Note on the Quinquisectional Equation. *By* L. J. ROGERS.
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It has been shown by Gauss and Lebesgue that the tri- and quadrisectional equations for a prime p can be made to depend upon the resolution of p into certain quadratic forms, the former depending on an identity $4p = M^2+27N^2$, and the latter on an identity $p = A^2+B^2$.

It is proposed here to establish a similar method for the derivation of the quinquisectional equation for primes of the form $5\lambda+1$.

In Vol. XVIII., p. 215, Prof. Lloyd Tanner adopts the following notation:—

$$F\omega \equiv X_0+\omega X_1+\omega^2 X_2+\omega^3 X_3+\omega^4 X_4,$$

where the X's are the roots of the quinquisectional equation and ω is a fifth root of unity. Thus $F\omega = (\omega^\lambda, r)$ of Bachmann's *Kreistheilungslehre.*

Moreover, $q\omega$ denotes $F\omega^2/(F\omega)^2$, whence, by the general theorem of Jacobi's,

$$q\omega \,.\, q\omega^4 = p.$$

Thus, putting $\quad q\omega = q_0+q_1\omega+q_2\omega^2+q_3\omega^3+q_4\omega^4,$

we have (see Vol. XVIII., p. 217)

$$\begin{aligned} q_0+q_1+q_2+q_3+q_4 &= -1, \\ q_0q_1+q_1q_2+q_2q_3+q_3q_4+q_4q_0 &= -\lambda, \\ q_0q_2+q_1q_3+q_2q_4+q_3q_0+q_4q_1 &= -\lambda, \\ q_0^2+q_1^2+q_2^2+q_3^2+q_4^2 &= 4\lambda+1. \end{aligned}$$

These give

$$\left.\begin{aligned} 16p = (4q_0-q_1-q_2-q_3-q_4)^2+5\,(q_1-q_2-q_3+q_4)^2 \\ +10\,(q_1-q_4)^2+10\,(q_2-q_3)^2 \\ \text{and } (4q_0-q_1-q_2-q_3-q_4)(q_1-q_2-q_3+q_4) \\ = -(q_1-q_4)^2-4\,(q_1-q_4)(q_2-q_3)+(q_2-q_3)^2 \end{aligned}\right\}, \quad (1)$$

$$\left.\begin{aligned} \textit{i.e.,} \qquad 16p &= a^2+5b^2+10\,(c^2+d^2) \\ \text{where} \qquad ab &= -c^2-4cd+d^2 \end{aligned}\right\}, \quad (2)$$

and
$$\begin{aligned} a &= 4q_0-q_1-q_2-q_3-q_4 = 5q_0+1,\\ b &= q_1-q_2-q_3+q_4,\\ c &= q_1-q_4,\\ d &= q_2-q_3. \end{aligned}$$

Equations (2), however, are not a unique representation of p in this form, since the cyclic symmetry in the q's shows that similar expressions may be obtained by rotating the suffixes, so that a might $= 5q_1+1$, $b = q_2-q_3-q_4+q_0$, &c. Moreover, if we were to treat the trisectional complexes in a similar way, we should only obtain the representation of $4p$ in the form m^2+3n^2, which is not definite enough to establish the trisectional equation.

Passing on to p. 221 in Vol. XVIII., we find a modification of the above results, depending on the fact that

$$X_0^2 = a_0X_0+a_1X_2+a_2X_2+a_3X_3+a_4X_4,$$

where the a's are all integers.

The connexion between the a's and the q's is shown on p. 222, viz.,

$$\begin{aligned} 5a_0 &= -2 \;\; +3q_0-4\lambda,\\ 5a_1 &= \;\;\; 2q_1+\;q_3-4\lambda,\\ 5a_2 &= \;\;\; 2q_2+\;q_1-4\lambda,\\ 5a_3 &= \;\;\; 2q_3+\;q_4-4\lambda,\\ 5a_4 &= \;\;\; 2q_4+\;q_2-4\lambda, \end{aligned}$$

which gives
$$\begin{aligned} q_1-q_2-q_3+q_4 &= 5\,(a_1-a_2-a_3+a_4) = 5l, \text{ say},\\ 2\,(q_1-q_4)-(q_2-q_3) &= 5\,(a_1-a_4) = 5m, \text{ say},\\ (q_1-q_4)+2\,(q_2-q_3) &= 5\,(a_2-a_3) = 5n, \text{ say}. \end{aligned}$$

Hence equation (2) becomes

$$\left.\begin{aligned} 16p &= a^2+125l^2+50\,(m^2+n^2)\\ al &= m^2-4mn-n^2 \end{aligned}\right\}. \qquad (3)$$

We see then that every prime of the form $5\lambda+1$ can be expressed as the sixteenth of the sum of such multiples of four squares, which are connected by a single further relation.

It remains only to show that such a relation is unique, and that, if it has once been obtained, by trial or otherwise, then the quinquisectional equation can be derived from it without ambiguity. When

p is not large, (3) can be found by inspection, but, in most cases, this paper does not profess to offer an essentially simpler method than those established by Prof. Lloyd Tanner.

To show that the representation is unique, we may observe that, if $\omega = e^{\frac{2}{5}\pi i}$,

$$\begin{aligned} 16p = \;& \{a+5\sqrt{5}l+2\sqrt{5}(\omega^2-\omega^3)m+2\sqrt{5}(\omega-\omega^4)n\} \\ & \times\{a+5\sqrt{5}l-2\sqrt{5}(\omega^2-\omega^3)m-2\sqrt{5}(\omega-\omega^4)n\} \\ =\;& a^2+125l^2+50(m^2+n^2)+10\sqrt{5}(al-m^2+4mn-n^2). \end{aligned}$$

Moreover, if α, β, γ, δ be any integers, we have identically

$$\begin{aligned} & \{a+5\sqrt{5}l+2\sqrt{5}(\omega^2-\omega^3)m+2\sqrt{5}(\omega-\omega^4)n\} \\ & \times\{\alpha+5\sqrt{5}\lambda-2\sqrt{5}(\omega^2-\omega^3)\mu-2\sqrt{5}(\omega-\omega^4)\nu\} \\ =\;& (a\alpha+125l\lambda+50m\mu+50n\nu) \\ & +5\sqrt{5}(a\lambda+\alpha l-2m\mu+4m\nu+4\mu\nu+2n\nu) \\ & +2\sqrt{5}(\omega^2-\omega^3)\{-a\mu+\alpha m+5l(\mu-2\nu)-5\lambda(m-2n)\} \\ & +2\sqrt{5}(\omega-\omega^4)\{-a\nu+\alpha n-5l(2\mu+\nu)+5\lambda(2m+n)\} \\ =\;& A+5\sqrt{5}\,.\,L+2\sqrt{5}(\omega^2-\omega^3)M+2\sqrt{5}(\omega-\omega^4)N, \text{ say.} \end{aligned} \tag{4}$$

If, then, also, $\quad 16p = \alpha^2+125\lambda^2+50(\mu^2+\nu^2),$

where $\quad \alpha\lambda = \mu^2-4\mu\nu-\nu^2,$

we see that, by interchanging italic and Greek letters in the left-hand side of (4) and multiplying the four factors together, we have

$$\begin{aligned} 256p^2 &= \{a^2+125l^2+50(m^2+n^2)\}\{\alpha^2+125\lambda^2+50(\mu^2+\nu^2)\} \\ &= A^2+125L^2+50(M^2+N^2), \end{aligned}$$

where $\quad AL = M^2-LMN-N^2.$

Again, if we change ω into ω^2, we get

$$\begin{aligned} 16p = \;& \{a-5\sqrt{5}l-2\sqrt{5}(\omega-\omega^4)m+2\sqrt{5}(\omega^2-\omega^3)n\} \\ & \times\{a-5\sqrt{5}l+2\sqrt{5}(\omega-\omega^4)m+2\sqrt{5}(\omega^2-\omega^3)n\}. \end{aligned}$$

Now, if instead of $\omega = e^{\frac{2}{5}\pi i}$, we suppose that

$$\omega^4+\omega^3+\omega^2+\omega+1 \equiv 0, \quad \text{mod } p,$$

and $\quad \sqrt{5} \equiv \omega+\omega^4-\omega^2-\omega^3,$

it is clear that one of the factors in each of the products representing $16p$ must $\equiv 0$.

Without loss of generality, we may assume

$$a+5\sqrt{5}\,l \equiv 2\sqrt{5}\,(\omega^2-\omega^3)\,m+2\sqrt{5}\,(\omega-\omega^4)\,n,$$
$$a-5\sqrt{5}\,l \equiv 2\sqrt{5}\,(\omega-\omega^4)\,m-2\sqrt{5}\,(\omega^2-\omega^3)\,n,$$

and, similarly,

$$\alpha+5\sqrt{5}\,\lambda \equiv 2\sqrt{5}\,(\omega^2-\omega^3)\,\mu+2\sqrt{5}\,(\omega-\omega^4)\,\nu,$$
$$\alpha-5\sqrt{5}\,\lambda \equiv 2\sqrt{5}\,(\omega-\omega^4)\,\mu-2\sqrt{5}\,(\omega^2-\omega^3)\,\nu.$$

From these four congruences, we see that

$$(a+5\sqrt{5}l)(\alpha+5\sqrt{5}\,\lambda)+(a-5\sqrt{5}\,l)(\alpha-5\sqrt{5}\,\lambda)$$
$$\equiv 5\left\{4\,(\omega^2-\omega^3)^2+4\,(\omega-\omega^4)^2\right\}(m\mu+n\nu),$$

i.e., $$a\alpha+125l\lambda+50\,(m\mu+n\nu)\equiv 0,$$

while $$(a+5\sqrt{5}\,l)(\alpha+5\sqrt{5}\,\lambda)-(a-5\sqrt{5}\,l)(\alpha-5\sqrt{5}\,l)$$
$$\equiv 40\,(\omega^2-\omega^3)(\omega-\omega^4)(m\nu+\mu n)$$
$$+20\left\{(\omega^2-\omega^3)^2-(\omega-\omega^4)^2\right\}(m\mu-n\nu)$$
$$\equiv -40\sqrt{5}\,(m\nu+\mu n)+20\sqrt{5}\,(m\mu-n\nu),$$

i.e., $$a\lambda+\alpha l+4\,(m\nu+\mu n)+2m\mu-2n\nu\equiv 0.$$

Thus $$A\equiv 0 \quad\text{and}\quad L\equiv 0,$$

and, since, in consequence,

$$M^2+N^2\equiv 0 \quad\text{and}\quad M^2-4MN-N^2\equiv 0,$$

we see that $$M\equiv 0,\quad N\equiv 0.$$

Let $$A=Up,\quad L=Xp,\quad M=Yp,\quad N=Zp,$$

so that $$256p^2=p^2\,(U^2+125X^2+50Y^2+50Z^2),$$
$$256=U^2+125X^2+50Y^2+50Z^2,$$

where $$UX=Y^2-4YZ-Z^2.$$

The only possible solution for these equations is

$$U=16,\quad X=Y=Z=0,$$

so that $$a\alpha+125l\lambda+50\,(m\mu+n\nu)=16p, \tag{5}$$

while $$L=M=N=0.$$

These last three equations give the ratios of α, λ, μ, ν uniquely in terms of a, l, m, n, and, by inspection, we see

$$\alpha=ka,\quad \lambda=kl,\quad \mu=km,\quad \nu=kn.$$

But this reduces (5) to

$$k(a^2+125l^2+50m^2+50n^2) = 16p;$$

therefore $$k = 1.$$

Hence we see that the representation of $16p$ is unique.

2. We may now obtain the quinquisectional equation in terms of a, l, m, n. On p. 227, Vol. XVIII., it is given in the form

$$Y^5-10p.Y^3-5p\Sigma q\omega\, Y^2+5p(p-\Sigma q\omega q\omega^2)Y-p\Sigma q\omega q\omega q\omega^2 = 0, \quad (1)$$

where Y has for roots $5X_0+1$, $5X_1+1$, &c.

Since $$q\omega\, q\omega^4 = p,$$

we may put $$q\omega = \sqrt{p}\, e^{\theta_1 i}, \quad q\omega_4 = \sqrt{p}\, e^{-\theta_1 i},$$

and, similarly, $$q\omega^2 = \sqrt{p}\, e^{\theta_2 i}, \quad q\omega^3 = \sqrt{p}\, e^{-\theta_2 i},$$

where we may suppose $\sqrt{p}$ to be the positive square root. Then

$$\left.\begin{aligned} \Sigma q\omega &= 2\sqrt{p}(\cos\theta_1+\cos\theta_2) \\ \Sigma q\omega q\omega^2 &= 4p\cos\theta_1\cos\theta_2 \\ \Sigma q\omega q\omega q\omega^2 &= 2p\sqrt{p}\{\cos(2\theta_1+\theta_2)+\cos(\theta_1-2\theta_2)\} \\ &= 2p\sqrt{p}\{(2\cos\theta_1\cos\theta_2-1)(\cos\theta_1+\cos\theta_2) \\ &\qquad -2\sin\theta_1\sin\theta_2(\cos\theta_1-\cos\theta_2)\} \end{aligned}\right\}. \quad (2)$$

Moreover, if $$F\omega = \sqrt{p}\, e^{\phi_1 i}, \quad F\omega^4 = \sqrt{p}\, e^{-\phi_1 i},$$
$$F\omega^2 = \sqrt{p}\, e^{\phi_2 i}, \quad F\omega^3 = \sqrt{p}\, e^{-\phi_2 i},$$

then equations (8) and (9) on p. 217 are equivalent to

$$2\phi_1 = \theta_1+\phi_2,$$
$$2\phi_2 = \theta_2-\phi_1,$$

whence $$\phi_1 = \tfrac{1}{5}(2\theta_1+\theta_2),$$
$$\phi_2 = \tfrac{1}{5}(2\theta_2-\theta_1),$$

while $$Y_r = \omega^{-r}F\omega+\omega^{-2r}F\omega^2+\omega^{-3r}F\omega^3+\omega^{-4r}F\omega^4$$
$$= 2\sqrt{p}\left\{\cos\left(\phi_1-\frac{2\pi}{5}r\right)+\cos\left(\phi_2-\frac{4\pi}{5}r\right)\right\}, \quad (3)$$

where $r = 0, 1, 2, 3, 4$.

This shows that any equation of form (1), whether quinquisectional or not, whose coefficients depend on (2), can be solved trigonometrically by the quinquisection of an angle $2\theta_1+\theta_2$ or $2\theta_2-\theta_1$ depending on the coefficients.

Now $\sqrt{p}\,e^{\theta_1 i} = q\omega$ and $\sqrt{p}\,e^{\theta_2 i} = q\omega^2$,

whence
$$\sqrt{p}\cos\theta_1 = q_0+(q_1+q_4)\frac{\sqrt{5}-1}{4}-(q_2+q_3)\frac{\sqrt{5}+1}{4}$$
$$= \tfrac{1}{4}(a+5\sqrt{5}\,l)$$
$$\sqrt{p}\cos\theta_2 = \tfrac{1}{4}(a-5\sqrt{5}\,l)$$
$$\sqrt{p}\sin\theta_1 = (q_1-q_4)\sin\frac{2\pi}{5}+(q_2-q_3)\sin\frac{4\pi}{5},$$
$$\sqrt{p}\sin\theta_2 = (q_1-q_4)\sin\frac{4\pi}{5}-(q_2-q_3)\sin\frac{2\pi}{5};$$

therefore $p\sin\theta_1\sin\theta_2 = \frac{\sqrt{5}}{4}(c^2-cd-d^2)$ [*v.* § 1 (2)]
$$= \frac{5\sqrt{5}}{4}(m^2+mn-n^2).$$

By these relations the equation (1) reduces to
$$Y^5-10pY^3-5paY^2+5p\left\{p-\tfrac{1}{4}(a^2-125l^2)\right\}Y$$
$$-p\left\{\frac{a}{8}(a^2-625l^2)-ap-\frac{625}{2}lmn\right\}=0. \quad (4)$$

We may now return to our supposition that numerical solutions for a, l, m, n have been found in the equations § 1 (3). In applying them to (4), it will readily be seen that a certain ambiguity arises as to the proper sign to be taken for a and lmn. The choice, however, can be easily determined from the following considerations. Since
$$a = 5q_0+1 \equiv 1, \quad \text{mod } 5,$$
there is no ambiguity in its value; and, if (l, m, n) is a correct solution of § 1 (3), the only other sets of solutions are $(l, -m, -n)$, $(-l, n, -m)$, and $(-l, -n, m)$, all of which yield the same product lmn, so that (4) is not altered by these alternative choices.

For instance, $16\times 331 = 61^2+125.1^2+50(5^2+2^2)$,

where $61 = 5^2+4.10-4$,

so that $a = 61 \equiv 1, \quad \text{mod } 5.$

Let then $l = 1, \quad m = 5, \quad n = -2,$

so that $lmn = -10;$

we see by the above reasoning that all other possible choices of values for l, m, n would have given the same value for the product.

3. It may be noticed that the representation of $16p$ in § 1 (3) depends upon the fact that p, a rational quantity, is found to be the product of two irrational factors $q\omega \cdot q\omega^4$, which in general, for unrestricted values of the q's, would involve terms containing the square root of 5.

The result may be generalized by supposing a case in which some rational quantity can be expressed as the product of two linear functions of a quadrisectional equation.

Let n be any prime of the form $4k+1$ whose quadrisectional roots are

$$\eta_0 = r + r^{g^4} + r^{g^8} + \dots,$$
$$\eta_1 = r^{g} + r^{g^5} + r^{g^9} + \dots,$$
$$\eta_2 = r^{g^2} + r^{g^6} + r^{g^{10}} + \dots,$$
$$\eta_3 = r^{g^3} + r^{g^7} + r^{g^{11}} + \dots,$$

where $$r = e^{(2\pi/n)i},$$

and g is a primitive root of n; and let us suppose that

$$P = (s_0\eta_0 + s_1\eta_1 + s_2\eta_2 + s_3\eta_3)(s_0\eta_2 + s_1\eta_3 + s_2\eta_0 + s_3\eta_1)$$

and the s's are rational.

Let $\eta_0 + \eta_1 i^m + \eta_2 i^{2m} + \eta_3 i^{3m}$ be written Fi^m, where

$$m \equiv 0, 1, 2, 3, \mod 4;$$

so that $$F1 = -1,$$

and
$$4\eta_0 = -1 + Fi + Fi^2 + Fi^3,$$
$$4\eta_1 = -1 - iFi - Fi^2 + Fi^3,$$
$$4\eta_2 = -1 - Fi + Fi^2 - Fi^3,$$
$$4\eta_3 = -1 + iFi - Fi^2 - Fi^3.$$

Moreover, if $s_0 + s_1 i^m + s_2 i^{2m} + s_3 i^{3m}$ be written si^m, where

$$m \equiv 0, 1, 2, 3, \mod 4,$$

we have

$$\begin{aligned} 16P &= 16\,(s_0\eta_0 + s_1\eta_1 + s_2\eta_2 + s_3\eta_3)(s_0\eta_2 + s_1\eta_3 + s_2\eta_0 + s_3\eta_1) \\ &= (-s1 + si^1Fi + si^2Fi^2 + siFi^3)(-s1 - si^3Fi + si^2Fi^2 - siFi^3) \\ &= (-s1 + si^2Fi^2)^2 - (si^3Fi + siFi^3)^2. \end{aligned}$$

Now Fi is the $(\omega^{\frac{1}{2}(n-1)}, g)$ of Bachmann's *Kreistheilungslehre*, p. 84, and

$$Fi^2 = \eta_0+\eta_2-\eta_1-\eta_3 = \sqrt{p},$$

while $$(Fi)^2 = (a+bi)\, Fi^2,$$

where $$a^2+b^2 = p,$$

and $$Fi \,.\, Fi^3 = (-1)^{\frac{1}{2}(n-1)}\, p.$$

Hence $$16P = (-s1+si^2\sqrt{p})^2-(si^3)^2\,(a+bi)\sqrt{p}-(si)^2\,(a-bi)\sqrt{p}$$
$$-si^3 \,.\, si\,(-1)^{\frac{1}{2}(n-1)}\,p.$$

Equating rational and irrational terms, we have

$$16P = (s_0+s_1+s_2+s_3)^2+n\,(s_0-s_1+s_2-s_3)^2$$
$$-2n\,(-1)^{\frac{1}{2}(n-1)}\left\{(s_0-s_2)^2+(s_1-s_3)^2\right\},$$

while

$$(s_0+s_1+s_2+s_3)(s_0-s_1+s_2-s_3)$$
$$= a\left\{(s_1-s_3)^2-(s_0-s_2)^2\right\}-2b\,(s_1-s_3)(s_0-s_2).$$

These equations may be written in the form

$$\left.\begin{aligned} 16P &= A^2+nB^2-2n\,(-1)^{\frac{1}{2}(n-1)}\,(C^2+D^2) \\ AB &= a\,(D^2-C^2)-2bCD \end{aligned}\right\}. \qquad (1)$$

Two cases of these relations obviously present themselves.

Firstly, we may take the product

$$(x-r\,)(x-r^{g^4})(x-r^{g^8})\,\ldots$$
$$\times\,(x-r^{g})(x-r^{g^5})(x-r^{g^9})\,\ldots,$$

which can be written in the form

$$s_0\eta_0+s_1\eta_1+s_2\eta_2+s_3\eta_3,$$

while $$(x-r^{g^2})(x-r^{g^6})\,\ldots$$
$$\times\,(x-r^{g^3})(x-r^{g^7})\,\ldots$$

becomes $$s_0\eta_2+s_1\eta_3+s_2\eta_0+s_3\eta_1,$$

where s_0, s_1, s_2, s_3 are rational integral functions of x. Hence $16\,(x^n-1)/(x-1)$ can be written as in (1). For instance,

$$16\,(x^{13}-1)/(x-1) = (4x^6+2x^5-5x^4-2x^3-5x^2+2x+4)^2$$
$$+\,13x^4\,(x+1)^4+26\left\{(x^5-x)^2+(x^5+x^4-x^2-x)^2\right\},$$

while $$13 = 3^2+2^2,$$

$$16\,(x^{17}-1)/(x-1) = (4x^8+2x^7+10x^6-3x^5+8x^4-3x^3+10x^2+2x+4)^2$$
$$+17\,(2x^6-x^5-x^3+2x^2)^2$$
$$-34\left\{(x^7+x^5-2x^4+x^3+x)^2+(x^7+2x^5-2x^4+2x^3+x)^2\right\},$$

while $$17 = 1^2+4^2.$$

Secondly, we may apply (1) to the theory of n-sectional equations in general, where $n = 4k+1$ and is prime, and more especially when $n \equiv 5$, mod 8. The results will have a close analogy to the theorems contained in chapter xx. of Bachmann's *Kreistheilungslehre*, though it is not easy to see how far the result of p. 290 *et seq.* can be generalized.

For instance, when $n \equiv 5$, mod 8, and P is a prime of form $kn+1$, we shall obtain a relation similar to (19), p. 287, viz.,

$$16P^{\frac{1}{2}(n-1)} = A^2+nB^2+2n\,(C^2+D^2),$$

where $$AB = a\,(D^2-C^2)-2bCD,$$

and it is probable that in all cases the numbers A, B, C, D contain as a common factor some power of P, reducing the identity to one analogous to (24) on p. 290, the representation being unique.

Again, in what we may call the two-square representation of primes or their powers treated of in this portion of Bachmann's work, the primes of form $4k+1$ are omitted as leading to forms which are ambiguous in consequence of the Pellian equation

$$1 = x^2-ny^2.$$

In the present four-square representation, we can include some such primes unambiguously, viz., those which $\equiv 5$, mod 8, but shall probably meet with a new Pellian ambiguity when the prime $\equiv 1$, mod 8, depending on possible integral solutions of

$$1 = A^2+nB^2-2n\,(C^2+D^2),$$

where $$AB = a\,(D^2-C^2)-2bCD.$$

It would be interesting to know how far the integral solution of these equations is possible, both when $n = a^2+b^2$ is prime and when it is composite, and whether any algorithm similar to continued fractions can be employed to determine such a solution.