AN EXPRESSION OF $(1-z)^{-1}$ BY MEANS OF POLYNOMIALS

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IT is well enough known (see, for instance, Borel, *Séries divergentes*, 1901, p. 164) how to deduce a Mittag-Leffler star expansion of a monogenic function from a given explicit expansion of $(1-z)^{-1}$ of the same character. Some readers may be interested in having such an expansion actually set forth; the method is, in part, merely a development of the original method of Runge (*Acta Mathematica*, VI, 1885, p. 237).

Given in the plane of the complex variable z any region of finite dimensions containing no point infinitely near to any point of the real axis from $z = 1$ to $z = +\infty$, it will be shewn how to form a series of polynomials converging uniformly in this region and representing $(1-z)^{-1}$ therein.

Taking first a complex variable $\zeta, = \xi + i\eta$, enclose the points $\zeta = 1$, $\zeta = 1+c$, wherein c is an arbitrary real positive quantity, by a closed curve consisting of (i) the straight lines $\eta = \pm a$, from $\xi = 1$ to $\xi = 1+c$, the quantity a being real and positive and arbitrary (< 1), (ii) a semicircle convex to the origin $\zeta = 0$ satisfying the equation

$$(\xi-1)^2+\eta^2 = a^2,$$

(iii) a semicircle concave to the origin, of equation $(\xi-1-c)^2+\eta^2 = a^2$. Keeping c and a fixed for the present, take a positive integer r so that c/ra is less than unity, $= \sigma$ say; it is supposed that a is less than c, so that $r > 1$; and take

$$c_0 = 1, \quad c_1 = 1+\frac{c}{r}, \quad c_2 = 1+\frac{2c}{r}, \quad \ldots, \quad c_r = 1+c,$$

so that the segment from $\zeta = 1$ to $\zeta = 1+c$ is divided into r equal parts. If $n_1, n_2, \ldots, n_r$ be positive integers, the rational function

$$\frac{1}{1-\zeta}\left\{1-\left(\frac{c_1-1}{c_1-\zeta}\right)^{n_1}\right\}$$

is finite at $\zeta=1$, but has a pole of order n_1 at $\zeta=c_1$; the rational function

$$\frac{1}{1-\zeta}\left\{1-\left(\frac{c_1-1}{c_1-\zeta}\right)^{n_1}\right\}\left\{1-\left(\frac{c_2-c_1}{c_2-\zeta}\right)^{n_2}\right\}^{n_1}$$

is finite at $\zeta=1$, $\zeta=c_1$, but has a pole of order $n_1 n_2$ at $\zeta=c_2$; in general, writing

$$x_s=\left(\frac{c_s-c_{s-1}}{c_s-\zeta}\right)^{n_s} \quad (s=1, \ldots, r),$$

the rational function of ζ expressed by the product

$$U=\frac{1}{1-\zeta}(1-x_1)(1-x_2)^{n_1}(1-x_3)^{n_1 n_2} \ldots (1-x_r)^{n_1 n_2 \ldots n_{r-1}},$$

has a pole only at $\zeta=1+c$, of order $n_1 n_2 \ldots n_r$.

The difference
$$\frac{1}{1-\zeta}-U$$
is of the form $(1-\zeta)^{-1}P$, wherein, if $\rho_1, \rho_2, \ldots, \rho_k$ denote, in turn, x_1, x_2 repeated n_1 times,* x_3 repeated $n_1 n_2$ times, and so on, P is of the form

$$1-(1-\rho_1)(1-\rho_2)\ldots(1-\rho_k),$$

that is of the form $\Sigma\rho_1-\Sigma\rho_1\rho_2+\Sigma\rho_1\rho_2\rho_3-\ldots$;

hence, if $r_i=|\rho_i|$, we have

$$|P| \leqslant \Sigma r_1+\Sigma r_1 r_2+\Sigma r_1 r_2 r_3+\ldots$$
$$\leqslant (1+r_1)(1+r_2)\ldots(1+r_k)-1.$$

But, when ζ is without the closed curve above described round the segment from $\zeta=1$ to $\zeta=1+c$, we have

$$\left|\frac{1}{1-\zeta}\right|<\frac{1}{a}, \qquad \left|\frac{c_s-c_{s-1}}{c_s-\zeta}\right|<\frac{c/r}{a}, \qquad <\sigma.$$

Wherefore

$$\left|\frac{1}{1-\zeta}-U\right|<\frac{1}{a}\left\{(1+\sigma^{n_1})(1+\sigma^{n_2})^{n_1}(1+\sigma^{n_3})^{n_1 n_2}\ldots(1+\sigma^{n_r})^{n_1 n_2 \ldots n_{r-1}}-1\right\}.$$

Now, let ϵ be an arbitrary real positive quantity; take then μ a positive number, such that $e^{\mu}-1<\epsilon a$; suppose further that the positive integers $n_1, \ldots, n_r$ previously used are chosen, so that

$$\sigma^{n_1}<\frac{\mu}{1+\mu}, \quad \text{or} \quad \frac{\sigma^{n_1}}{1-\sigma^{n_1}}<\mu,$$

$$\sigma^{n_2}<\frac{1}{n_1}\sigma^{2n_1}, \quad \sigma^{n_3}<\frac{1}{n_1 n_2}\sigma^{3n_1}, \quad \ldots, \quad \sigma^{n_r}<\frac{1}{n_1 n_2 \ldots n_{r-1}}\sigma^{rn_1};$$

* So that $(1-x_2)^{n_1}$ is replaced by $(1-\rho_2)(1-\rho_3)\ldots(1-\rho_{n_1+1})$.

then, as $1+x < e^x$, the modulus of $(1-\zeta)^{-1}-U$, is less than

$$\frac{1}{a}\{\exp(\sigma^{n_1}+n_1\sigma^{n_2}+n_1n_2\sigma^{n_3}+\ldots+n_1n_2\ldots n_{r-1}\sigma^{n_r})-1\},$$

and thus less than $\frac{1}{a}\{\exp(\sigma^{n_1}+\sigma^{2n_1}+\ldots+\sigma^{rn_1})-1\}$,

which is less than $\frac{1}{a}\left\{\exp\left(\frac{\sigma^{n_1}}{1-\sigma^{n_1}}\right)-1\right\}$;

and therefore less than the arbitrary quantity ϵ.

Now make a change of independent variable, putting

$$z = \frac{c\zeta}{1+c-\zeta},$$

equivalent with
$$\zeta = \frac{(1+c)z}{c+z};$$

thereby the points $\zeta = 0$, $\zeta = 1$, $\zeta = 1+c$ become respectively the points $z = 0$, $z = 1$, $z = \infty$, z being real and positive so long as ζ is real and positive and $\zeta < 1+c$; also

$$(1-z)^{-1} = \frac{c}{1+c}(1-\zeta)^{-1}+\frac{1}{1+c};$$

the function U becomes a rational function of z with a pole only at $z=\infty$, that is, it is a polynomial in z; this we write in the form

$$U = \frac{1+c}{c}H-\frac{1}{c},$$

equivalent with
$$H = \frac{c}{1+c}U+\frac{1}{1+c},$$

so that H is also a polynomial in z; thence

$$(1-z)^{-1}-H = \frac{c}{1+c}\{(1-\zeta)^{-1}-U\}.$$

Further, the extreme points of the closed curve, namely,

$$\xi = 1-a,\ \eta = 0, \quad \text{and} \quad \xi = 1+c+a,\ \eta = 0,$$

become respectively, if $z = x+iy$, the points

$$x = c(1-a)/(c+a),\ y = 0, \quad \text{and} \quad x = -c(1+c+a)/a,\ y = 0;$$

as a approaches to zero the limiting positions of these are respectively $x = 1$, $y = 0$, and $x = -\infty$, $y = 0$. The lines $\eta = \pm a$ become portions respectively of the two circles

$$(x+c)^2+y^2 = \pm \frac{c(c+1)}{a} y,$$

of which the former can be written

$$y = \frac{(x+c)^2}{2\mu} + \frac{(x+c)^4}{2\mu} \{\mu + [\mu^2 - (x+c)^2]^{\frac{1}{2}}\}^{-2},$$

wherein $\mu = \frac{1}{2}c(c+1)/a$; thus, *for a given value of* x, by taking a sufficiently small, c being kept fixed, the ordinate y can be supposed arbitrarily small.

With these materials we can now recapitulate as follows:—Take in the plane of z any region of finite dimensions which does not include any point infinitely near to any point of the real axis from $z = 1$ to $z = +\infty$; take c an arbitrary real positive quantity (<1); take a, also real and positive, so small that the region enclosed as above by portions of the circles

$$(x+c)^2+y^2 = \pm c(c+1)\, y/a,$$

and curves passing through the points

$$z = c(1-a)/(c+a), \qquad z = -c(1+c+a)/a,$$

does not include any point of the originally given region; then take r so that $\sigma = c/ra$ is less than unity; and, taking an arbitrary real positive ϵ, take the positive integers $n_1, n_2, \ldots, n_r$ by the rules previously given. We can then form a polynomial in z, say H, of order $n_1 n_2 \ldots n_r$, such that throughout the originally given region

$$|(1-z)^{-1} - H| < \epsilon.$$

If, then, $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$ be an aggregate of real positive numbers with zero as their limit, and the polynomial H corresponding to the case when ϵ is replaced by ϵ_m be denoted by H_m, the series of polynomials

$$H_1 + (H_2 - H_1) + (H_3 - H_2) + \ldots,$$

whose sum to m terms is H_m, has a sum converging to $(1-z)^{-1}$ uniformly for the whole interior of the given region.