, p. 457).

Thus Hardy's extension of Cantor's argument, in which elements corresponding to *higher* number classes are successively removed, and from which follows, by the Schröder-Bernstein theorem, that any cardinal number is either an Aleph or is greater than all Alephs (follows them all when arranged in order of magnitude), cannot be replaced, with any advantage, by an extension of the former method.

It is one of the greatest triumphs of the symbolic logic developed, for the most part, by Peano and Russell to have showed clearly that there is an assumption involved in what is popularly expressed as the performance of any infinite series of acts of arbitrary selection. Before Russell had noticed this, the need of such an assumption to make such arguments rigorous had, indeed, been pointed out by Zermelo, Beppo Levi, and Bernstein,* and, less clearly, Borel; obscurely felt by Hobson and Harward; and disregarded by all others who have written on the subject.

12.

Let us examine more closely the first stage of Hardy's argument. If M be a given transfinite aggregate, we can always, by what precedes, remove an enumerable aggregate from M. Hence, by the Schröder-Bernstein theorem,

$$\mathfrak{m} \geqslant \aleph_0.$$

If we dismiss the case of equality, we are to prove that

$$\mathfrak{m} \geqslant \aleph_1. \tag{5}$$

For this purpose, we remark that, firstly, if γ is any fixed number of the second number class, there is always a part of M similar to the (enumerable) aggregate of ordinal numbers less than γ; secondly, if γ' is any number of the second number class greater than γ, there is always a part of M similar to the aggregate of ordinal numbers less than γ', *and which contains the former part as a proper part.* This last condition is easily seen to be essential, since we cannot prove (5) from the knowledge that M has an enumerable part, by arguing that this *same* part can always be reordered in the type of any number of the second class.

Now, if ϕ is a definite one-one correspondence which images a definite part u of M on the numbers less than γ, there is not only one, but *an infinity*, of correspondences ψ which fulfil the two conditions which we require: (i.) $\psi(u) = \phi(u)$, (ii.) ψ images a part (v) of M on the numbers

* Cf. Bernstein, "Bemerkung zur Mengenlehre," *Gött. Nachr.*, 1904, pp. 1–4.

less than γ';* from which we deduce that u is a proper part of v, and, if z is that part of v which is left when u is taken away, $\psi(z)$ is the aggregate of ordinal numbers ζ, such that

$$\gamma' > \zeta \geqslant \gamma.$$

We have to pick out *one* definite ψ for *each* γ' in order to define a part of M which is of cardinal number $\aleph_1$. That this is possible, since for each γ' there is an infinite choice of such ψ's, though no pre-eminent one (for example, there is not, in general, a "first" or a "last" ψ for each γ'), may be true, and, in fact, has frequently been considered as obvious; that it is a supposition, which needs either a proof or a new axiom asserting it, becomes evident on a closer logical consideration.

13.

If we have a class u defined as "the entities x such that some propositional function (ϕx) is true of them," we cannot derive the legitimacy of some proposition "x is a u" without the premiss "u is not the null-class."† The latter premiss, which may also be expressed: "the class u exists" (or, more popularly, "the class u has at least one member"), plays a very important part in Russell's work, since it is one of his chief objects to prove the "existence theorems of mathematics," that is to say, to prove that the various classes (such as numbers and types) defined in mathematics are not null.

In order to define the product of any class (not necessarily finite) of cardinal numbers, we form, with Whitehead,‡ the conception of the "multiplicative class" of a class of classes no two of which have any term in common. Let k be such a class of exclusive classes; the multiplicative class of k is the class each of whose terms is a class formed by choosing one and only one term from each of the classes (p) which are terms of k. Then the cardinal number of terms in the multiplicative class of k is defined to be the product of all the numbers of

* That there *are* such imagings as ψ is evident from the consideration that, if (ii.) could not be fulfilled at the same time as (i.), there would be a *lowest* ordinal number amongst those equal to or greater than γ and less than *or equal to* γ', to which *no* imaging ψ such that $\psi(u) = \phi(u)$ can correlate an element of M. But this is impossible.

We cannot say, instead of (ii.): "ψ images a part of M on the numbers less than ω_1"; for, although we can prove that there is no number less than ω_1 which is the lowest of those to which no imaging χ such that $\chi(u) = \phi(u)$ corresponds, we cannot prove that ω_1 itself is not.

† Cf. Whitehead, "On Cardinal Numbers," *Amer. Jour. of Math.*, Vol. XXIV., 1902, p. 373.

‡ *Loc. cit.*, pp. 369, 383, 385.

the various classes composing k, and is denoted:

$$\prod_{p \epsilon k} p.$$

The theorem or axiom of the existence of the multiplicative class was necessary to the validity of many of Whitehead's proofs, but was probably considered by him as not needing a proof. A multiplicative class occurs, as we have seen, in the proof that any aggregate has either a cardinal number which is an Aleph or a part similar to the aggregate of ordinal numbers (or Alephs). In this case our supposition may be expressed as follows (an axiom which is equivalent to Zermelo's):—

Consider as argument of a function an aggregate of existent classes (not necessarily exclusive*); we imagine a *many*-valued function defined for the whole of this argument-aggregate in such a way that, if of the argument-classes, $f(x)$ is a class which is any x. Then our axiom is that there exists† a *one*-valued function F of the same argument, such that $F(x)$ is a member of the class $f(x)$.‡

14.

It remains to consider the evidence for the truth of the axiom. It is that this type of argument has been independently used and considered valid by almost all writers on the theory of aggregates. We have seen it used by Cantor, Zermelo, Hardy, and Whitehead; it is involved in some theorems and conceptions of Schoenflies§ and König,‖ and Bernstein¶ has, in consequence of a remark made by Beppo Levi,** explicitly formulated and adopted the axiom, which was involved in his proof that the class of closed aggregates is similar to the continuum; and we may remark that Levi's proof that there is a *definite* way of putting all closed aggregates in a one-one correspondence with the continuum if metrical properties (which Bernstein avoids) are used, supports the truth of the axiom.

* They must be exclusive for Whitehead's purpose.

† That is to say, can be defined "sub specie aeternitatis," if I may so express myself.

‡ We do not assume that *all* such functions F form a class, but merely that the statement that F is a function of the kind required is not false for every F.

§ *Op. cit.*, pp. 9, 13–14, 26, 41.

‖ "Zum Kontinuum-Problem," *Verhandl. des dritten internat. Math.-Kongr.*, 1905, pp. 144–147.

¶ "Bemerkung zur Mengenlehre," *Gött. Nachr.*, 1904, pp. 1–4.

** "Intorno alla teoria degli aggregati," *Lomb. Ist. Rend.*, (2), t. xxxv., 1902, pp. 863–869; cf. Bernstein in *Math. Ann.*, Bd. LXI., 1905, pp. 132, 146.

Also my proof* of the equality†

$$\aleph_\gamma \cdot \aleph_\gamma = \aleph_\gamma ;$$

a proof which, I have learnt,‡ is supported by the authority of Cantor, utilises this axiom; as also does my proof§ that a series with no first term contains a part of type

$$*\omega,$$

a very evident theorem.

[*Note added April* 30*th*, 1906.

Since writing the above, some papers have appeared which must be briefly referred to here.

Russell|| has discussed Hobson's paper, and many of his criticisms are, in substance, the same as mine.¶ In particular, Russell and I were led independently to the observation that certain propositional functions do not define classes: he, from working on his (cardinal) contradiction; I, from working on the ordinal contradiction of Burali-Forti. My own result is implicitly contained in my first paper of 1903,** in which I maintained that W has no type. This point of view is obviously different from the one attributed to me by Russell,†† that the alternative that the type of W does not exist is neglected, while the alternative that W does not exist is accepted. In fact, it has always seemed to me absurd to suppose that there is no such thing as W; for we can, without contradiction, define classes such that there are parts of them similar to the class of ordinal numbers, and the latter class would not exist if W did not.

What justifies the naming of my own theory (with some others, like Cantor's) the "theory of the limitation of size" is that I differ from upholders of the "zig-zag theory," in not admitting that a contradictory "class," like "the x's such that x is not an x," can become an irreproachable class, like "the class of all classes," by adding new members to it. For this reason, then, a supporter of any "limitation of size" theory must

* *Phil. Mag.*, March, 1904, pp. 295–301.

† This equality must be proved before it can be proved that the Alephs form a series (*ibid.*, Jan., 1904, p. 74).

‡ Bernstein (*Math. Ann.*, Bd. LXI., 1905, pp. 150, 151) states that Cantor had communicated this result to him, having proved it in what is, apparently, the method I followed.

§ *Phil. Mag.*, Jan., 1904, p. 65.

|| "On some Difficulties in the Theory of Transfinite Numbers and Order Types," *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 4, pp. 29–53.

¶ *Ibid.*, pp. 30, 31, 40–43.

** *Phil. Mag.*, Jan., 1904, p. 66.

†† *Loc. cit.*, p. 44.

deny both the notion of a class of all classes and the definition of numbers as classes.*

With regard to the "multiplicative axiom" and the more general Zermelo's axiom,† the following remark may be made. Zermelo's axiom is equivalent to the axiom that, if u is a class and v is the class of all non-null classes which are contained in u, then the complex of propositions: w is a class; w is contained in u;‡ x is a v implies, for every such x, that x and w have one, and only one, term in common;—is not false for every value of w. This axiom (cf. § 13 above), which I am accustomed to call the "multiplicative axiom," has the advantage of being quite analogous to the "multiplicative axiom" of Russell, which is used in the conception of the product of an infinity of cardinal numbers, and which results from the above, if we substitute "v is any class of *mutually exclusive* classes" and "w is contained in the logical sum of v."

Hobson§ has devoted a paper to showing that König's distinction between "finitely defined" entities and those which are not so defined is not valid. This is not the place to examine Hobson's arguments, but I may remark that his method‖ is the same as the one I have given in § 10 of my above paper for showing that Hobson's requirement of a "norm"¶ is too narrow.

Finally, Schönflies** suggests that Russell's contradiction is really the statement: "the class of all classes is a member of itself." I fail to see that this is contradiction, and Schönflies†† only assumes that it is so. Also Schönflies'‡‡ remark that such "classes" as Russell's are contradictory can hardly be regarded, as he states, as a solution of the contradiction.]

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* See Russell, *loc. cit.*, p. 39.

† *Ibid.*, pp. 47–53.

‡ Here u is the "logical sum of v."

§ "On the Arithmetic Continuum," *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 4, 1906, pp. 21–28.

‖ *Loc. cit.*, pp. 24, 25.

¶ *Ibid.*, Vol. 3, 1905, p. 180.

** "Über die logischen Paradoxien der Mengenlehre," *Jahresber. d. Deutsch. Math.-Ver.*, Bd. xv., 1906, pp. 19–25.

†† *Loc. cit.*, p. 22.

‡‡ *Loc. cit.*, p. 20.