A NOTE ON THE CONTINUITY OR DISCONTINUITY OF A FUNCTION DEFINED BY AN INFINITE PRODUCT

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1. Abel's well known theorem on the continuity of power series naturally suggests the question as to whether a similar theorem holds for infinite products. Does the convergence of the product

(1) $$P = \prod_0^\infty (1+a_n)$$

involve the absolute convergence of the product

(2) $$P_1(x) = \prod_0^\infty (1+a_n x^n)$$

for all values of x whose modulus is less than unity, and the truth of the equation

$$P_1(x) \to P$$

as $x \to 1$? The path along which $x \to 1$ is here supposed to be any such path as is permitted in Stolz's extension of Abel's theorem, that is to say, any path which lies inside the unit circle, has a tangent at every point, and does not touch the circle. Such a path we shall describe for brevity as a *standard* path.*

This question immediately suggests another: does the convergence of (1) involve that of

(3) $$P_2(x) = \prod_0^\infty (1+a_n x)$$

for all values of x, and the truth of the equation

(4) $$P_2(x) \to P$$

when $x \to 1$ in any manner?

2. If the product (1) is absolutely convergent, that is to say, if the series Σa_n is absolutely convergent, it can be shown at once that all these questions must be answered in the affirmative.

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* Paths which have no tangents may be dismissed from consideration; no interest attaches to them, and the only result of admitting them is a little unnecessary complication of our definitions.

The question stated above was, if I remember rightly, first suggested to me personally by Prof. V. Ramaswami Aiyar of Gooty, India, in a letter which I received from him a year or two ago.

The simplest proof of this depends upon what Mr. Bromwich has called Tannery's theorem—viz., that, if

$$|g_n(x)| < M_n,$$

where M_n is independent of x, and ΣM_n is convergent, throughout any region D of values of x, then

$$\Pi\{1+g_n(x)\} \tag{5}$$

is uniformly convergent throughout D. If every $g_n(x)$ is continuous, the product is, of course, also continuous.

We shall denote by D_1 any region bounded by a standard curve beginning and ending at the point $x = 1$; and by D_2 any region bounded by a closed curve and including the point $x = 1$. Then

$$|a_n x^n| \leqslant |a_n|$$

throughout D_1; and

$$|a_n x| \leqslant R|a_n|$$

throughout D_2, R being the greatest distance of any point of D_2 from the origin; and it follows at once that $P_1(x)$ and $P_2(x)$ are uniformly convergent throughout D_1 and D_2 respectively, the boundaries of the regions included.

A theorem similar to Tannery's, rather more general, but rather less simple and natural, was given by Arzelà.* This theorem asserts that, if

(i.) $g_n(x)$ tends uniformly to zero, as $n \to \infty$, for all values of x in D;

(ii.) $\sum_{\nu=0}^{n} |g_\nu(x)| < K$ for all values of n and x;†

then the uniform convergence of $\Sigma g_n(x)$ is a *sufficient* condition for that of $\Pi\{1+g_n(x)\}$.‡

* *Mem. di Bologna*, ser. 4, t. IV. (1883), p. 427; Stolz und Gmeiner, *Einleitung in die Funktionentheorie*, bd. II, p. 431.

† If Tannery's condition $|g_n(x)| < M_n$, where ΣM_n is convergent, is satisfied, it is evident that Arzelà's two conditions are satisfied. The converse is not true. Suppose, for example, that

$$g_0(x) = 1 \quad (0 \leqslant x < \tfrac{1}{2}), \qquad g_0(x) = 0 \quad (\textit{otherwise});$$
$$g_1(x) = \tfrac{1}{2} \quad (\tfrac{1}{2} \leqslant x < \tfrac{3}{4}), \qquad g_1(x) = 0 \quad (\textit{otherwise});$$
$$g_2(x) = \tfrac{1}{3} \quad (\tfrac{3}{4} \leqslant x < \tfrac{7}{8}), \qquad g_2(x) = 0 \quad (\textit{otherwise});$$
$$\cdots \quad \cdots \quad \cdots \qquad \cdots \quad \cdots \quad \cdots$$
$$g_n(x) = \frac{1}{n+1} \quad (1-2^{-n} \leqslant x < 1-2^{-n-1}), \qquad g_n(x) = 0 \ (\textit{otherwise}).$$

Then it is clear that $\sum_{\nu=0}^{n} |g_\nu(x)| < 1$, and that $g_n(x) \to 0$ uniformly; but Tannery's condition is not satisfied, since $\Sigma(1/n)$ is divergent. There is no difficulty in constructing a similar example in which every g_n is continuous.

‡ If x and $g_n(x)$ are restricted to be real, the condition is also *necessary*.

It is plain that our results concerning the products $\Pi(1+a_n x^n)$, $\Pi(1+a_n x)$ can be deduced at once from Arzelà's as well as from Tannery's theorem.

3. We may now pass on to consider the more interesting case in which the product (1) is only *conditionally* convergent. So far as I am aware, the only general tests of any importance that have ever been given for the conditional convergence of a product are Cauchy's test—viz.,

the product $\Pi(1+a_n)$ *is convergent if* Σa_n *is convergent and* Σa_n^2 *absolutely convergent;*

and Pringsheim's extension of Cauchy's test—viz.,

the product is convergent if Σa_n, Σa_n^2, ..., Σa_n^{k-1} *are convergent and* Σa_n^k *absolutely convergent.**

A product which is convergent in virtue of Cauchy's or Pringsheim's tests we shall call a *regularly convergent* product.

4. THEOREM A.—*If the product* (1) *is regularly convergent, all the questions of* § 1 *may be answered in the affirmative.*

This result I shall deduce from the following general theorem:—

THEOREM B.—*If the series*†

$$\Sigma g_n(x),\ \Sigma g_n^2(x),\ \ldots,\ \Sigma g_n^{k-1}(x),\ \Sigma|g_n^k(x)|$$

are uniformly convergent throughout any region D *in the plane of* x, *then the product*

$$\Pi\{1+g_n(x)\}$$

is uniformly convergent throughout D.

This theorem is very easy to prove. We can choose n_0 so that, for $n \geqslant n_0$, $|g_n| < \delta < 1$, for all values of x in question. We can then ignore the first n_0 factors, so that nothing is lost by supposing $|g_n| < \delta$

* Pringsheim, *Math. Annalen*, bd. XXII., p. 482; Stolz und Gmeiner, *l.c.*, p. 436. The latter test may be stated in the more general form "the product is convergent if

$$\Sigma\left(a_n - \tfrac{1}{2}a_n^2 + \ldots \pm \frac{1}{k-1}\, a_n^{k-1}\right)$$

is convergent and Σa_n^k absolutely convergent," but the extension seems of but little interest.

If a_n is real, Σa_n^2 can, of course, only converge absolutely or diverge to $+\infty$; the product converges or diverges to 0 accordingly. In this case Pringsheim's extension cannot be needed.

† More generally, if $\Sigma\left(g_n - \tfrac{1}{2}g_n^2 + \ldots \pm \frac{1}{k-1}\, g_n^{k-1}\right)$, $\Sigma|g_n^k|$ are uniformly convergent.

for all values of n and x. Then

$$\log(1+g_n) = g_n - \tfrac{1}{2}g_n^2 + \ldots + \frac{(-1)^k}{k-1} g_n^{k-1} + (-1)^{k+1}\phi_n,$$

where

$$\phi_n = \frac{g_n^k}{k} - \frac{g_n^{k+1}}{k+1} + \ldots,$$

so that

$$|\phi_n| < \frac{|g_n|^k}{k}(1+\delta+\delta^2+\ldots) < \frac{|g_n^k|}{k(1-\delta)}.$$

Hence $\Sigma\phi_n$ is uniformly convergent, and so therefore is $\Sigma\log(1+g_n)$.

5. From Theorem B the truth of Theorem A follows almost immediately.

(i.) Let $g_n = a_n x^n$, and let D be the region D_1 of § 2. Then

$$\Sigma a_n,\ \Sigma a_n^2,\ \ldots,\ \Sigma a_n^{k-1},\ \Sigma|a_n^k|$$

are convergent, and therefore, by Stolz's extension of Abel's theorem,

$$\Sigma a_n x^n,\ \Sigma a_n^2 x^{2n},\ \ldots,\ \Sigma a_n^{k-1} x^{(k-1)n},\ \Sigma|a_n^k||x^{kn}|$$

are uniformly convergent throughout D_1.

(ii.) Let $g_n = a_n x$, and let D be the region D_2 of § 2. Then

$$\Sigma a_n x,\quad \Sigma a_n^2 x^2,\quad \ldots,\quad \Sigma a_n^{k-1} x^{k-1},\quad \Sigma\,|a_n^k|\,|x^k|$$

are uniformly convergent throughout D_2.

It is easy to deduce, from Theorem B and from the known extensions of Abel's theorem, more general results concerning products of the types

$$\Pi\{1+a_n f_n(x)\};$$

but the cases in which $f_n(x) = x^n$ or x seem so much the most interesting that it is hardly worth while to set any others out at length.

6. There still remains the case in which the product (1) is convergent but not regularly convergent, or, as we may say, *irregularly convergent*.

As regards such irregular convergence one may distinguish two possibilities.

(*a*) It is possible that the product $\Pi(1+a_n)$ may be convergent, although the series Σa_n is not convergent, and indeed even if the series diverges to $+\infty$. The following examples of this are interesting, and we shall have occasion to make use of them later on.

(i.) Consider the product*

(6)
$$\prod_0^{\infty} \left\{ \left(1+\frac{1}{\sqrt{n-\frac{1}{2}}}\right)\left(1-\frac{1}{\sqrt{n+\frac{1}{2}}}\right) \right\},$$

for which
$$a_{2\nu}=\frac{1}{\sqrt{\nu-\frac{1}{2}}}, \quad a_{2\nu+1}=-\frac{1}{\sqrt{\nu+\frac{1}{2}}}.$$

Here $(1+a_{2\nu})(1+a_{2\nu+1})=1$, so that the product is convergent, and has the value 1, although $\Sigma(a_{2\nu}+a_{2\nu+1})$ or

$$\Sigma \frac{1}{\nu-\frac{1}{4}}$$

diverges to $+\infty$.

(ii.) The product
$$\Pi\left(1+\frac{e^{n\theta i}}{\sqrt{n}}\right)$$

is convergent, unless θ is a multiple of π,† since the series

$$\Sigma\frac{e^{n\theta i}}{\sqrt{n}}, \quad \Sigma\frac{e^{2n\theta i}}{n}, \quad \Sigma\frac{1}{n\sqrt{n}}$$

are convergent.

It follows that

(7)
$$\Pi\left|1+\frac{e^{n\theta i}}{\sqrt{n}}\right|^2 = \Pi\left(1+\frac{2\cos n\theta}{\sqrt{n}}+\frac{1}{n}\right)$$

is convergent. But
$$\Sigma\left(\frac{2\cos n\theta}{\sqrt{n}}+\frac{1}{n}\right)$$

plainly diverges to $+\infty$.

The product may also converge when some of the later members of the sequence of series
$$\Sigma a_n, \quad \Sigma a_n^2, \quad \ldots, \quad \Sigma a_n^{k-1}$$

oscillate or diverge. All such cases afford illustrations of our first possibility with respect to irregular convergence.

(*b*) The second possibility with respect to irregular convergence is that it should not be possible to find a value of k for which

$$\Sigma\,|a_n|^k$$

is convergent. Suppose, for example, that

$$a_n=\frac{e^{n\theta i}}{\log n},$$

where θ/π is *irrational.* Then Σa_n^k is convergent for *all* values of k, but

* This product is used by Pringsheim (*Math. Annalen*, bd. XXXIII., p. 154) for another purpose.

† The product diverges to infinity if θ is a multiple of 2π, to 0 if θ is an odd multiple of π.

never absolutely. And, so far as the tests at our disposal at present go, the question of the convergence of the product

$$\Pi\left(1+\frac{e^{n\theta i}}{\log n}\right) \tag{8}$$

remains open.

7. I shall return to the second possibility in a moment. But first I wish to show, by means of examples drawn from the first class of irregularly convergent products, that the *mere* convergence of the product (1) is *not* sufficient to ensure an affirmative answer to the questions of § 1.

(i.) *The convergence of* (1) *does not necessarily involve the convergence of* $\Pi(1+a_n x)$ *for any value of* x *other than* $x=0$ *and* $x=1$. We saw above that the product

$$\prod_0^\infty \left\{\left(1+\frac{1}{\sqrt{n-\frac{1}{2}}}\right)\left(1-\frac{1}{\sqrt{n+\frac{1}{2}}}\right)\right\}$$

is convergent. But

$$\prod_0^\infty \left\{\left(1+\frac{x}{\sqrt{n-\frac{1}{2}}}\right)\left(1-\frac{x}{\sqrt{n+\frac{1}{2}}}\right)\right\}$$

is convergent only if $\displaystyle\prod_{\nu=0}^{\infty}\left(1+\frac{a_\nu}{\nu-\frac{1}{4}}\right)$

is convergent, where $a_\nu = \frac{1}{4}-(x-\frac{1}{2})^2$;

and this is so only if $a_\nu = 0$, *i.e.*, if $x=0$ or $x=1$.

(ii.) The convergence of (1) does, of course, imply the absolute convergence of $\Pi(1+a_n x^n)$ for any value of x numerically less than unity. *But it does not imply the truth of the equation*

$$\Pi(1+a_n x^n) \to \Pi(1+a_n)$$

as $x \to 1$, *even by real values and when* a_n *is real.*

We saw in § 6 that the product

$$\prod_1^\infty \left(1+\frac{2\cos n\theta}{\sqrt{n}}+\frac{1}{n}\right) \tag{7}$$

is convergent, provided θ is not a multiple of π. Let us denote its value by ϖ. Then I shall prove that if

$$a_n = \frac{2\cos n\theta}{\sqrt{n}}+\frac{1}{n}$$

then

(8) $$\prod_1^\infty (1+a_n x^n) \to 2\varpi$$

as $x \to 1$ by real values.

In order to prove this we observe that since $\Pi\left(1+\frac{e^{n\theta i}}{\sqrt{n}}\right)$ is *regularly* convergent

$$\Pi\left(1+\frac{e^{n\theta i}}{\sqrt{n}}x^n\right) \to \Pi\left(1+\frac{e^{n\theta i}}{\sqrt{n}}\right)$$

as $x \to 1$. It follows that the same relation holds between the products formed by taking the modulus of every factor, and therefore that

$$\Pi\left(1+\frac{2x^n \cos n\theta}{\sqrt{n}}+\frac{x^{2n}}{n}\right) \to \varpi.$$

Our conclusion will therefore be established if we prove that

$$\prod_1^\infty \left\{\frac{1+\frac{2x^n \cos n\theta}{\sqrt{n}}+\frac{x^n}{n}}{1+\frac{2x^n \cos n\theta}{\sqrt{n}}+\frac{x^{2n}}{n}}\right\} = \prod_1^\infty (1+\beta_n),$$

where $$\beta_n = \frac{x^n(1-x^n)}{n+2x^n \cos n\theta\sqrt{n}+x^{2n}},$$

tends to the limit 2 as $x \to 1$. Now β_n is positive and less than K/n, and so the series $\Sigma\beta_n^2$ is uniformly convergent for $0 \leqslant x \leqslant 1$. Hence also the series

$$\Sigma\{\log(1+\beta_n)-\beta_n\}$$

is uniformly convergent, and so

$$\log \Pi(1+\beta_n)-\Sigma\beta_n \to 0.$$

We therefore require only to show that

$$\Sigma\beta_n \to \log 2.$$

But, if $$\gamma_n = x^n(1-x^n)/n,$$

we have $$\gamma_n-\beta_n = \frac{x^n(1-x^n)}{n}\,\frac{2x^n \cos n\theta\sqrt{n}+x^{2n}}{n+2x^n \cos n\theta\sqrt{n}+x^{2n}};$$

so that $$|\gamma_n-\beta_n| < Kn^{-\frac{3}{2}}.$$

Hence $\Sigma(\gamma_n-\beta_n)$ is absolutely and uniformly convergent, and so

$$\Sigma\gamma_n-\Sigma\beta_n \to 0.$$

But
$$\Sigma \gamma_n = \Sigma \frac{x^n}{n} - \Sigma \frac{x^{2n}}{n} = \log\left(\frac{1}{1-x}\right) - \log\left(\frac{1}{1-x^2}\right)$$
$$= \log(1+x) \to \log 2;$$

and so our conclusion follows, viz., that
$$\Pi(1+a_n x^n) \to 2\Pi(1+a_n)$$
as $x \to 1$.

8. In conclusion, I wish to say something about the second possibility mentioned in § 6. This possibility seems to me very interesting. But I know of no example of such a product, and I am unable to construct one. Indeed, I cannot determine whether the product (8) is ever convergent or not; and the considerations which follow show, I think, that the question is not one which can be settled without considerable difficulty.

Let us consider first a simpler product, viz.,
$$\prod_1^\infty \left(1+\frac{e^{n\theta i}}{n^a}\right) \quad (0 < a < 1).$$

Let k be the least integer such that $ka > 1$. The series Σa_n^k is absolutely convergent, the series
$$\Sigma a_n, \quad \Sigma a_n^2, \quad \ldots, \quad \Sigma a_n^{k-1}$$
are convergent unless one of θ, 2θ, …, $(k-1)\theta$ is a multiple of 2π. Thus the product is regularly convergent unless θ/π has one of a limited number of rational values; if, *e.g.*, $a = \frac{1}{2}$, $k = 3$, it is regularly convergent unless θ/π is an integer.

Now, let us consider the product (8), for which
$$a_n = e^{n\theta i}/\log n;$$
and let us suppose that θ/π is a rational fraction p/q. The series
$$\Sigma a_n^k = \Sigma \frac{e^{kn\theta i}}{(\log n)^k}$$
is conditionally convergent except for such values of k as make $k\theta/\pi$ an even integer.

First suppose p even. Then q is odd, and Σa_n^k is convergent, except for
$$k = q,\ 2q,\ 3q,\ \ldots,$$
while
$$a_n^q = (\log n)^{-q}.$$
Also
$$\log(1+a_n) = a_n - \tfrac{1}{2}a_n^2 + \ldots - \frac{1}{q-1}a_n^{q-1} + \frac{1}{q}a_n^q(1+\epsilon_n),$$
where $\epsilon_n \to 0$ with n. It follows at once that *the product* (8) *diverges* (*its*

associated product of moduli diverging to $+\infty$) *whenever* θ/π *is a rational fraction with an even numerator.*

Secondly, suppose p odd. Then Σa_n^k is convergent, except for

$$k = 2q,\ 4q,\ 6q,\ \dots;$$

and an argument similar to that used above shows that *the product* (8) *diverges to* 0 *whenever* θ/π *is a rational fraction with an odd numerator.*

Thus the product (8) is certainly never convergent when θ/π is rational. Is it ever convergent? That, it seems to me, is a very interesting question; but I must confess myself entirely unable to answer it. I can only suggest the problem: *to find a product* $\Pi(1+a_n)$, *such that* Σa_n^k *is always convergent, but never absolutely, and whose convergence, divergence, or oscillation is capable of proof.*

It is hardly necessary to point out that the argument given above applies to any product $\Pi(1+a_n e^{n\theta i})$, where a_n is a positive function of n which tends steadily to 0 as $n \to \infty$, and which is such that

$$\Sigma a_n^{-k}$$

is divergent for all values of k.