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LXXIII. On the operator ∇ in combination with homogeneous functions

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It must, however, be remembered that although viscosity does not, to any appreciable extent, affect the forms of these ripples, it does very rapidly damp them out. Thus the amplitude of the ripple in fig. 1 is halved in less than one fifth of a second, so that it must be sought within a very few centimetres of the generating source. But, if the ripples of fig. 4 could be produced, they might be expected to travel some twenty or thirty centimetres without any serious diminution of amplitude.

LXXIII. *On the Operator ∇ in Combination with Homogeneous Functions.* By FRANK L. HITCHCOCK, *Ph.D.**

1. AMONG the uses of the Hamiltonian operator ∇ there are three which are particularly remarkable. First is the use of ∇ to distinguish the character of fields of force, fluid motion, and other vector fields. Second is its use to express integral relations having to do with space-integration over surfaces and volumes. Third, when ∇ is combined with functions which are homogeneous in the point-vector ρ, many new results are obtained.

To recall the leading facts under the first category:—If a vector function F of the point-vector ρ satisfies the relation $V\nabla F=0$, its rotation vector or "curl" is zero, and its distribution is lamellar. If $S\nabla F=0$, the "divergence" is zero, and the distribution solenoidal. If both these relations hold, so that $\nabla F=0$, the distribution is Laplacean. If F is everywhere at right angles to its own curl, we have $SF\nabla F=0$; as I am not aware of any name for such a distribution, I shall venture to call it ORTHOGYRAL †. The most significant property of an orthogyral vector is that it becomes lamellar when multiplied by a suitably chosen variable scalar ‡.

Under the second category fall the quaternionic forms of the theorems of Gauss and of Stokes on multiple integrals, which have been greatly extended by the late Profs. Tait and C. J. Joly and by Dr. Alex. McAulay.

My present object is to develop somewhat further the uses

* Communicated by the Author.

† Pronounced ortho jī'ral.

‡ Such a characterization of vector fields by means of differential operators may be greatly extended. Thus the four fields to which names are above given are characterized by the linear operators $V\nabla$, $S\nabla$, ∇, and $SF\nabla$, special cases of the general linear quaternion function of ∇, which in these combinations is, analytically, both vector and differentiator. I have considered the general question in a former paper ("The Double Nature of Nabla," Phil. Mag. Jan. 1909).

of ∇ in the third of the above ways,—in combination with homogeneous functional operations. A few facts are already known; chief of which is Euler's theorem, written in quaternion form as

$$S\rho\nabla \,.\, F\rho = -nF\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $F\rho$ is any function of ρ (scalar or vector), homogeneous of degree n in ρ. Aside from Euler's theorem, most of the known results on homogeneous functions in connexion with ∇ are combinations of ∇ with linear vector operators, and are due to the writers above mentioned. For example, if $dF\rho = \phi d\rho$, and if ϕ' is the linear vector operator conjugate to ϕ, then

$$\phi'\alpha - \phi\alpha = V\alpha V\nabla F\rho, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where α is any vector not acted on by ∇*.

2. Before proceeding to the proof of new theorems, it will be necessary to enter briefly into a few elementary considerations. First, with regard to notation, I shall write, for brevity, $T\rho = r$ and $U\rho = u$, so that $\rho = ru$.

Next, as to the definition of a homogeneous function, it is most natural for a vector algebraist to write

$$F\rho = r^n Fu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

as the definition of homogeneity, either for scalar or vector. This is of course precisely equivalent to the usual definition, and much more available. In words: A homogeneous function of ρ is one that can be factored into a power of r (that is $T\rho$), and a function of u (that is $U\rho$), alone.

The differentials of $T\rho$ and of $U\rho$ are important, and may be expanded in many forms (Tait, Art. 140). For the present purpose we may take as most convenient for the tensor of ρ,

$$dr = -Sud\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and for the unit vector

$$du = r^{-1}(d\rho + uSud\rho). \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Again, we often need to apply ∇ to a function of u alone. This is achieved by writing

$$dF\rho = \phi d\rho. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

We then have

$$\begin{aligned} dFu &= \phi du \\ &= \phi(d\rho + uSud\rho)r^{-1}, \text{ by (5).} \end{aligned}$$

* Tait, 'Quaternions,' 3rd Ed., Arts. 185, 186.

We now obtain $\nabla \mathrm{F}u$ from $d\mathrm{F}u$ by changing $d\rho$ into ρ', at the same time writing ∇' to the left of the whole. That is

$$\nabla \mathrm{F}u = \nabla'\phi(\rho' + u\mathrm{S}u\rho')r^{-1}, \quad . \quad . \quad . \quad . \quad (7)$$

where the accents indicate that ∇ acts only on the accented ρ. The expression $\nabla \mathrm{F}u$ therefore stands always for a function homogeneous of degree -1. This holds when $\mathrm{F}\rho$ is either scalar or vector, since the foregoing identities depend only on the linear character of ϕ and not upon its dimensionality.

Finally, if $\mathrm{F}\rho$ is homogeneous of degree n in ρ, $\nabla \mathrm{F}\rho$ is homogeneous of degree $n-1$. For

$$\nabla \mathrm{F}\rho = \nabla (r^n \mathrm{F}u), \text{ by definition,}$$
$$= nr^{n-1}\nabla r \, . \, \mathrm{F}u + r^n \nabla \mathrm{F}u, \text{ by distributing } \nabla;$$

but, in the first term on the right, $\nabla r = u$ (by Tait, Art. 145), and in the second term, $\nabla \mathrm{F}u$, as has been shown, is homogeneous of degree -1. Hence the right side may be factored into r^{n-1} and a function of u alone. It is therefore, by definition, homogeneous of degree $n-1$. This, also, holds for scalar or for vector.

3. I shall now prove the following theorem in regard to solenoidal vectors:—

Any homogeneous vector may be rendered solenoidal by adding to it a term of the form ρt, where t is a properly chosen scalar; exception must be made of vectors of degree -2.

For consider the effect of ∇ upon the vector $\rho\mathrm{S}\nabla\mathrm{F}\rho$, where $\mathrm{F}\rho$ is a vector homogeneous of degree n in ρ. We have

$$\nabla(\rho\mathrm{S}\nabla\mathrm{F}\rho) = \nabla\rho \, . \, \mathrm{S}\nabla\mathrm{F}\rho + \nabla\mathrm{S}\nabla\mathrm{F}\rho \, . \, \rho, \text{ by distributing } \nabla.$$

But, in the first term on the right, $\nabla\rho = -3$. Furthermore, scalars are commutative, so that if we take the scalar part of both sides we may write

$$\mathrm{S}\nabla(\rho\mathrm{S}\nabla\mathrm{F}\rho) = -3\mathrm{S}\nabla\mathrm{F}\rho + \mathrm{S}\rho\nabla \, . \, \mathrm{S}\nabla\mathrm{F}\rho.$$

Now the scalar $\mathrm{S}\nabla\mathrm{F}\rho$, as already pointed out, is of degree $n-1$. We may therefore apply Euler's theorem to the right-hand term, and have, (by (1)),

$$\mathrm{S}\rho\nabla \, . \, \mathrm{S}\nabla\mathrm{F}\rho = -(n-1)\mathrm{S}\nabla\mathrm{F}\rho.$$

By combining terms, therefore,

$$\mathrm{S}\nabla(\rho\mathrm{S}\nabla\mathrm{F}\rho) = -(n+2)\mathrm{S}\nabla\mathrm{F}\rho. \quad . \quad . \quad . \quad (8)$$

The following identity will now be evident,

$$S\nabla\left\{F\rho+\frac{\rho S\nabla F\rho}{n+2}\right\}=0, \quad . \quad . \quad . \quad . \quad (9)$$

$F\rho$ being a vector homogeneous in ρ of degree other than -2. This identity proves the theorem and shows how to find the scalar t.

The term ρt is uniquely determined. For if there were two values, their difference would be a scalar multiple of ρ and would be solenoidal. Call this difference ρt_1. But by the same order of reasoning as above, $S\nabla(\rho t_1)=-(n+2)t_1$, which cannot vanish unless $t_1=0$ or $n=-2$*.

4. As a simple, but important, extension of the foregoing theorem, let us suppose (what is frequently the case) that a non-homogeneous vector can be written as the sum of several vector terms, each homogeneous in its own degree, *e. g.* let

$$F\rho=F_1\rho+F_2\rho+\ldots+F_n\rho+\ldots,$$

where the subscripts denote the degrees of their terms. By applying the theorem to the separate terms, we see that $F\rho$ may be rendered solenoidal by adding the vector

$$\rho S\nabla\left\{\frac{F_1\rho}{3}+\frac{F_2\rho}{4}+\ldots+\frac{F_n\rho}{n+2}+\cdots\right\}.$$

The series concerned may be infinite, provided they are convergent.

Conversely (as an example of integration with ∇), if the convergence of a vector, $(S\nabla F\rho)$, be given, we can write down a value for the vector itself, which shall be a scalar multiple of ρ, provided we can expand the convergence as a sum of homogeneous scalar functions of ρ lacking a term of degree -3. For example, if we have given

$$\text{convergence}=S\nabla F\rho=t_0+t_1+t_2+\ldots+,$$

where the subscripts denote the degrees of their terms, then a possible value of $F\rho$ having this convergence is

$$F\rho=-\rho\left\{\frac{t_0}{3}+\frac{t_1}{4}+\frac{t_2}{5}+\ldots\right\},$$

a flux directed toward the origin.

5. These very simple results on the solenoidal character of vector fields may naturally lead us to inquire whether there

* In a similar manner we may show that a term of the form ρt, if t is a scalar of degree -3 in ρ, is always solenoidal.

are not analogous facts in regard to lamellar vectors. The following is the case :—

Any homogeneous vector may be rendered lamellar by adding to it a term of the form $V\rho\tau$, where τ is a properly chosen vector ; exception must be made of vectors of degree -1.

For consider the well-known vector identity (Tait, Art. 90),

$$V\alpha V\beta\gamma = \gamma S\alpha\beta - \beta S\alpha\gamma.$$

Writing ρ for α, ∇ for β, and $F\rho$ for γ, this identity becomes

$$V\rho V\nabla F\rho = F\rho' . S\rho\nabla' - \nabla' S\rho F\rho', \quad . \quad . \quad (10)$$

where, on the right, accents indicate that ∇ acts only on $F\rho$. By Euler's theorem, $F\rho' . S\rho\nabla' = -nF\rho$. Also,

$$\nabla S\rho F\rho = \nabla S\rho' F\rho + \nabla' S\rho F\rho', \quad . \quad . \quad . \quad (11)$$

by distributing ∇. (Unaccented ∇ acts on all that follows in the same term.) But the first term on the right of (11) is the same as $-F\rho$, by Tait, Art. 146. Whence (11) becomes

$$\nabla S\rho F\rho = -F\rho + \nabla' S\rho F\rho'. \quad . \quad . \quad . \quad . \quad (12)$$

By adding (10) and (12) and solving for $F\rho$ we therefore have the identity

$$F\rho = -\frac{V\rho V\nabla F\rho}{n+1} - \frac{\nabla S\rho F\rho}{n+1}, \quad . \quad . \quad . \quad (13)$$

$F\rho$ being any vector homogeneous in ρ of degree other than -1. The right-hand term is obviously lamellar. Stated in words, (13) shows that any homogeneous vector field (exception noted) may be taken as the sum of two fields, one lamellar (irrotational), the other at right angles to the point vector. By transposing, and operating with $V\nabla$, (13) becomes

$$V\nabla \left\{ F\rho + \frac{V\rho V\nabla F\rho}{n+1} \right\} = 0. \quad . \quad . \quad . \quad (14)$$

This latter identity proves the proposition, and shows how to find the vector τ. The identity (14) may be verified by direct operation*.

* The method used above for obtaining (14) is not quite parallel to that by which the analogous (9) was proved. Indeed, (9) might have been proved by first establishing the identity, analogous to (13),

$$(n+2)F\rho = -\rho S\nabla F\rho - V\nabla V\rho F\rho,$$

by expanding the last term on the right by the formula, Phil. Mag. June 1902, p. 579, (6). We then have (9) by the operator $S\nabla$; or we have (14) by writing $V\nabla F\rho$ in place of $F\rho$ which is any homogeneous vector, so making n become $n-1$.

The term $V\rho\tau$ is uniquely determined. For if there were two possible values their difference would be of the form $V\rho\tau_1$, and would be lamellar. Now the term $V\rho\tau_1$ is of degree n, hence τ_1 is of degree $n-1$. We therefore have identically*

$$V\nabla V\rho\tau_1 = -(n+1)\tau_1 - \rho S\nabla\tau_1 \quad . \quad . \quad . \quad (15)$$

If n does not equal -1, τ_1 does not have degree -2, and may be rendered solenoidal by a term in ρ without altering the value of $V\rho\tau_1$. Hence we may suppose $S\nabla\tau_1=0$, and the right side of (15) cannot vanish if τ_1 does not vanish and n is not -1; that is, the term $V\rho\tau$ is uniquely determined.

6. From the identity (14) may be easily deduced a second example of inverse operation (integration) with ∇. Suppose a rotation vector, $(V\nabla F\rho)$, to be known at all points of a given region, and to be expressible as a sum of vectors each homogeneous in ρ, lacking a term of degree -2. For example, let

$$\text{rotation vector} = V\nabla F\rho = \tau_0 + \tau_1 + \tau_2 + \ldots,$$

where the subscripts denote the degrees of their terms. A possible value of $F\rho$ then is

$$F\rho = -V\rho\left\{\frac{\tau_0}{2} + \frac{\tau_1}{3} + \frac{\tau_2}{4} + \ldots\right\}, \quad . \quad . \quad (16)$$

a vector everywhere at right angles to ρ. The vector $F\rho$ is often called the vector potential of its derived vector $V\nabla F\rho$. Thus (16) shows how to write down a possible vector potential for any assigned solenoidal vector whose components are either polynomials or other sums of homogeneous terms (exception noted). (16) may be directly verified by expanding the right side with the aid of identities like (15)†.

* Phil. Mag. *loc. cit.*

† It is well known that possible values for a required vector potential can be found by partial integration with respect to the point-coordinates x, y, and z. The above method illustrates how ∇ may replace partial integration,—a principle probably more far-reaching than any application which has yet been made of it. As another illustration, let $Xdx+Ydy+Zdz=dP=0$ be an exact differential equation. X, Y, and Z are components of the vector ∇P. Suppose $P=S\rho F\rho$. If ∇P can be written as a sum of homogeneous vectors

$$\nabla P = \sigma_0 + \sigma_1 + \sigma_2 + \ldots$$

we may write down P by the formula

$$P = -S\rho\left\{\frac{\sigma_0}{1} + \frac{\sigma_1}{2} + \frac{\sigma_2}{3} + \ldots\right\},$$

proved by multiplying both sides of (13) by ρ and taking scalars.

7. Most of the foregoing results are extensions of familiar properties of linear vector functions. The identity (13), expressing a homogeneous vector at the sum of an irrotational vector and a vector perpendicular to ρ, appears in the linear case as the familiar

$$\phi\rho=\omega\rho+\mathrm{V}\epsilon\rho \text{ (Tait, Art. 186)}, \quad . \quad . \quad . \quad (17)$$

where ω is a self-conjugate linear function and ϵ is a constant vector. Here $\omega\rho=\frac{\phi\rho+\phi'\rho}{2}$, and $\mathrm{V}\epsilon\rho=\frac{\phi\rho-\phi'\rho}{2}$, whence the identity (17) may be written

$$\phi\rho=\frac{\phi\rho-\phi'\rho}{2}+\frac{\phi\rho+\phi'\rho}{2}. \quad . \quad . \quad . \quad . \quad (18)$$

To bring out the analogy between (13) and (18), put, as before, $d\mathrm{F}\rho=\phi d\rho$, so that $\phi\rho=n\mathrm{F}\rho$ by Euler's theorem, and $\nabla\mathrm{S}\rho\mathrm{F}\rho=-\phi'\rho-\mathrm{F}\rho$ by (12). By (2), $\mathrm{V}\rho\mathrm{V}\nabla\mathrm{F}\rho=(\phi'-\phi)\rho$. By substitution of these values, (13) becomes

$$\mathrm{F}\rho=\frac{(\phi-\phi')\rho}{n+1}+\frac{\phi\rho+n\phi'\rho}{n(n+1)}, \quad . \quad . \quad . \quad (19)$$

which evidently reduces to (18) for the case $n=1$. The right-hand term may be taken as an extended $\omega\rho$; and, just as $\omega\rho$ is at all points of space normal to the general family of central quadric surfaces $\mathrm{S}\rho\phi\rho=\text{const.}$, so this term is normal to the cubic or higher surfaces $\mathrm{S}\rho\mathrm{F}\rho=\text{const.}$ Again, just as the axes of $\omega\rho$ possess the special property of being mutually at right angles, so the axes of its analogue have a specific configuration; but the consideration of axes lies outside the scope of the present paper.

8. The identity (13) also throws a good deal of light on the nature of orthogyral vectors. To distinguish these sharply from other vectors, we may say that an orthogyral vector is one satisfying the two following conditions :—

1. Neither the vector nor its curl vanishes identically.
2. The scalar product of the vector and its curl vanishes identically.

Lamellar vectors are thus excluded from the company of orthogyral vectors. It will also be convenient to distinguish two cases, according as $\mathrm{S}\rho\mathrm{F}\rho$ does, or does not, vanish identically. If an orthogyral vector $\mathrm{F}\rho$ is everywhere at right angles to ρ, the family of surfaces normal to $\mathrm{F}\rho$ consists of cones. The right-hand term of (13) disappears, and the vector may be said to be *conical*. If, on the other hand, $\mathrm{S}\rho\mathrm{F}\rho$ does not vanish identically, an orthogyral $F\rho$ may be said to be *mixed.* These two types are clearly not restricted to homogeneous vectors. I shall speak of the scalar $S\rho F\rho$ as the *associated scalar* of the vector $F\rho$.

From (13) we may now deduce a variety of simple consequences. First, *any* homogeneous vector whose associated scalar vanishes identically is orthogyral. This is evident from the mere form of (13) except when $n=-1$. In this case it can be shown by actual expansion of $V\nabla V\rho\tau$, as in (15).

Second, if a homogeneous orthogyral vector be divided by its associated scalar, the resulting vector is lamellar, for we have

$$V\nabla\left\{\frac{V\rho V\nabla F\rho+\nabla S\rho F\rho}{S\rho F\rho}\right\}=0,$$

by actual expansion, if $SF\nabla F=0$, *i. e.* if F is orthogyral. Exception must be made of conical vectors.

It further appears from (13) that the theory of orthogyral vectors may be connected with that of algebraic plane curves. Suppose n, the degree of the homogeneous vector $F\rho$, to be a positive integer, and let $F\rho$ be orthogyral of mixed type, and let its components be polynomials. Let x, y, and z, the usual coordinates of a point ρ in space, define a point in a plane in homogeneous coordinates. Then the associated scalar, $S\rho F\rho$, will define, by its vanishing, a plane curve, of degree $n+1$. I shall now show that if $F\rho$ is orthogyral the curve defined by its associated scalar must have n double points. The condition that $F\rho$ shall be orthogyral is that the scalar product of the two vectors $V\nabla F\rho$ and $\nabla S\rho F\rho$ shall vanish identically. Call the components of $\nabla S\rho F\rho$ X, Y, and Z, and those of $V\nabla F\rho$ X_1, Y_1, and Z_1. Expanding the scalar product we have the condition

$$XX_1+YY_1+ZZ_1=0, \text{ identically.}$$

These six components define six curves. Wherever X and Y both vanish, either Z or Z_1 must vanish. But Z_1 meets X or Y at most in $n(n-1)$ points, the product of their degrees*. Hence at the remaining $n^2-n(n-1)$ intersections of X and Y we must also have Z vanish. That is, $\nabla S\rho F\rho$ vanishes at n points, and the curve $S\rho F\rho=0$ has n double points.

For example, let $n=1$. The associated scalar, being of the second degree, defines a conic. It must have one double

* If Z_1, X, and Y happen to have a common factor, we may make a new choice of our coordinate system. It is easy to complete the formal proof.

point, hence consists of two straight lines. We may then put

$$S\rho F\rho = S\alpha\rho S\beta\rho,$$

where α and β are constant vectors. Operating by ∇ gives

$$\nabla S\rho F\rho = -\alpha S\beta\rho - \beta S\alpha\rho.$$

The vector $V\nabla F\rho$, being now of degree zero, is constant, and must be parallel to $V\alpha\beta$. Hence the most general orthogyral linear vector is of the form

$$aV\rho V\alpha\beta + b(\alpha S\beta\rho + \beta S\alpha\rho),$$

where a and b are constant scalars.

If we let n equal 2, the associated scalar defines a cubic with two double points, hence degenerate.

If $n=3$, the most general orthogyral vector has for its associated scalar a quartic of deficiency zero.

In a similar manner, if we start with any *two* homogeneous scalars we may write down orthogyral vectors in the form (13). For the vector $V\nabla u\nabla v$ is solenoidal, whatever scalars u and v may be. Hence

$$aV\rho V\nabla u\nabla v + b\nabla(uv)$$

is orthogyral if u and v are homogeneous, a and b being constants. That is, to any pair of algebraic plane curves corresponds a two-parameter family of orthogyral vectors.

9. In conclusion it may be said that the differential and integral relations of this paper are extensions to space of the one-dimensional formulas for $\frac{d(x^n)}{dx}$ and $\int x^n dx$. In fact, most of the preceding results reduce to these, or to identities, if we put $\rho = ix$ and $\nabla = i\frac{d}{dx}$. That a calculus with ∇ is worthy of systematic and extensive development there can be no doubt. We should naturally expect greater variety and complexity in proportion as the geometry of space is many-sided in comparison with that of one dimension. It would be essential to consider next the values of n treated above as exceptional cases—not a difficult matter, but leading to logarithms and other non-homogeneous functions, beyond the special domain of the present paper.