On the Vibrations of an Elastic Sphere. By HORACE LAMB, M.A.

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The following paper contains an examination into the nature of the fundamental modes of vibration of an elastic sphere by the method employed in a previous communication, "On the Oscillations of a Viscous Spheroid."* The problem here considered is one of considerable theoretic interest, being as yet the only case in which the vibrations of an elastic solid whose dimensions are all finite have been discussed with any attempt at completeness. I have therefore thought it worth while to go into considerable detail in the interpretation of the results, and have endeavoured, by numerical calculations and the construction of diagrams, to present these in as definite and intelligible a form as possible. I find that some of my results (the most important being the general classification of the fundamental modes given in § 5 below) have been already obtained by Paul Jaerisch, of Breslau, in *Crelle*, t. lxxxviii. (1879); but the methods employed, as well as the form in which the results in question are expressed, are entirely different in the two investigations.

1. For convenience of reference, I place together at the outset several analytical results which will be required.

It has been shown (*Proc. Lond. Math. Soc.*, Dec. 1881), that the solution of the following system of equations—

$$\left.\begin{array}{c}(\nabla^2+k^2)\,u=0,\quad(\nabla^2+k^2)\,v=0,\quad(\nabla^2+k^2)\,w=0\\ \dfrac{du}{dx}+\dfrac{dv}{dy}+\dfrac{dw}{dz}=0\end{array}\right\}\quad\ldots\ldots(1),$$

where $\nabla^2=d^2/dx^2+d^2/dy^2+d^2/dz^2$, and k is a constant, is given by the formulæ

$$\left.\begin{aligned}u&=\Sigma\left\{\psi_n(kr)\left(y\frac{d\chi_n}{dz}-z\frac{d\chi_n}{dy}\right)+\psi_{n-1}(kr)\frac{d\phi_n}{dx}\right.\\&\qquad\left.-\frac{n}{n+1}\,\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\,\psi_{n+1}(kr)\,\frac{d}{dx}\,\frac{\phi_n}{r^{2n+1}}\right\},\\ v&=\Sigma\left\{\psi_n(kr)\left(z\frac{d\chi_n}{dx}-x\frac{d\chi_n}{dz}\right)+\psi_{n-1}(kr)\frac{d\phi_n}{dy}\right.\\&\qquad\left.-\frac{n}{n+1}\,\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\,\psi_{n+1}(kr)\,\frac{d}{dy}\,\frac{\phi_n}{r^{2n+1}}\right\},\\ w&=\Sigma\left\{\psi_n(kr)\left(x\frac{d\chi_n}{dy}-y\frac{d\chi_n}{dx}\right)+\psi_{n-1}(kr)\frac{d\phi_n}{dz}\right.\\&\qquad\left.-\frac{n}{n+1}\,\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\,\psi_{n+1}(kr)\,\frac{d}{dz}\,\frac{\phi_n}{r^{2n+1}}\right\}\end{aligned}\right\}\quad\ldots\ldots(2).$$

* "Proceedings of the London Mathematical Society," Vol. xiii., pp. 51—66, Nos. 183, 184.

The symbols χ_n, ϕ_n here stand for solid harmonics of algebraical degree n; $r = (x^2+y^2+z^2)^{\frac{1}{2}}$; and ψ_n is defined by the equation

$$\psi_n(\theta) = 1 - \frac{\theta^2}{2 \,.\, 2n+3} + \frac{\theta^4}{2 \,.\, 4 \,.\, 2n+3 \,.\, 2n+5} - \&c. \quad\text{.........(3).}$$

This function possesses the following properties, which will be of frequent application

$$\psi_n'(\theta) = -\frac{\theta}{2n+3}\psi_{n+1}(\theta) \quad\text{.....................(4),}$$

$$\psi_n(\theta) + \frac{\theta}{2n+1}\psi_n'(\theta) = \psi_{n-1}(\theta) \quad\text{..................(5),}$$

$$\psi_n(\theta) - \psi_{n-1}(\theta) = \frac{\theta^2}{2n+1 \,.\, 2n+3}\psi_{n+1}(\theta) \quad\text{............(6).}$$

In order that (2) should give the solution of the system (1) with complete generality, it is necessary that the sign Σ of summation should embrace both positive and negative values of n; but in the present application the condition that the motion must be finite at the centre of the sphere (which is taken as origin of coordinates) restricts us to positive values of n. We then have

$$\psi_n(\theta) = (-)^n \,.\, 1 \,.\, 3 \,.\, 5 \,...\, 2n+1 \,.\, \left(\frac{1}{\theta}\frac{d}{d\theta}\right)^n \frac{\sin\theta}{\theta} \quad\text{.........(7).*}$$

Another formula for ψ_n is

$$\psi_n(\theta) = \frac{1 \,.\, 3 \,.\, 5 \,...\, 2n+1}{\theta^{n+1}}\left\{H_n \sin\left(\theta - \frac{n\pi}{2}\right) + K_n \cos\left(\theta - \frac{n\pi}{2}\right)\right\} \quad\text{...(8),}$$

where $$H_n = 1 - \frac{n-1 \,.\, n \,.\, n+1 \,.\, n+2}{2 \,.\, 4 \,.\, \theta^2} + \&c.,$$

$$K_n = \frac{n \,.\, n+1}{2 \,.\, \theta} - \frac{n-2 \,.\, n-1 \,.\, n \,.\, n+1 \,.\, n+2 \,.\, n+3}{2 \,.\, 4 \,.\, 6 \,.\, \theta^3} + \&c.$$

If we multiply the three equations (2) by x, y, z in order, and add, we find

$$xu + yv + zw = \Sigma n\psi_n(kr) \,.\, \phi_n \quad\text{.....................(9),}$$

the reduction being effected by means of (6), and of known properties of spherical harmonics.

Again, if we write

$$2\xi = \frac{dw}{dy} - \frac{dv}{dz}, \quad 2\eta = \frac{du}{dz} - \frac{dw}{dx}, \quad 2\zeta = \frac{dv}{dx} - \frac{du}{dy} \quad\text{.........(10),}$$

we find

$$2\xi = -\Sigma\frac{k^2}{2n+1}\psi_n(kr)\left(y\frac{d\phi_n}{dz} - z\frac{d\phi_n}{dy}\right)$$

$$-\Sigma(n+1)\left\{\psi_{n-1}(kr)\frac{d\chi_n}{dx} - \frac{n}{n+1}\frac{k^2r^{2n+3}}{2n+1 \,.\, 2n+3}\psi_{n+1}(kr)\frac{d}{dx}\frac{\chi_n}{r^{2n+1}}\right\} \quad\text{............(11),}$$

* A formula equivalent to this is given by Heine, *Kugelfunctionen*, t. i., p. 240. The formula (8) has been given in slightly different forms by various writers: it appears to be due substantially to Poisson.

with symmetrical formulæ for 2η, 2ζ. In the reduction, use has been made of (4), (5), and of the following formula of Spherical Harmonics,

$$x\phi_n = \frac{r^2}{2n+1}\left\{\frac{d\phi_n}{dx} - r^{2n+1}\frac{d}{dx}\frac{\phi_n}{r^{2n+1}}\right\} \quad \text{...............(12).}$$

The exchange of *rôles* between the harmonics ϕ_n, χ_n, in passing from (2) to (11), is noticeable.

By comparing (2), (10), and (11), we see that (2) may be written in the forms

$$u = \frac{dN}{dy} - \frac{dM}{dz},\quad v = \frac{dL}{dz} - \frac{dN}{dx},\quad w = \frac{dM}{dx} - \frac{dL}{dy} \quad \text{.........(13),}$$

where $$L = 2\xi/k^2,\quad M = 2\eta/k^2,\quad N = 2\zeta/k^2 \quad \text{..................(14).}$$

2. The equations of motion of a homogeneous isotropic elastic solid free from external force may be written

$$\rho\frac{d^2\alpha}{dt^2} = \mathfrak{m}\frac{d\delta}{dx} + \mathfrak{n}\nabla^2\alpha,\quad \rho\frac{d^2\beta}{dt^2} = \mathfrak{m}\frac{d\delta}{dy} + \mathfrak{n}\nabla^2\beta,\quad \rho\frac{d^2\gamma}{dt^2} = \mathfrak{m}\frac{d\delta}{dz} + \mathfrak{n}\nabla^2\gamma \text{...(15).}$$

The notation is that of Thomson and Tait, except that $\mathfrak{m}$, $\mathfrak{n}$ are written for the m, n of these writers; viz., α, β, γ are the component displacements, and δ, $= d\alpha/dx + d\beta/dy + d\gamma/dz$, is the dilatation, at the point (x, y, z) of the solid, ρ is the density, $\mathfrak{n}$ the rigidity, and $\mathfrak{m}$ a constant, such that $\mathfrak{m} - \frac{1}{3}\mathfrak{n}$ is the resilience of volume. If the state of stress at the point (x, y, z) be expressed in the usual way by the symbols p_{xx}, p_{xy}, p_{xz}, &c., we have

$$\left.\begin{aligned} p_{xx} &= (\mathfrak{m}-\mathfrak{n})\,\delta + 2\mathfrak{n}\frac{d\alpha}{dx}, & p_{yz} &= \mathfrak{n}\left(\frac{d\beta}{dz} + \frac{d\gamma}{dy}\right),\\ p_{yy} &= (\mathfrak{m}-\mathfrak{n})\,\delta + 2\mathfrak{n}\frac{d\beta}{dy}, & p_{zx} &= \mathfrak{n}\left(\frac{d\gamma}{dx} + \frac{d\alpha}{dz}\right),\\ p_{zz} &= (\mathfrak{m}-\mathfrak{n})\,\delta + 2\mathfrak{n}\frac{d\gamma}{dz}, & p_{xy} &= \mathfrak{n}\left(\frac{d\alpha}{dy} + \frac{d\beta}{dx}\right), \end{aligned}\right\} \quad \text{.........(16).}$$

If the centre of the sphere be taken as origin of coordinates, the conditions that the surface should be free from stress are

$$xp_{xx} + yp_{xy} + zp_{xz} = 0,\quad xp_{yx} + yp_{yy} + zp_{yz} = 0,\quad xp_{zx} + yp_{zy} + zp_{zz} = 0 \text{...(17),}$$

when $r = a$, the radius of the sphere.

If we substitute from (16), these may be put in the form

$$\frac{\mathfrak{m}-\mathfrak{n}}{\mathfrak{n}}\,x\delta + \left(r\frac{d}{dr} - 1\right)\alpha + \frac{d}{dx}(x\alpha + y\beta + z\gamma) = 0,\ \text{\&c. \&c.} \quad \text{...(18),}$$

where d/dr denotes a differentiation along the radius vector.

3. To determine the fundamental modes, we assume, as usual, that

α, β, γ all vary as e^{ipt}, where $i^2 = -1$. The equations (15) then become

$$-p^2\rho\alpha = \mathfrak{m}\frac{d\delta}{dx}+\mathfrak{n}\nabla^2\alpha, \quad -p^2\rho\beta = \mathfrak{m}\frac{d\delta}{dy}+\mathfrak{n}\nabla^2\beta,$$

$$-p^2\rho\gamma = \mathfrak{m}\frac{d\delta}{dz}+\mathfrak{n}\nabla^2\gamma \quad \text{.......................(19).}$$

Differentiating these with respect to x, y, z in order, and adding, we find

$$(\nabla^2+h^2)\,\delta = 0 \quad \text{........................... (20),}$$

where

$$h^2 = p^2\rho/(\mathfrak{m}+\mathfrak{n}) \quad \text{.............................(21).}$$

The solution of (20) is

$$\delta = \Sigma\psi_n(hr)\,.\,\omega_n \quad \text{.............. ....(22),}$$

where ω_n stands for the general solid harmonic of positive degree n.

Now, if we put

$$k^2 = p^2\rho/\mathfrak{n} \quad \text{............................ ..(23),}$$

the equations (19) may be written

$$(\nabla^2+k^2)\,\alpha = \left(1-\frac{k^2}{h^2}\right)\frac{d\delta}{dx}, \text{ \&c. \&c.} \quad \text{.................(24).}$$

A particular solution of (24) is easily seen to be

$$\alpha = -\frac{1}{h^2}\frac{d\delta}{dx}, \quad \beta = -\frac{1}{h^2}\frac{d\delta}{dy}, \quad \gamma = -\frac{1}{h^2}\frac{d\delta}{dz} \quad \text{.........(25),}$$

where δ has the value (22). These values of α, β, γ satisfy the relation

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz} = \delta.$$

Hence the complete solution of (23) is given by

$$\alpha = -\frac{1}{h^2}\frac{d\delta}{dx}+u, \quad \beta = -\frac{1}{h^2}\frac{d\delta}{dy}+v, \quad \gamma = -\frac{1}{h^2}\frac{d\delta}{dz}+w \quad \text{...(26),}$$

where u, v, w are any functions, which satisfy the system (1), *i.e.*, whose values are given by (2).

4. It remains to substitute from (26) in the surface-conditions (18). In this process the formula (12) is required. It will be found that the conditions (18) assume the forms

$$\left.\begin{aligned}
&\Sigma\left\{P_n\left(y\frac{d\chi_n}{dz}-z\frac{d\chi_n}{dy}\right)+A_n\frac{d\omega_n}{dx}+B_n\frac{d}{dx}\frac{\omega_n}{r^{2n+1}}+C_n\frac{d\phi_n}{dx}+D_n\frac{d}{dx}\frac{\phi_n}{r^{2n+1}}\right\}=0\\
&\Sigma\left\{P_n\left(z\frac{d\chi_n}{dx}-x\frac{d\chi_n}{dz}\right)+A_n\frac{d\omega_n}{dy}+B_n\frac{d}{dy}\frac{\omega_n}{r^{2n+1}}+C_n\frac{d\phi_n}{dy}+D_n\frac{d}{dy}\frac{\phi_n}{r^{2n+1}}\right\}=0\\
&\Sigma\left\{P_n\left(x\frac{d\chi_n}{dy}-y\frac{d\chi_n}{dx}\right)+A_n\frac{d\omega_n}{dz}+B_n\frac{d}{dz}\frac{\omega_n}{r^{2n+1}}+C_n\frac{d\phi_n}{dz}+D_n\frac{d}{dz}\frac{\phi_n}{r^{2n+1}}\right\}=0
\end{aligned}\right\} \text{...(27),}$$

where P_n, A_n, B_n, C_n, D_n are certain constants whose precise values will be investigated further on. These conditions are to hold over the surface of the sphere $r = a$. I proceed to show that they require that

$$P_n = 0 \quad \text{.........................(28)},$$

$$A_n \omega_n + C_n \phi_n = 0 \quad \text{.........................(29)},$$

and

$$B_n \omega_n + D_n \phi_n = 0 \quad \text{.........................(30)}.$$

In the first place, if we multiply (27) by x, y, z in order, and add, we obtain, equating separately to zero the terms involving surface-harmonics of the same order n,

$$n(A_n \omega_n + C_n \phi_n) - (n+1)(B_n \omega_n + D_n \phi_n) = 0 \text{...........(31)}.$$

Again, consider the function

$$U = \Sigma \left\{ P_n \left(y \frac{d\chi_n}{dz} - z \frac{d\chi_n}{dy} \right) + A_n \frac{d\omega_n}{dx} + C_n \frac{d\phi_n}{dx} + \left(\frac{r}{a} \right)^{2n+3} \left(B_n \frac{d}{dx} \frac{\omega_n}{r^{2n+1}} + D_n \frac{d}{dx} \frac{\omega_n}{r^{2n+1}} \right) \right\}.$$

This is finite and continuous, and satisfies the equation $\nabla^2 U = 0$ throughout the interior of the sphere $r = a$, and vanishes [in virtue of (27)] at the surface. Its first derivatives are also finite and continuous. Hence we must have $U = 0$ throughout the sphere. In like manner, we have $V = 0$, $W = 0$, where V, W are obtained from U by cyclical interchange of the letters x, y, z. Hence, differentiating the three equations $U = 0$. $V = 0$, $W = 0$ with respect to x, y, z in order, and adding, we find $\Sigma (2n+3)(n+1)(B_n \omega_n + D_n \phi_n) = 0$, and thence (30). Substituting in (31), we see that (29) must also hold, and the original equations (27) then show that $P_n = 0$.

5. It appears from the foregoing argument that the fundamental modes of vibration fall naturally into two distinct classes.

The modes of the first class are obtained by making $n = 0, 1, 2, 3,$ &c., in the formulæ

$$\left. \begin{aligned} \alpha &= \psi_n(kr) \left(y \frac{d\chi_n}{dz} - z \frac{d\chi_n}{dx} \right), \\ \beta &= \psi_n(kr) \left(z \frac{d\chi_n}{dx} - x \frac{d\chi_n}{dz} \right), \\ \gamma &= \psi_n(kr) \left(x \frac{d\chi_n}{dy} - y \frac{d\chi_n}{dx} \right) \end{aligned} \right\} \quad \text{..................(32)}.$$

For any given value of n the admissible values of k are determined by the superficial conditions (18), which yield an equation in ka (written above for shortness in the form $P_n = 0$), whose precise form will be ascertained presently.

When the values of k are known, the corresponding frequencies

$(p/2\pi)$ are determined by (23). In any mode of this class, the lines of motion (to borrow a term from Hydrodynamics) lie on a series of spherical surfaces having their common centre at the origin; viz., they are the lines in which these surfaces are intersected by the cones $\chi_n/r^n = \text{const.}$; in other words, the system of lines of motion lying on any one of the spherical surfaces in question are the contour-lines of the harmonic χ_n on that surface. The motions on any two such surfaces are geometrically similar, differing only in amplitude. The amplitude of vibration at any point is proportional to the corresponding value of $\psi_n(kr)\,.\,d\chi_n/d\epsilon$, where $d\chi_n$ is the change in the value of χ_n produced by an infinitesimal displacement in a direction at right angles to the line of motion and to the radius vector, the displacement being measured by the angle $d\epsilon$ which it subtends at the centre of the sphere. The nodal surfaces consist of the spheres defined by $\psi_n(kr) = 0$, and of the cones for which $d\chi_n/d\epsilon = 0$. A closer examination of the nature of the motion in one or two of the more interesting cases will be made below.

The vibrations of the second class are of a more complex character. The fundamental modes are obtained by making $n = 0, 1, 2, 3$, &c., in the formulæ

$$\left.\begin{aligned} \alpha &= -\frac{1}{h^2}\frac{d\delta_n}{dx} + \psi_{n-1}(kr)\frac{d\phi_n}{dx} - \frac{n}{n+1}\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\psi_{n+1}(kr)\frac{d}{dx}\frac{\phi_n}{r^{2n+1}} \\ \beta &= -\frac{1}{h^2}\frac{d\delta_n}{dy} + \psi_{n-1}(kr)\frac{d\phi_n}{dy} - \frac{n}{n+1}\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\psi_{n+1}(kr)\frac{d}{dy}\frac{\phi_n}{r^{2n+1}} \\ \gamma &= -\frac{1}{h^2}\frac{d\delta_n}{dz} + \psi_{n-1}(kr)\frac{d\phi_n}{dz} - \frac{n}{n+1}\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\psi_{n+1}(kr)\frac{d}{dz}\frac{\phi_n}{r^{2n+1}} \end{aligned}\right\}\ \ldots(33),$$

where $$\delta_n = \psi_n(hr)\,.\,\omega_n \quad \ldots\ldots\ldots\ldots(34).$$

The superficial conditions yield two equations, of which (29) and (30) are the abbreviated forms. The elimination of the ratio ω_n/ϕ_n between these leads to an equation involving both ha and ka; but, since the ratio h/k is determined by the elastic properties of the substance, viz., it $= \sqrt{\{\mathfrak{n}/(\mathfrak{m}+\mathfrak{n})\}}$, there is virtually but one unknown. The vibration at any point P may be resolved into two components, one in the direction of the radius vector, the other at right angles to it. The value of the radical component is

$$(x\alpha + y\beta + z\gamma)/r = -\frac{1}{h^2}\frac{d\delta_n}{dr} + \frac{n}{r}\psi_n(kr)\,\phi_n \ldots\ldots\ldots(34a).$$

The second or tangential component is at right angles to that cone of the system $\phi_n/r^n = \text{const.}$ which passes through P, and its value is $F(r)\,.\,d\phi_n/d\epsilon$, where $F(r)$ is a certain function of r, and $d\epsilon$ has the same meaning as before.* Except in the species $n = 0$ there are no nodal surfaces (*i.e.*, surfaces at every point of which there is absolute rest), but there may be nodal *lines*.† There are, however, certain surfaces which are of great use in helping us to understand the general character of the vibration in any given case, viz., the spheres and cones at every point of which the normal motion is zero. The spherical surfaces of this kind are obtained by equating the right-hand side of (34a) to zero; the cones are those for which $d\phi_n/d\epsilon = 0$.

6. It may be well to point out, at this stage, that the results of our analysis are not in accordance with the views advanced by Lamé‡ as to the nature of the fundamental modes of vibration of elastic solids in general. Expressed in modern terminology, Lamé's reasoning is substantially as follows. From the equations (15), we deduce by differentiation

$$\frac{d^2\xi}{dt^2} = \frac{\mathfrak{n}}{\rho}\nabla^2\xi,\quad \frac{d^2\eta}{dt^2} = \frac{\mathfrak{n}}{\rho}\nabla^2\eta,\quad \frac{d^2\zeta}{dt^2} = \frac{\mathfrak{n}}{\rho}\nabla^2\zeta \;\ldots\ldots\ldots\ldots(35),$$

and

$$\frac{d^2\delta}{dt^2} = \frac{\mathfrak{m}+\mathfrak{n}}{\rho}\nabla^2\delta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(36),$$

where δ denotes, as before, the dilatation, and ξ, η, ζ are the components of the elementary rotation, at the point (x, y, z); viz.,

$$2\xi = d\gamma/dy - d\beta/dz, \text{ \&c.}$$

From these equations, and from a known theorem of Kinematics, it results,§ that any arbitrary disturbance, originated in any part of an elastic solid, breaks up in general into two distinct waves which travel with different velocities. We have, in the first place, a wave of pure rotation, unaccompanied by dilatation, propagated with a velocity $= \sqrt{(\mathfrak{n}/\rho)}$; and superposed on this there is a wave of pure dilatation, without rotation, travelling with the velocity $\sqrt{\{(\mathfrak{m}+\mathfrak{n})/\rho\}}$. In *plane* waves of these two kinds, the vibrations are respectively parallel and perpendicular to the wave front, and a similar statement holds good approximately in the general case. Hence the two kinds of waves are

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* The simplest way of verifying these statements is by a special choice of axes of coordinates, *e.g.*, take x in the direction of the radius vector, and y parallel to the normal at P to the cone $\phi_n/r^n = \text{const.}$ The value of $F(r)$ is

$$F(r) = \frac{1}{r}\left\{-\Upsilon\psi_n(hr) + \frac{(2n+1)\,\psi_{n-1}(kr) - n\psi_n(kr)}{n+1}\right\},$$

where Υ denotes the ratio of ω_n/h^2 to ϕ_n, obtained from (29) or (30).

† The nodal lines are of two classes, viz., (1) those formed by the intersections of the spheres and cones whose description follows above; and (2) those formed by the intersections of the spheres $F(r) = 0$ with the cones $\phi_n = 0$. The distinction between the two classes is illustrated in Fig. 4.

‡ *Théorie Mathématique de l'Élasticité*, 11[me] leçon.

§ Stokes, *Camb. Trans.*, t. ix., p. 12 (1849).

conveniently characterized as waves of *transverse* and of *longitudinal* vibration respectively. Now any vibration of a finite elastic solid may be conceived as made up of systems of waves travelling to and fro in it, and undergoing continual reflection at the boundary. Lamé proceeds to infer that the fundamental modes will in all cases fall into two distinct classes, in which the constituent waves are of transverse and of longitudinal displacement, respectively. In other words, in the modes of the former class there would be no change of volume, whilst in those of the latter class the motion would be irrotational. Now, although in the classification of § 5, the vibrations of the first class are purely "transversal," those of the second class are, if we except the particular case of the radial vibrations [obtained by making $n=0$ in the formulæ (33), (34)], not irrotational. The error in Lamé's reasoning consists in the tacit assumption that a wave undergoes no change of character on reflection at the boundary of the solid. It was pointed out by Green,* in 1837, that a wave of transversal vibrations, falling on the boundary of an elastic medium, will in general give rise to *two* reflected waves, one of transversal, the other of longitudinal displacement, and the complementary statement is of course also true.

The fact that the radial vibrations are the only purely "longitudinal" modes (in Lamé's sense of the word) of which an elastic sphere is capable, seems to have been first recognized by Jaerisch. The same writer has pointed out the existence of the modes which I have termed "of the second class," and has given at the end of his paper formulæ (in polar coordinates) which can be shown to be equivalent to (33), (34).

Vibrations of the First Class (Transversal Vibrations).

7. The formulæ (32) make $\delta = 0$, and $x\alpha + y\beta + z\gamma = 0$, so that the superficial conditions (18) reduce to $(rd/dr - 1)\,\alpha = 0$, &c., &c., whence

$$ka\psi_n'(ka) + (n-1)\,\psi_n(ka) = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (37).\dagger$$

This agrees with a result of Herr Jaerisch. I proceed to examine some of the simpler cases.

Species $n = 1$ (*Rotatory Vibrations*).

If we take the axis of the harmonic χ_1 as axis of x, writing $\chi_1 = x$,‡ the formulæ (31) become

$$\alpha = 0, \quad \beta = \psi_1(kr) \, . \, z, \quad \gamma = -\,\psi_1(kr) \, . \, y.$$

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* *Camb. Trans.*, t. vii., pp. 15, 16; or *Collected Papers*, p. 260.

† Since ψ_n is identical with the infinite series which occurs in the expression for the Bessel's Function of order $n+\frac{1}{2}$, it follows from known properties of these functions that the roots of this equation in ka are all real. See Todhunter, *Functions of Laplace, &c.*, § 400.

‡ The time-factor is omitted here (and elsewhere in what follows) for shortness.

Each of the infinitely thin concentric spherical strata of which the sphere may be supposed made up oscillates as a whole about the axis of the harmonic χ_1 as axis of rotation. The amplitude of the angular displacement of any such stratum is proportional to the corresponding value of $\psi_1(kr)$. The equation (37) reduces to $ka\psi_1'(ka) = 0$, *i.e.*, by (4), to $k^2a^2\psi_2(ka) = 0$. The first root is $ka = 0$; this gives an infinitely long period, and corresponds to the rotation of the sphere as a whole. The remaining roots are to be obtained from $\psi_2(ka) = 0$. By means of (7) or (8), we bring this into the form

$$\tan\theta = \frac{3\theta}{3-\theta^2} \quad\text{.............................(38)},$$

where $\theta = ka$. The approximate positions of the roots of (38) can be found by a well-known graphical method. When s is at all large, the value of the s^{th} root* only falls a little short of $(s+1)\pi$. The calculation of the roots can be carried to any desired degree of accuracy by the method explained by Fourier,† for the equation (40) below; or we may (in the case of the higher roots) make use of a series, as in Rayleigh's *Sound*, t. ii., p. 233. I find, for the first six roots,

$$ka/\pi = 1{\cdot}8346,\ 2{\cdot}8950,\ 3{\cdot}9225,\ 4{\cdot}9385,\ 5{\cdot}9489,\ 6{\cdot}9563\ \ldots(39).$$

It may be remarked that the value of ka/π for any fundamental mode is equal to T_0/τ, where τ is the period of oscillation, and T_0 is the time which would be occupied by a wave of *transversal* displacement in travelling a distance equal to the diameter of the sphere. In the mode corresponding to any one of the roots (39) after the first, the roots of lower order give the positions of the internal spherical *loop-surfaces*, *i.e.*, surfaces across which there is no stress. Thus in the second mode there is a spherical loop whose radius r is given by

$$r/a = (1{\cdot}8346)/(2{\cdot}8950),$$

or $r = {\cdot}6337a$. The positions of the spherical *nodes* are given by $\psi_1(kr) = 0$, or

$$\tan\theta = \theta \quad\text{.................................(40)},$$

where $\theta = kr$. The solutions of this are known‡ to be

$$kr/\pi = 1{\cdot}4303,\ 2{\cdot}4590,\ 3{\cdot}4709,\ 4{\cdot}4774,\ 5{\cdot}4818,\ 6{\cdot}4844\ \ldots(41).$$

Species $n = 2$.

The equation (37) becomes in this case

$$ka\,\psi_2'(ka) + \psi_2(ka) = 0 \quad\text{.......................(42)}.$$

* We do not count the zero root, which is irrelevant, having been introduced by the algebraical manipulation. The same remark applies to other equations of a similar character which occur further on.

† *Theory of Heat* (Freeman's translation), p. 271.

‡ Schwerd, quoted by Verdet, *Optique Physique*, t. i., p. 266.

Substituting the value of ψ_2 from (7) or (8), and reducing, we obtain

$$\cot\theta = \frac{5\theta^2-12}{\theta^3-12\theta} \quad\text{..........................(43)},$$

where $\theta = ka$. We readily find that the first root of this lies between $\pi/2$ and π, and that, when s is large, the s^{th} root falls a little short of $(s+\frac{1}{2})\pi$. The calculation of the first root is a little troublesome, and is best effected by one of the methods explained by Lord Rayleigh in the Society's *Proceedings*, t. v., p. 119. I find, for the first six roots,

$$ka/\pi = {\cdot}7961,\ 2{\cdot}2715,\ 3{\cdot}3469,\ 4{\cdot}3837,\ 5{\cdot}4059,\ 6{\cdot}4209\ \text{...(44)}.$$

As before, in the modes corresponding to any one of these roots after the first, the inferior roots give the positions of the internal spherical loops. The positions of the spherical nodes are given by $\psi_2(kr) = 0$, the roots of which have been already stated in (39).

The character of the vibration depends on the form of χ_2. The fundamental types of spherical harmonic of the second degree are the zonal harmonic, and the sectorial harmonic with two nodal diametral planes. Taking first the case of the zonal harmonic, we write

$$\chi_2 = 2x^2-y^2-z^2;$$

the formulæ (32) then become

$$\alpha = 0,\quad \beta = 6\psi_2(kr)\,.\,xz,\quad \gamma = -\,6\psi_2(kr)\,.\,xy.$$

Every particle oscillates in a direction at right angles to the meridional plane in which it is situate, the amplitude of the angular displacement about the axis of the harmonic being proportional to the corresponding value of $\psi_2(kr)\,.\,x$. The sign of the displacement is reversed as we cross the equatorial plane, which is nodal. In the particular case where there is no spherical node, the motion in its main features somewhat resembles the gravest torsional vibration of a solid cylinder having about equal lateral and longitudinal dimensions. The comparative slowness of the vibration in this case [its frequency is proportional to the slowest of the roots (44)] is thus easily understood.*

The modes of vibration corresponding to the sectorial harmonic may be examined by making $\chi_2 = yz$. The motion is symmetrical with respect to each of the planes $y = 0$, $z = 0$; and the lines of motion corresponding to any surface having its centre at the origin form four systems of oval curves lying in the four lunes into which the surface is divided by the two planes in question.†

* For a cylinder of length $2a$ the corresponding value of ka is $\cdot 5$. From the results given further on it appears probable that the mode in question is the gravest of all the fundamental modes of the sphere.

† This statement may be generalized so as to apply to all forms of the harmonic χ_2, provided we replace the two planes referred to by two planes drawn through the centre of the sphere at right angles to the two axes of χ_2. It is interesting to trace in this way the transition to the case of the zonal harmonic, when the two axes coincide.

I have not thought it necessary to carry the discussion of the transversal vibrations any further. With the help of the properties stated in § 5, a clear idea of the nature of the motion in some varieties of the third and fourth species may be gathered from Plates VI. to IX., at the end of the first volume of Maxwell's *Electricity*.

VIBRATIONS OF THE SECOND CLASS.

8. These are defined by the equations (33), which we write for shortness in the form

$$\alpha = -\frac{1}{h^2}\frac{d\delta_n}{dx}+\alpha', \quad \beta = -\frac{1}{h^2}\frac{d\delta_n}{dy}+\beta', \quad \gamma = -\frac{1}{h^2}\frac{d\delta_n}{dz}+\gamma' \quad \ldots(45).$$

Since $k^2/h^2 = (\mathfrak{m}+\mathfrak{n})/\mathfrak{n}$, the first of the surface conditions (18) may be written

$$\left(\frac{k^2}{h^2}-2\right)x\delta_n+\left(r\frac{d}{dr}-1\right)\alpha+\frac{d}{dx}(x\alpha+y\beta+z\gamma) = 0 \quad \ldots\ldots(46).$$

Let us first substitute in this the parts of α, β, γ which involve δ_n. We obtain

$$\left(\frac{k^2}{h^2}-2\right)x\delta_n-\frac{2}{h^2}\,r\frac{d}{dr}\cdot\frac{d\delta_n}{dx} \quad \ldots\ldots(47).$$

Now,

$$x\delta_n = \psi_n(hr)\cdot\frac{r^2}{2n+1}\left(\frac{d\omega_n}{dx}-r^{2n+1}\frac{d}{dx}\frac{\omega_n}{r^{2n+1}}\right) \quad \ldots\ldots(48),$$

and

$$\frac{d\delta_n}{dx} = \psi_{n-1}(hr)\frac{d\omega_n}{dx}+\frac{h^2r^{2n+2}}{2n+1\,.\,2n+3}\,\psi_{n+1}(hr)\frac{d}{dx}\frac{\omega_n}{r^{2n+1}} \quad \ldots\ldots(49),$$

the reductions being effected by means of (12) and of the relations (4), (5). Hence

$$r\frac{d}{dr}\frac{d\delta_n}{dx} = \{hr\psi'_{n-1}(hr)+(n-1)\psi_{n-1}(hr)\}\frac{d\omega_n}{dx}$$
$$+\frac{h^2r^{2n+2}}{2n+1\,.\,2n+3}\{hr\psi'_{n+1}(hr)+(n+1)\psi_{n+1}(hr)\}\frac{d}{dx}\frac{\omega_n}{r^{2n+1}} \quad \ldots(50).$$

Substituting from (48) and (50) in (47), and making $r = a$, we obtain, after a little reduction,

$$A_n\frac{d\omega_n}{dx}+B_n\frac{d}{dx}\frac{\omega_n}{r^{2n+1}} \quad \ldots\ldots(51),$$

where

$$A_n = \frac{1}{h^2}\left\{\frac{k^2a^2}{2n+1}\psi_n(ha)-2(n-1)\psi_{n-1}(ha)\right\} \quad \ldots\ldots(52),$$

$$B_n = -\frac{1}{h^2}\cdot\frac{k^2a^{2n+3}}{2n+1}\left\{\psi_n(ha)+\frac{2(n+2)}{k^2a^2}ha\psi'_n(ha)\right\} \quad \ldots\ldots(53).$$

It remains to calculate the part of (46) depending on α', β', γ'. We have, by (9),

$$x\alpha'+y\beta'+z\gamma' = n\psi_n(kr)\,.\,\phi_n,$$

and thence, as in (49),

$$\frac{d}{dx}(x\alpha'+y\beta'+z\gamma') = n\psi_{n-1}(kr)\frac{d\phi_n}{dx}+n\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\psi_{n+1}(kr)\frac{d}{dx}\frac{\phi_n}{r^{2n+1}}.$$

Again,

$$\left(r\frac{d}{dr}-1\right)\alpha' = \{kr\psi'_{n-1}(kr)+(n-2)\,\psi_{n-1}(kr)\}\frac{d\phi_n}{dx}$$
$$-\frac{n}{n+1}\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\{kr\psi'_{n+1}(kr)+n\psi_{n+1}(kr)\}\frac{d}{dx}\frac{\phi_n}{r^{2n+1}}.$$

Hence the remaining part of the surface condition (46) is found to be

$$C_n\frac{d\phi_n}{dx}+D_n\frac{d}{dx}\frac{\phi_n}{r^{2n+1}} \quad\text{..........................(54)},$$

where
$$C_n = ka\psi'_{n+1}(ka)+2\,(n-1)\,\psi_{n-1}(ka)$$
$$= -\left\{\frac{k^2a^2}{2n+1}\psi_n(ka)-2\,(n-1)\,\psi_{n-1}(ka)\right\} \quad\text{......(55)},$$

and
$$D_n = -\frac{n}{n+1}\frac{k^2a^{2n+3}}{2n+1\,.\,2n+3}\{ka\psi'_{n+1}(ka)-\psi_{n+1}(ka)\}$$
$$= -\frac{n}{n+1}\frac{k^2a^{2n+3}}{2n+1}\left\{\psi_n(ka)+\frac{2\,(n+2)}{k^2a^2}ka\psi'_n(ka)\right\} \quad\text{......(56)},$$

the transformations being effected by means of the relations (4), (5).

It follows from the reasoning of § 4, that the final equation to determine the frequencies of vibration is to be found* by eliminating the ratio ω_n/ϕ_n between the equations

$$A_n\omega_n+C_n\phi_n = 0 \quad\text{..................(29)},$$
$$B_n\omega_n+D_n\phi_n = 0 \quad\text{..................(30)},$$

viz., it is
$$B_n/A_n = D_n/C_n \quad\text{..........................(57)},$$

where the values of A_n, B_n, C_n, D_n are to be substituted from (52), (53), (55), (56).

I proceed to examine some of the more important cases.

Species $n = 0$ (*Radial Vibrations*).

9. In this case we have α', β', $\gamma' = 0$, so that

$$\alpha = -\frac{1}{h^2}\frac{d\delta_0}{dx},\quad \beta = -\frac{1}{h^2}\frac{d\delta_0}{dy},\quad \gamma = -\frac{1}{h^2}\frac{d\delta_0}{dz}$$

* It is here assumed that $n > 0$. When $n = 0$ we have α', β', $\gamma' = 0$, and the process becomes much simpler. See § 9.

where δ_0 is proportional to $\psi_0(hr)$. The motion is therefore symmetrical about the centre of the sphere, being everywhere in the direction of the radius vector. The expression (51) reduces to

$$B_0 \frac{d}{dx} \frac{\omega_0}{r},$$

where ω_0 is a constant, so that the surface conditions yield simply $B_0 = 0$. This may be written

$$\psi_0(\theta) + \frac{4}{\lambda^2 \theta} \psi_0'(\theta) = 0 \text{.......................} (58),^*$$

where $\theta = ha$, and λ^2, $= h^2/k^2$, is a constant depending on the elastic properties of the substance of the sphere, viz., it $= \mathfrak{n}/(\mathfrak{m}+\mathfrak{n})$. If σ denote the ratio of lateral contraction to longitudinal extension when a bar of the substance is stretched lengthwise, we have $\sigma = (\mathfrak{m}-\mathfrak{n})/2\mathfrak{m}$,

and thence
$$\lambda^2 = \frac{\frac{1}{2}-\sigma}{1-\sigma}.$$

For actual solids the values of σ may, for all we know, range from 0 to $\frac{1}{2}$,† and therefore λ^2 may range between $\frac{1}{2}$ and 0.

There is no difficulty in showing that, for any admissible value of λ^2, the roots of (58) are all real, and are separated by the roots of $\psi_0(\theta) = 0$, which are $\theta = s\pi$. Writing for $\psi_0(\theta)$ its value $\sin\theta/\theta$, we bring (58) into the form

$$\tan\theta = \frac{4\theta}{4 - \dfrac{\theta^2}{\lambda^2}} \text{.............................} (59).$$

For any assigned value of λ^2 the roots of this can be calculated by methods already indicated. I have computed the values of ha/π corresponding to the first six roots in each of the following cases:—

$$\sigma = 0,\ \tfrac{1}{4},\ \tfrac{3}{10},\ \tfrac{1}{3}.\ ‡$$

The corresponding values of λ^2 are

$$\lambda^2 = \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{2}{7},\ \tfrac{1}{4}.$$

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* The theory of the radial vibrations has been treated by Clebsch. Allowing for difference of notation, the equation (58) will be found to agree with that numbered (48) at p. 69 of his *Theorie der Elasticität.*

† The case $\sigma = 0$ seems to be realized approximately in cork. The value $\sigma = \frac{1}{2}$ corresponds to a substance whose resilience of volume is infinite in comparison with its rigidity.

‡ The values $\sigma = \frac{1}{4}$, $\sigma = \frac{1}{3}$ are those assumed by Poisson and Wertheim respectively as applicable to all truly isotropic substances. The value $\sigma = \frac{3}{10}$ lies between the values (·294 and ·310) found experimentally by Kirchhoff and Everett, respectively, for *steel.* See Everett, *Units and Physical Constants*, p. 64.

The results of the calculation are given in the following table:—

s	$\sigma = 0$	$\sigma = \frac{1}{4}$	$\sigma = \frac{3}{10}$	$\sigma = \frac{1}{3}$
1	·6626	·8160	·8500	·8733
2	1·8909	1·9285	1·9391	1·9470
3	2·9303	2·9539	2·9606	2·9656
4	3·9485	3·9658	3·9707	3·9744
5	4·9590	4·9728	4·9767	4·9796
6	5·9660	5·9774	5·9806	5·9830

The physical meaning of the constant ha/π is similar to that of ka/π, viz., we have $ha/\pi = T_1/\tau$, where T_1 is the time occupied by a wave of *longitudinal* vibration in traversing a space equal to the diameter of the sphere, and τ denotes as before the time of oscillation.

To compare the frequencies of vibration in the above cases with those of the various transversal modes, we should have to calculate the corresponding values of ka/π. These would be obtained by multiplying the numbers in the several columns of the above table by the factors

$$1·4142, \quad 1·7321, \quad 1·8708, \quad 2,$$

respectively. It appears that, for any particular mode, the value of ka increases as λ^2 diminishes. This is in accordance with a general principle formulated by Lord Rayleigh.* If, keeping $\mathfrak{n}$ constant, we increase $\mathfrak{m}$, the potential energy corresponding to any deformation which involves a change of density will be increased. It follows, by the principle referred to, that all the modes of the second class will rise in frequency.

In the mode corresponding to any root of (59) after the first, the inferior roots give the positions of the spherical loops. The positions of the spherical nodes are determined by $d\delta_0/dr = 0$, or $\tan hr = hr$. The corresponding values of hr/π are the numbers quoted in (41).

10. In dealing with the higher species, it is convenient to discuss separately, in the first instance, the case where the substance of the sphere is *incompressible* (*i.e.*, $\mathfrak{m} = \infty$, $\lambda^2 = 0$, $\sigma = \frac{1}{2}$). This is of course an extreme case, but it is very amenable to mathematical treatment, and the results will help us to understand the nature of the various modes in other cases where the actual calculation would be much more difficult. Moreover, as we have just seen, the values of ka obtained on the supposition of incompressibility will be superior limits to the values of ka for the corresponding modes in the general case.

Making, then, $h = 0$ in the formulæ for A_n, B_n, C_n, D_n, we find that

* *Theory of Sound*, t. i., p. 85.

the equations (29), (30) may be written

$$\left\{\frac{\theta^2}{2n+1}-2(n-1)\right\}\frac{\omega_n}{h^2}-\left\{\frac{\theta^2}{2n+1}\psi_n-2(n-1)\psi_{n-1}\right\}\phi_n=0\ldots(60),$$

$$\frac{\omega_n}{h^2}+\frac{n}{n+1}\left\{\psi+\frac{2(n+2)}{\theta}\psi_n'\right\}\phi_n=0\ldots\ldots\ldots\ldots\ldots\ldots(61),$$

where $\theta = ka$, and ψ_n is written in place of $\psi_n(\theta)$ for shortness. It is to be remarked that, though h is zero, ω_n/h^2 is finite; viz., if Π denote the average normal *tension* about the point (x, y, z), we have $\omega_n/h^2 = \Pi/p^2\rho$. Eliminating the ratio ω_n/ϕ_n between (60), (61), we obtain an equation to determine θ. This equation can, by means of the relations (4), (5), be put into a great variety of forms, of which the following is perhaps the simplest:—

$$(2n-1)\psi_{n-2}-(2n+1)\psi_{n-1}+2n\psi_n-\frac{4n\,.\,n-1\,.\,n+2}{2n+1\,.\,2n+3}\psi_{n+1}=0\ldots(62).$$

When expanded in rising powers of θ, this equation becomes, after considerable reduction,

$$\Sigma\frac{8m^3+m^2(16n+8)+m(16n^2+4n-10)+4n^3+4n^2-2n-6}{2\,.\,4\ldots2m\,.\,2n+1\,.\,2n+3\ldots2n+2m+3}(-)^m\,.\,\theta^{2m}$$
$$=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(63),$$

the first term being

$$\frac{(n-1)(4n^2+8n+6)}{2n+1\,.\,2n+3}.$$

In the actual calculation of the roots, I have, however, preferred to work out the formulæ for each value of n separately, using the equations (62), (63) only by way of verification.

For any assigned value of n, the character of the vibration will depend of course on the order of the root and on the form of the harmonic ϕ_n.* In the important case of the *zonal* harmonic, we may borrow the method of the Stream-function from Hydrodynamics. If the motion of an incompressible substance take place in a series of planes through the axis of x, and be the same in each such plane, and if u, v, w denote the components of the displacement at the point (x, y, z), we may write†

$$u=\frac{1}{\varpi}\frac{d\Psi}{d\varpi},\quad \mathfrak{v}=-\frac{1}{\varpi}\frac{d\Psi}{dx}\ldots\ldots\ldots\ldots\ldots\ldots(64),$$

where $\varpi = \sqrt{(y^2+z^2)}$, and $\mathfrak{v}, = (yv+zw)/\varpi$, is the displacement in the direction of ϖ. The function Ψ represents 2π times the total volume displaced across any surface bounded by the circle whose coordinates

* Or ω_n, since these two harmonics are proportional.

† *Motion of Fluids*, § 103.

are (x, ϖ). Let us apply this to the case where

$$u = \psi_{n-1}(kr)\frac{d\phi_n}{dx} - \frac{n}{n+1}\,\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\,\psi_{n+1}(kr)\frac{d}{dx}\frac{\phi_n}{r^{2n+1}}, \text{ \&c., \&c.,}$$

ϕ_n being a zonal harmonic with the x-line as axis. The corresponding value (Ψ_1 say) of Ψ is*

$$\Psi_1 = yN - zM \quad\text{.............................(65)},$$

where M, N are to be found from (14) and (11), by putting $\chi_n = 0$, and omitting the sign Σ; viz.,

$$M = -\frac{1}{n+1}\psi_n(kr)\left(z\frac{d\phi_n}{dx} - x\frac{d\phi_n}{dz}\right),$$

$$N = -\frac{1}{n+1}\psi_n(kr)\left(x\frac{d\phi_n}{dy} - y\frac{d\phi_n}{dx}\right).$$

Substituting in (65), we find

$$\Psi_1 = -\frac{1}{n+1}\psi_n(kr)\left(nx\phi_n - r^2\frac{d\phi_n}{dx}\right) \quad\text{..............(66)}.$$

The stream-function (Ψ_2 say) corresponding to the case

$$u = -\frac{1}{h^2}\frac{d\omega_n}{dx}, \quad v = -\frac{1}{h^2}\frac{d\omega_n}{dy}, \quad w = -\frac{1}{h^2}\frac{d\omega_n}{dz},$$

where ω_n also is a zonal harmonic with the x-line as axis, is found from this by making $k = 0$ and writing $-\omega_n/h^2$ for ϕ_n, viz., it is

$$\Psi_2 = \frac{1}{h^2}\,\frac{1}{n+1}\left(nx\omega_n - r^2\frac{d\omega_n}{dx}\right) \quad\text{.....................(67)}.$$

Hence for the fundamental modes of species n we have, when the harmonic ϕ_n is zonal,

$$\Psi = \Psi_1 + \Psi_2 = -\frac{1}{n+1}\left(nx\phi_n - r^2\frac{d\phi_n}{dx}\right)\{\psi_n(kr) - \Upsilon\} \quad\text{......(68)},$$

where Υ denotes the value of the ratio of ω_n/h^2 to ϕ_n, to be obtained from (60) or (61). This result may also be expressed in the form

$$\Psi = -\frac{1}{n+1}\varpi\frac{d\phi_n}{d\epsilon}\{\psi_n(kr) - \Upsilon\} \quad\text{.................(68a)},$$

where ϵ is the co-latitude. By assigning to Ψ a series of constant values, we get the equations of the lines of motion.

From (34) and (34a) we obtain, since h is now $= 0$,

$$x\alpha + y\beta + z\gamma = n\{\psi_n(kr) - \Upsilon\}\phi_n \quad\text{.................(69)}.$$

This formula holds independently of the form of ϕ_n. We see that, over

* *Ibid.*, §§ 141, 142.

the spherical surfaces for which $\psi_n(kr) = \Upsilon$, the motion is wholly tangential.

We can now proceed to the examination of particular cases.

Species $n = 1$. $[\sigma = \frac{1}{4}.]$

11. The equations (60), (61) become

$$\frac{\omega_1}{h^2} - \psi_1(\theta) \cdot \phi_1 = 0 \quad \text{.......................(70),*}$$

$$\frac{\omega_1}{h^2} + \tfrac{1}{2}\left\{\psi_1(\theta) + \frac{6}{\theta}\psi_1'(\theta)\right\}\phi_1 = 0 \quad \text{..................(71).}$$

Eliminating the ratio ω_1/ϕ_1, we find

$$1 + \frac{2\psi_1'(\theta)}{\theta\psi_1(\theta)} = 0 \quad \text{..........................(72).}$$

It is easy to prove that the roots of this are all real, and are separated by the roots of $\psi_1(\theta) = 0$, which have been given in (41). If we substitute the value of $\psi_1(\theta)$ from (7) or (8), (72) becomes

$$\cot\theta = \frac{3\theta^2 - 6}{\theta^3 - 6\theta} \quad \text{..............................(73).}$$

I find, for the first six roots of this equation,

$$ka/\pi = 1{\cdot}2319,\ 2{\cdot}3692,\ 3{\cdot}4101,\ 4{\cdot}4310,\ 5{\cdot}4439,\ 6{\cdot}4528 \ \ldots(74)$$

The motion is evidently symmetrical about the axis of the harmonic ϕ_1 which we may conveniently take as axis of x, writing $\phi_1 = x$. From (70) we have $\Upsilon = \psi_1(ka)$, so that (68) becomes

$$\Psi = \frac{\varpi^2}{2}\{\psi_1(kr) - \psi_1(ka)\} \quad \text{.....................(75).}$$

It appears that at the surface of the sphere the motion is wholly tangential. I have traced, in Figs. 1 and 2, the lines of motion corresponding to equidistant values of Ψ, for the fundamental modes corresponding to the first and second roots of (73). In the mode corresponding to the s^{th} root, there are $s-1$ internal spherical surfaces of zero normal motion whose radii are given by

$$\psi_1(kr) = \psi_1(ka) \quad \text{.......................(76) ;†}$$

there are no spherical nodes (in the strict sense of the term), but there are s nodal *circles* in the equatorial plane. The radii of these circles

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* This equation may be otherwise obtained by expressing that the total momentum of the sphere is zero.

† For $s = 2$, I find $kr/\pi = 1{\cdot}465$, whence $r/a = {\cdot}618$.

are defined by $d\Psi/d\varpi = 0$, $x = 0$; *i.e.*, by

$$\psi_1(k\varpi) + \frac{k\varpi}{2}\psi_1'(k\varpi) = \psi_1(ka) \quad \text{.................(77).*}$$

There are no internal spherical loops.

The amplitude of vibration at the centre of the sphere is (on the scale of our formulæ)

$$a_0 = 1 - \psi_1(ka),$$

and the amplitude at any point on the equator is (on the same scale)

$$a_E = \tfrac{1}{2}ka\psi_1'(ka).$$

For the first mode I find $a_E/a_0 = -\cdot 4865$.

Species $n = 2$. $[\sigma = \frac{1}{3}.]$

Writing $n = 2$ in the formulæ (60), (61), we have

$$\left\{\frac{\theta^2}{5} - 2\right\}\frac{\omega_2}{h^2} - \left\{\frac{\theta^2}{5}\psi_2(\theta) - 2\psi_1(\theta)\right\}\phi_2 = 0 \quad \text{.........(78),}$$

$$\frac{\omega_2}{h^2} + \frac{2}{3}\left\{\psi_2(\theta) + \frac{8}{\theta}\psi_2'(\theta)\right\}\phi_2 = 0 \quad \text{............(79),}$$

where $\theta = ka$ as before. Eliminating ω_2/ϕ_2, and substituting the values of ψ_1, ψ_2 from (7) or (8), we obtain, after some reduction,

$$\tan\theta = \frac{-5\theta^5 + 92\theta^3 - 480\theta}{\theta^6 - 25\theta^4 + 252\theta^2 - 480} \quad \text{....................(80).}$$

Solving this, I find

$ka/\pi = \cdot 8485,\ 1\cdot 7420,\ 2\cdot 8257,\ 3\cdot 8709,\ 4\cdot 8974,\ 5\cdot 9148$, &c....(81).

The frequency of the first mode is therefore much lower than in the species $n = 1$.

The radii of the spherical surfaces of zero normal motion are to be found from

$$\psi_2(kr) = \Upsilon \quad \text{..............................(82),}$$

where

$$\Upsilon = \frac{k^2a^2\psi_2(ka) - 10\psi_1(ka)}{k^2a^2 - 10} \quad \text{..................(83),†}$$

by (78).

The character of the vibration will in other respects depend upon the form of ϕ_2. For the case of the zonal harmonic we may utilize the formulæ (68). If we write $\phi_2 = 2x^2 - y^2 - z^2$, this becomes

$$\Psi = \tfrac{2}{3}x\varpi^2\{\psi_2(kr) - \Upsilon\} \quad \text{.......................(84).}$$

* For $s = 1$, I find $k\varpi/\pi = \cdot 792$, or $\varpi/a = \cdot 643$. For $s = 2$, $k\varpi/\pi = \cdot 885$ and $1\cdot 935$, whence $\varpi/a = \cdot 374$ and $\cdot 817$. The spheres on which the nodal circles are great circles cut the lines of motion at right angles.

† For the first and second roots, I find $\Upsilon = \cdot 1171$ and $\Upsilon = \cdot 0789$, respectively.

I have traced, in Figs. 3 and 4, the forms of the lines of motion in the first two modes. The first mode is, from one point of view, the most important of all the fundamental modes of the sphere, for it corresponds to the gravest longitudinal vibration of a bar, or other elongated body. It is interesting to trace in imagination the changes in the forms of the lines of motion as we pass by gradual stages from such a body to a sphere.

It may easily be shown that, in all the modes at present under consideration, the amplitudes of vibration at two points equidistant from the centre, and situate, one in the axis of the harmonic ϕ_2, the other in the equatorial plane, are in the ratio of two to one.

The nature of the vibrations when the harmonic ϕ_2 is sectorial may be examined by making $\phi_2 = y^2 - z^2$. In the gravest mode, the y-diameter expands whilst the z-diameter contracts, and *vice versa*, all points in the x-diameter remaining at rest.

The species of higher orders ($n = 3$, 4, &c.) seem hardly of sufficient interest to call for a detailed examination. The main difficulty, in attempting to form a clear idea of the nature of the various fundamental modes, arises from our want of familiarity with the nature and properties of the various types of spherical harmonic which are involved. When these are thoroughly comprehended, the statements made near the end of § 5 give a fair indication of the characters of the corresponding vibrations.

12. It remains to examine in what way the conclusions of § 11 must be modified when the substance of the sphere is not incompressible.

Species $n = 1$. $[\sigma < \tfrac{1}{2}.]$

The equations (29), (30) become

$$\psi_1(ha)\cdot\frac{\omega_1}{h^2} - \psi_1(ka)\cdot\phi_1 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(85),^*$$

$$\left\{\psi_1(ha) + \frac{6}{k^2a^2}\,ha\psi_1'(ha)\right\}\frac{\omega_1}{h^2} + \tfrac{1}{2}\left\{\psi_1(ka) + \frac{6}{ka}\,\psi_1'(ka)\right\}\phi_1 = 0 \;\ldots(86).$$

Eliminating the ratio ω_1/ϕ_1, we obtain

$$1 + \frac{4\lambda\psi_1'(\lambda\theta)}{\theta\psi_1(\lambda\theta)} + \frac{2\psi_1'(\theta)}{\theta\psi_1(\theta)} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(87),$$

where $\theta = ka$, and $\lambda = h/k, = \sqrt{\{n/(m+n)\}}$, as before. It is readily found that the expression on the left-hand side of (87) is positive for small values of θ, and again that the fraction $\psi_1'(\theta)/\psi_1(\theta)$ changes

* See the footnote to equation (70).

from $-\infty$ to $+\infty$ whenever θ, increasing,* passes through a root of $\psi_1(\theta) = 0$. Hence, if $\vartheta_1, \vartheta_2, \vartheta_3$, &c., be the combined system of roots of the two equations $\psi_1(\theta) = 0$, $\psi_1(\lambda\theta) = 0$, or (say) the roots of

$$\psi_1(\theta) \,.\, \psi_1(\lambda\theta) = 0,$$

arranged in ascending order of magnitude, it appears that one root of (87) lies between 0 and ϑ_1, a second between ϑ_1 and ϑ_2, a third between ϑ_2 and ϑ_3, and so on. It can thence be shown, by much the same method as in Todhunter's *Functions of Laplace, &c.*, § 396, that the roots of (87) are all real, and that only one root lies within each of the intervals specified. The actual calculation of the roots is laborious. A good approximation to the first root may be made by the method of Lord Rayleigh's paper already cited. The form of the method which I have adopted is that ascribed by Prof. Cayley (in a note at the end of the paper in question) to Encke. If the equation (87) be cleared of fractions, and expanded in rising powers of θ, it assumes, after division by θ^8, the form

$$P - Q\theta^2 + R\theta^4 - S\theta^6 + T\theta^8 - \&c. = 0 \;\ldots\ldots\ldots\ldots\ldots (88).$$

I find, for the values of the coefficients,

$$P = 3 - 4\lambda^2,$$

$$Q = \frac{5}{14} - \frac{1}{10}\lambda^2 - \frac{2}{7}\lambda^4,$$

$$R = \frac{1}{72} + \frac{3}{140}\lambda^2 - \frac{1}{56}\lambda^4 - \frac{1}{126}\lambda^6,$$

$$S = \frac{1}{3696} + \frac{17}{15120}\lambda^2 + \frac{1}{3920}\lambda^4 - \frac{1}{1680}\lambda^6 - \frac{1}{8316}\lambda^8,$$

$$T = \frac{1}{314496} + \frac{1}{41580}\lambda^2 + \frac{13}{423360}\lambda^4 - \frac{1}{211680}\lambda^6 - \frac{13}{1330560}\lambda^8 - \frac{1}{864864}\lambda^{10}.$$

Now, let $a = Q/P$, $b = R/P$, $c = S/P$, &c., so that (88) may be written

$$1 - a\theta^2 + b\theta^4 - c\theta^6 + d\theta^8 - \&c. = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (89).$$

If the equation whose roots are the squares of the roots of this be

$$1 - a_1\theta^2 + b_1\theta^4 - \&c. = 0,$$

we have $\quad a_1 = a^2 - 2b, \quad b_1 = b^2 - 2ac + 2d,$ &c.,

and we may suppose a_2, b_2, &c. derived from a_1, b_1, &c., in like manner. I have calculated the value of a_2 for the cases

$$\lambda^2 = \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{7}, \tfrac{1}{4}, 0,$$

which correspond to $\quad \sigma = 0, \tfrac{1}{4}, \tfrac{3}{10}, \tfrac{1}{3}, \tfrac{1}{2}.$

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* We are here considering only positive values of θ. The roots of (87) occur of course in pairs, of equal magnitude and opposite sign.

If $\theta_1, \theta_2, \theta_3$, &c. denote the roots of (89), we have $a_3 = \Sigma_1^\infty \theta^{-8}$. By equating a_3 to θ_1^{-8}, we obtain an approximate value of θ_1.* To obtain a more accurate value, we may apply a correction for the higher roots (which need not be known very accurately) in the manner explained in the paper referred to. The numbers given in the first line of the following table were calculated in this way, and afterwards verified on the equation obtained from (87), by substituting the values of $\psi_1(\theta)$, $\psi_1(\lambda\theta)$ from (8). The table gives the values of ka/π for the first three roots in each of the above-mentioned cases. The values for the second and third roots were found by tentative methods, except in the case $\sigma = \frac{1}{2}$, where they are repeated from (74).

s	$\sigma = 0$	$\sigma = \frac{1}{4}$	$\sigma = \frac{3}{10}$	$\sigma = \frac{1}{3}$	$\sigma = \frac{1}{2}$
1	·900	1·090	1·124	1·145	1·232
2	1·770	2·155	2·260	2·305	2·369
3	2·387	2·465	2·571	2·721	3·410

This table is a further illustration of Lord Rayleigh's principle cited above under the head of the radial vibrations. It appears that the value of $\mathfrak{m}$ has a very considerable influence on the frequency.

The radial component of the vibration is, by (34a),

$$-\frac{1}{h^2}\frac{d\delta_1}{dr} + \frac{1}{r}\psi_1(kr)\,\phi_1.$$

Writing $\phi_1 = x$, and making use of (34) and (85), we put this into the form

$$\left[\psi_1(kr) - \frac{\psi_1(ka)}{\psi_1(ha)}\{\psi_1(hr) + hr\psi_1'(hr)\}\right]\frac{x}{r}.$$

The radii of the spherical surfaces of zero normal motion are the values of r which make this expression vanish. When $r = a$ the value of the expression within square brackets becomes

$$= -\frac{ha\psi_1'(ha)}{\psi_1(ha)}\psi_1(ka)$$

$$= \frac{ka}{4}\{ka\psi_1(ka) + 2\psi_1'(ka)\} \quad\text{......................(90)},$$

the transformation being effected by means of (87). We verify, from either of those forms, that in the particular case $\sigma = \frac{1}{2}$ the radial motion is zero at the boundary. For other values of σ, there is in general a finite component of vibration in the direction of the normal. It is easily seen that the expression

$$\theta\psi_1(\theta) + 2\psi_1'(\theta)$$

* The values so obtained are in fact too small by about seven or eight units in the fourth place of decimals.

is positive for small values of θ; and the values of θ/π for which it changes sign have been given in (74). Hence, and from the foregoing table, we learn that (for values of $\sigma < \frac{1}{2}$) the signs of the expression (90) are, for the first three modes, +, −, +, respectively. We infer, with certainty as to the second mode, and with great probability as to the third, that for all values of σ there are in these cases one and two internal spherical surfaces, respectively, over which the radial motion is zero.* A rough idea of the arrangement of the lines of motion in these two modes may (at all events for values of σ only a little less than $\frac{1}{2}$) be obtained by drawing, in Figs. 1 and 2, a circle concentric with, and a little *inside*, the external boundary, and by taking the circle so drawn to represent the surface of the sphere.

Some further indications as to the influence of the value of σ on the character of the vibration may be obtained by comparing the signs of the vibration at the centre of the sphere and at points on the surface on the equator of the harmonic ϕ_1. The amplitude at the centre is

$$a_0 = 1 - \frac{\psi_1(ka)}{\psi_1(ha)},$$

and the amplitude at any point of the equator is

$$a_x = -\frac{k^2a^2}{10}\psi_2(ka).$$

It is readily found, for the first three modes, that a_0 is +, and that the signs of a_x are as given by the following scheme—

s	$\sigma = 0$	$\sigma = \frac{1}{4}$	$\sigma = \frac{3}{10}$	$\sigma = \frac{1}{3}$	$\sigma = \frac{1}{2}$
1	−	−	−	−	−
2	−	+	+	+	+
3	+	+	+	+	−

From this we gather that in the *first* mode there is always a nodal circle; that in the *second* mode there are two nodal circles if σ is not less than $\frac{1}{4}$, but that the second nodal circle disappears for some value of σ between 0 and $\frac{1}{4}$; and that in the *third* mode, although for $\sigma = \frac{1}{3}$ there are three nodal circles, yet the third of these disappears for some value of σ between $\frac{1}{3}$ and $\frac{1}{2}$.

Species $n = 2$. $[\sigma < \frac{1}{2}.]$

The equations (29), (30) become, when $n = 2$,

$$\left\{\frac{k^2a^2}{5}\psi_2(ha) - 2\psi_1(ha)\right\}\frac{\omega_2}{h^2} - \left\{\frac{k^2a^2}{5}\psi_2(ka) - 2\psi_1(ka)\right\}\phi_2 = 0 \;\ldots(91),$$

* We cannot, however, generalize this statement, and assert that in the s^{th} mode there are always $s-1$ such surfaces. There will in general be less than this number.

$$\left\{\psi_2(ha)+\frac{8}{k^2a^2}ha\psi_2'(ha)\right\}\frac{\omega_2}{h^2}+\tfrac{2}{3}\left\{\psi_2(ka)+\frac{8}{k^2a^2}ka\psi_2'(ka)\right\}\phi_2=0 \quad \text{..(92).}$$

The equation derived by eliminating ω_2/ϕ_2 is considerably more complicated than in the former species. I have therefore confined myself to the calculation of the lowest root, by Encke's method, for the same values of σ as before. When cleared of fractions and expanded in rising powers of θ, the equation in question assumes the form

$$P-Q\theta^2+R\theta^4-S\theta^6+T\theta^8-\&c.=0,$$

where $\theta=ka$. I find, for the values of the coefficients,

$$P=19-24\lambda^2,$$

$$Q=\frac{37}{9}-\frac{7}{2}\lambda^2-\frac{4}{3}\lambda^4,$$

$$R=\frac{193}{792}+\frac{5}{126}\lambda^2-\frac{13}{56}\lambda^4-\frac{1}{33}\lambda^6,$$

$$S=\frac{25}{3861}+\frac{125}{11088}\lambda^2-\frac{1}{168}\lambda^4-\frac{185}{33264}\lambda^6-\frac{1}{2574}\lambda^8,$$

$$T=\frac{719}{7413120}+\frac{163}{432432}\lambda^2+\frac{19}{133056}\lambda^4-\frac{59}{299376}\lambda^6$$
$$-\frac{253}{3459456}\lambda^8-\frac{1}{308880}\lambda^{10},$$

where λ^2 has the same meaning as before. When the value of a_2 for any assigned value λ^2 has been computed, the correction for the higher roots may be roughly estimated from its known value in the case $\lambda^2=0$. In this way I have obtained the following values of ka/π, which are probably accurate to the last figure given:—

$\sigma=0$	$\sigma=\frac{1}{4}$	$\sigma=\frac{3}{10}$	$\sigma=\frac{1}{3}$	$\sigma=\frac{1}{2}$
·823	·840	·842	·843	·848

It appears that the influence of the value of m on the frequency is (especially within such limits of the ratio m/n as include most actual solids) very slight.

13. As an application of the preceding results we may calculate the frequency of vibration of a steel ball one centimetre in radius, for the slowest of those fundamental modes in which the surface oscillates in the form of a harmonic spheroid of the second order. In § 12 we obtained for this case $ka/\pi=$ ·842. Now, $ka/\pi=T_0/\tau$, where τ is the period, and $T_0=2a/\sqrt{(n\rho^{-1})}$. Making then $a=1$, and adopting

from Everett* the values $n = 8\cdot19\times10^{11}$, $\rho = 7\cdot85$, in C. G. S. measure, I find that the frequency $\tau^{-1} = 136000$, about. For a steel globe of any other dimensions, this result must be divided by the radius in centimetres. For a globe of the size of the earth [$a = 6\cdot37\times10^{8}$], I find that the period $\tau = 1$ hr. 18 m.†

On some Formulæ arising from the Differentiation of Elliptic Functions with regard to the Modulus. By Rev. M. M. U. Wilkinson.

[*Read June 8th*, 1882.]

1. If we take the identity,

$$2\,\mathrm{cn}(\alpha-\beta)\,\mathrm{cn}(\beta-\gamma)\,\mathrm{cn}(\gamma-\alpha)=2-\mathrm{sn}^2(\alpha-\beta)-\mathrm{sn}^2(\beta-\gamma)-\mathrm{sn}^2(\gamma-\alpha)$$
$$+k^2\,\mathrm{sn}^2(\alpha-\beta)\,\mathrm{sn}^2(\beta-\gamma)\,\mathrm{sn}^2(\gamma-\alpha),$$

and differentiate with respect to k, we have

$$2k\,\mathrm{sn}^2(\alpha-\beta)\,\mathrm{sn}^2(\beta-\gamma)\,\mathrm{sn}^2(\gamma-\alpha)$$
$$=\{-2\,\mathrm{sn}(\alpha-\beta)\,\mathrm{cn}(\beta-\gamma)\,\mathrm{cn}(\gamma-\alpha)+2\,\mathrm{sn}(\alpha-\beta)\,\mathrm{cn}(\alpha-\beta)$$
$$-2k^2\,\mathrm{sn}(\alpha-\beta)\,\mathrm{cn}(\alpha-\beta)\,\mathrm{sn}^2(\beta-\gamma)\,\mathrm{sn}^2(\gamma-\alpha)\}\,\frac{d\,.\,\mathrm{am}(\alpha-\beta)}{dk}$$
$$=2\,\{-\mathrm{dn}(\alpha-\beta)-k^2\,\mathrm{sn}(\beta-\gamma)\,\mathrm{sn}(\gamma-\alpha)\}$$
$$\times\,\mathrm{sn}(\alpha-\beta)\,\mathrm{sn}(\beta-\gamma)\,\mathrm{sn}(\gamma-\alpha)\,\frac{d\,.\,\mathrm{am}(\alpha-\beta)}{dk}$$
$$=-2\,\mathrm{dn}(\beta-\gamma)\,\mathrm{dn}(\gamma-\alpha)\,\mathrm{sn}(\alpha-\beta)\,\mathrm{sn}(\beta-\gamma)\,\mathrm{sn}(\gamma-\alpha)\,\frac{d.\mathrm{am}(\alpha-\beta)}{dk};$$

whence

$$\frac{1}{\mathrm{dn}(\alpha-\beta)}\,\frac{d.\mathrm{am}(\alpha-\beta)}{dk}+\frac{1}{\mathrm{dn}(\beta-\gamma)}\,\frac{d.\mathrm{am}(\beta-\gamma)}{dk}+\frac{1}{\mathrm{dn}(\gamma-\alpha)}\,\frac{d.\mathrm{am}(\gamma-\alpha)}{dk}$$
$$=-\frac{k\,\mathrm{sn}(\alpha-\beta)\,\mathrm{sn}(\beta-\gamma)\,\mathrm{sn}(\gamma-\alpha)}{\mathrm{dn}(\alpha-\beta)\,\mathrm{dn}(\beta-\gamma)\,\mathrm{dn}(\gamma-\alpha)}.$$

2. It follows from this, that if

$$x_1+x_2+x_3+x_4=0,$$

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* *Units and Physical Constants*, p. 53.

† *Cf.* Thomson, *Phil. Trans.*, 1863, p. 573.