

$$\begin{aligned}
 \text{Hence} \quad & r^{-(n-1)} \frac{\partial}{\partial x_1} r^n Y(n, n_1, n_2, \dots, n_{p-1}) \\
 &= (n-n_1) \{2n+p-1-(n-n_1+1)\} Y(n-1, n_1, n_2, \dots, n_{p-1}) \\
 &= (n-n_1)(n+n_1+p-2) Y(n-1, n_1, n_2, \dots, n_{p-1}).
 \end{aligned}$$

The differentiations with respect to x_2, x_3, \dots may be similarly effected, but the results are not so simple.

Note on the Electric Capacity of a Conductor in the form of Two Intersecting Spheres. By W. D. NIVEN. Received and communicated November 12th, 1896. Received in revised form February 27th, 1897.

1. In the course of an interesting solution of the problem of "The Electrical Distribution on a Conductor bounded by two Spherical Surfaces cutting at any angle" (*Proc. Lond. Math. Soc.*, Vol. XXVI., p. 156), the author, Mr. H. M. Macdonald, has alluded to a paper of mine contained in Vol. XII., and has called attention to the discrepancies between his results and mine. He says: "In the paper by W. D. Niven mentioned above, an attempt is made to deduce the capacity of such a conductor from the solution of a functional equation for a particular value of one of the variables, but the result obtained does not seem in the case of the spherical bowl to agree with Lord Kelvin's. The results obtained hereafter also differ from those given by Niven."

In answer to this, I have to say (1) that the capacity of a hemispherical bowl as given by me agrees with Mr. Macdonald's own result, but (2) that in the other case where the results come into comparison, viz., when the conductor is bounded by a hemisphere and its base, I have to plead guilty to numerical mistakes in reducing this particular case from the general formulæ. In the latter case the accurate expression for the capacity is that given by Mr. Macdonald, viz., $2a \left(1 - \frac{1}{\sqrt{3}}\right)$, where a is the radius.

That the difference between the general expressions obtained by us for the capacity is only one of form is, in fact, shown by Mr. Macdonald in a note which follows this paper (see p. 214).

2. On a revision of my previous paper, I find that the work was presented in a needlessly difficult form, and that the proof of the main proposition only applied for an inclination of the two planes enclosing the electrified point less than two right angles. The results themselves may also be expressed more simply and compactly. I therefore take the opportunity of supplementing that paper by a new proof which is applicable to either case. I shall consider specially the case when the inclination of the planes (γ) in the region containing the electrified point C is greater than two right angles.

Let the plane of the paper pass through the inducing point C , and cut the two conducting planes in OA , OB .

Let $\angle COA = \alpha$, $OC = R$,

and let the inducing charge be $-R$.

The potential at any point P due to the electricity induced in OA must be a function of P 's coordinates z, r, θ , where z is the ordinate perpendicular to the plane of the paper, and r, θ are the polar coordinates in that plane, with O as pole; θ being measured from OA in the direction of the hands of a watch.

Denoting this potential by $f(z, r, \theta)$, and in like manner the potential due to the induced charge on OB by $g(z, r, \theta')$, where θ' denotes the same point P 's angular distance measured in the same direction from OB , so that

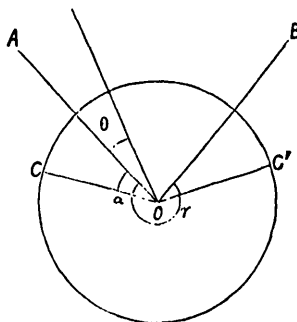
$$\theta' = \gamma + \theta,$$

we shall have, throughout the region $\theta = 0$ to $2\pi - \gamma$,

$$f(z, r, \theta) + g(z, r, \gamma + \theta) = R/CP. \quad (1)$$

If, instead of being placed at C , the charge be placed at C' , so situated that its angular distance from OB is equal to that of C from OA , the effect will be to produce the same distributions on OB , OA , respectively, as formerly existed on OA , OB . The potentials at z, r, θ will then be connected by the equation

$$g(z, r, \theta) + f(z, r, \gamma + \theta) = R/C'P. \quad (2)$$



3. If we denote $e^{\partial/\partial\theta}$ by \mathcal{E} , so that

$$f(\gamma + \theta) = \mathcal{E}f(\theta),$$

then, by the addition of (1) and (2), we obtain

$$f(z, r, \theta) + g(z, r, \theta) = \frac{R}{1 + \mathcal{E}r} \left(\frac{1}{CP} + \frac{1}{C'P} \right), \quad (3)$$

and, by the subtraction of (2) from (1),

$$f(z, r, \theta) - g(z, r, \theta) = \frac{R}{1 - \mathcal{E}r} \left(\frac{1}{CP} - \frac{1}{C'P} \right). \quad (4)$$

Since \mathcal{E} is a differential operator with regard to θ , the expressions on the right hand of (3) and (4) will, when substituted for V , satisfy Laplace's equation

$$\frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} = 0;$$

and therefore, so also with $f(z, r, \theta)$ and $g(z, r, \theta)$.

4. In preparation for the application of the operators in (3) and (4), it becomes necessary to find expressions for $1/CP$ and $1/C'P$ which will lead to finite results. In my former paper this was effected for the particular values $z = 0$, $r = R$, but Mehler has expressed $1/CP$ in a form which is as easy for the operations in the general as in the particular case [Heine, II., § 75 (32)]. The required expression may be established as follows:—

$$\begin{aligned} \text{Writing} \quad t^2 &= z^2 + R^2 + r^2 - 2Rr \cos \theta \\ &= 2Rr (\cosh \sigma - \cos \theta), \end{aligned}$$

$$\text{where} \quad 2Rr \cosh \sigma = z^2 + R^2 + r^2,$$

$$\begin{aligned} \text{we have} \quad \frac{R}{t} &= \sqrt{\frac{R}{2r}} \frac{1}{\sqrt{\cosh \sigma - \cos \theta}} \\ &= \frac{1}{2} \sqrt{\frac{R}{r}} \frac{1}{\sqrt{\sinh^2 \frac{1}{2} \sigma + \sin^2 \frac{1}{2} \theta}} \\ &= \frac{1}{2\pi} \sqrt{\frac{R}{r}} \int_0^\pi \frac{d\phi}{\sin \frac{1}{2} \theta + i \sinh \frac{1}{2} \sigma \cos \phi} \\ &= \frac{1}{\pi} \sqrt{\frac{R}{r}} \int_0^\pi \frac{\sin \frac{1}{2} \theta \, d\phi}{\sin^2 \frac{1}{2} \theta + \sinh^2 \frac{1}{2} \sigma \cos^2 \phi}. \end{aligned}$$

Now, putting $\sinh \frac{1}{2}\sigma \cos \phi = \sinh \frac{1}{2}v$,

$$d\phi = \frac{1}{\sqrt{2}} \frac{\cosh \frac{1}{2}v}{\sqrt{\cosh \sigma - \cosh v}} dv,$$

and observing that

$$\sin^2 \frac{1}{2}\theta + \sinh^2 \frac{1}{2}v = \sin^2 \frac{1}{2}(\theta + iv) \sin^2 \frac{1}{2}(\theta - iv),$$

we obtain

$$\frac{R}{t} = \frac{1}{2\sqrt{2}\pi} \sqrt{\frac{R}{r}} \int_0^\sigma [\operatorname{cosec} \frac{1}{2}(\theta + iv) + \operatorname{cosec} \frac{1}{2}(\theta - iv)] \times \frac{dv}{\sqrt{\cosh \sigma - \cosh v}}.$$

But, if z be a complex number, then (Hobson's *Trigonometry*, § 295)

$$\begin{aligned} \pi \operatorname{cosec} \pi z &= \frac{1}{z} - \frac{1}{z+1} - \frac{1}{z-1} + \frac{1}{z+2} + \frac{1}{z-2} - \&c. \\ &= \int_0^1 \frac{x^{z-1} + x^{-z}}{1+x} dx, \end{aligned}$$

where, it is clear, the real part of z must be less than unity.

On writing $x = e^{-2\pi\xi}$,

this becomes $\operatorname{cosec} \pi z = 2 \int_0^\infty \frac{\cosh(1-2z)\pi\xi}{\cosh \pi\xi} d\xi$.

Hence $\operatorname{cosec} \frac{1}{2}(\theta + iv) + \operatorname{cosec} \frac{1}{2}(\theta - iv) = 4 \int_0^\infty \frac{\cosh(\pi - \theta)\xi \cos v\xi}{\cosh \pi\xi} d\xi$,

$$\begin{aligned} \text{and } \frac{R}{t} &= \frac{\sqrt{2}}{\pi} \sqrt{\frac{R}{r}} \int_0^\sigma \frac{\cosh(\pi - \theta)\xi}{\cosh \pi\xi} \int_0^\sigma \frac{\cos \xi v}{\sqrt{\cosh \sigma - \cosh v}} dv d\xi \\ &= \sqrt{\frac{R}{r}} \int_0^\sigma \frac{\cosh(\pi - \theta)\xi}{\cosh \pi\xi} k_\pi(\cosh \sigma) d\xi, \end{aligned} \quad (5)$$

where $\frac{\pi}{\sqrt{2}} k_\pi(\cosh \sigma) = \int_0^\sigma \frac{\cos \xi v}{\sqrt{\cosh \sigma - \cosh v}} dv$.

5. Heine, by the employment of Cauchy's method, has found two other forms of the integral expressing $\frac{\pi}{\sqrt{2}} k_\pi(\cosh \sigma)$, or, more briefly, $\frac{\pi}{\sqrt{2}} k_\pi$, one of which, viz.,

$$\coth \xi \int_0^\infty \frac{\sin v\xi}{\sqrt{\cosh v - \cosh \sigma}} dv,$$

may be established by the method of the preceding article.

6. The longitudinal angles of any point P in the region not containing the charge relatively to C , C' being $\theta + \alpha$, $\theta + \gamma - \alpha$, we have thus

$$R \left(\frac{1}{CP} + \frac{1}{C'P} \right) = 2 \sqrt{\frac{R}{r}} \int_0^\infty \frac{\cosh(\theta + \frac{1}{2}\gamma - \pi) \xi \cosh(\frac{1}{2}\gamma - \alpha) \xi}{\cosh \pi \xi} k_r d\xi,$$

$$R \left(\frac{1}{CP} - \frac{1}{C'P} \right) = -2 \sqrt{\frac{R}{r}} \int_0^\infty \frac{\sinh(\theta + \frac{1}{2}\gamma - \alpha) \xi \sinh(\frac{1}{2}\gamma - \alpha) \xi}{\cosh \pi \xi} k_r d\xi.$$

Now, observing that

$$\frac{1}{1 + D^2} \cosh \theta \xi = \frac{\cosh(\theta - \frac{1}{2}\gamma) \xi}{2 \cosh \frac{1}{2}\gamma \xi},$$

and
$$\frac{1}{1 - D^2} \sinh \theta \xi = -\frac{\cosh(\theta - \frac{1}{2}\gamma) \xi}{2 \sinh \frac{1}{2}\gamma \xi},$$

on referring back to (3) and (4), we see that

$$f(z, r, \theta) + g(z, r, \theta) = \sqrt{\frac{R}{r}} \int_0^\infty \frac{\cosh(\theta - \pi) \xi \cosh(\frac{1}{2}\gamma - \alpha) \xi}{\cosh \pi \xi \cosh \frac{1}{2}\gamma \xi} k_r d\xi,$$

and
$$f(z, r, \theta) - g(z, r, \theta) = \sqrt{\frac{R}{r}} \int_0^\infty \frac{\cosh(\theta - \pi) \xi \sinh(\frac{1}{2}\gamma - \alpha) \xi}{\cosh \pi \xi \sinh \frac{1}{2}\gamma \xi} k_r d\xi.$$

From these, by addition and subtraction, we obtain

$$f(z, r, \theta) = \sqrt{\frac{R}{r}} \int_0^\infty \frac{\cosh(\pi - \theta) \xi \sinh(\gamma - \alpha) \xi}{\cosh \pi \xi \sinh \gamma \xi} k_r d\xi, \quad (6)$$

$$g(z, r, \theta) = \sqrt{\frac{R}{r}} \int_0^\infty \frac{\cosh(\pi - \theta) \xi \sinh \alpha \xi}{\cosh \pi \xi \sinh \gamma \xi} k_r d\xi. \quad (7)$$

Since $\sinh(\gamma - \alpha) \xi$ and $\sinh \alpha \xi$ are each of them less than $\sinh \gamma \xi$, the expressions just found are more convergent than the original integrals from which they have been derived.

7. It will be observed that f and g are unchanged when $2\pi - \theta$ is put for θ , so that, as was to be expected, the potentials at points similarly situated with respect to the two sides of either plane are equal as regards the charge on that plane.

The potential U due to the induced charges at any point z, r, θ in the region containing the electrified point is therefore given by

$$U = f(z, r, \theta) + g(z, r, \theta - 2\pi + \gamma),$$

where θ may have any value between $2\pi - \gamma$ and 2π . This gives

$$U = \sqrt{\frac{R}{r}} \int_0^\infty \frac{N}{\cosh \pi \xi \sinh \gamma \xi} k_r d\xi, \quad (8)$$

$$\begin{aligned}
\text{where } N &= \cosh (\pi-\theta) \xi \sinh (\gamma-\alpha) \xi + \cosh (3\pi-\theta-\gamma) \sinh \alpha \xi \\
&= \frac{1}{2} \{ \sinh (\pi+\gamma-\alpha-\theta) \xi + \sinh (-\pi+\gamma-\alpha+\theta) \xi \\
&\quad + \sinh (3\pi-\gamma+\alpha-\theta) \xi - \sinh (3\pi-\gamma-\alpha-\theta) \xi \} \\
&= \sinh \pi \xi \cosh (2\pi-\gamma+\alpha-\theta) \xi \\
&\quad + \sinh (\gamma-\pi) \xi \cosh (2\pi-\alpha-\theta) \xi.
\end{aligned}$$

The value of U thus found is finite and continuous, vanishes at an infinite distance, k_t being then zero, and satisfies the boundary conditions. It is therefore the solution required.

Case of $\gamma < \pi$.

8. The proof contained in the foregoing articles is expressed for the case when the angle in the region containing the electrified point is greater than two right angles. Taking, however, the figure in § 3, and placing the electrified point O in the acute angle, we shall have the same proof identically, provided angles are now measured positively when described from OA in the direction opposite to the motion of the hands of a watch.

Writing for a moment θ' for the longitudinal angle, we have the same expressions for $f(z, r, \theta')$ and $g(z, r, \theta')$, and the potential U due to the induced charges in the region containing the electrified point will be given by

$$U = f(z, r, \theta') + g(z, r, \theta' - 2\pi + \gamma),$$

γ now being the acute angle AOB , and θ' the angle measured positively from OA to a point in the acute angle.

Putting θ for $2\pi - \theta'$ and observing that

$$f(z, r, 2\pi - \theta) = f(z, r, \theta),$$

we have then $U = f(z, r, \theta) + g(z, r, \gamma - \theta)$,

where θ is now measured positively from OA in the direction of the hands of a watch, so that, in this case,

$$U = \sqrt{\frac{R}{r}} \int_0^\infty \frac{N}{\cosh \pi \xi \sinh \gamma \xi} k_t d\xi, \quad (9)$$

where $N = \cosh (\pi-\theta) \xi \sinh (\gamma-\alpha) \xi + \cosh (\pi-\gamma+\theta) \xi \sinh \alpha \xi$,

or, by transformation,

$$\sinh \pi \xi \cosh (\gamma-\alpha-\theta) \xi - \sinh (\pi-\gamma) \xi \cosh (\theta-\alpha) \xi.$$

Case of two Parallel Planes.

9. An interesting deduction from the general case in § 8 is that of two parallel planes with an electrified point between them.

In the general case, let the point O be moved to infinity, so that γ becomes infinitely small, and therefore also θ and α , and put

$$\frac{a}{\gamma} = \frac{a}{a+b},$$

$$\frac{\theta}{\gamma} = \frac{x}{a+b},$$

where a, b are the distances of the charge from OA, OB respectively.

If, in the expression for U , we put

$$\gamma\xi = (a+b)\zeta,$$

and then make γ small, we shall obtain

$$U = \frac{a+b}{\gamma} \int_0^\infty \frac{N'}{\sinh(a+b)\zeta} k_{(c)} d\zeta, \quad (10)$$

where $N' = e^{-x\zeta} \sinh b\zeta + e^{-(a+b-x)\zeta} \sinh a\zeta,$

or, on the removal of the origin to the electrified point, by putting

$$x = X+a,$$

we have, by an easy transformation,

$$N' = \cosh X\zeta [\sinh(a+b)\zeta - 2 \sinh a\zeta \sinh b\zeta] + \sinh X\zeta \sinh(a-b)\zeta. \quad (11)$$

10. In regard to k_ϵ , if σ be small, we may put

$$\cosh \sigma = 1 + \sigma^2/2,$$

and

$$\frac{\pi}{\sqrt{2}} k_\epsilon = \sqrt{2} \int_0^\sigma \frac{\cos \xi v}{\sqrt{\sigma^2 - v^2}} dv.$$

Therefore

$$k_\epsilon = J_0(\xi\sigma). \quad (\text{Heine, II., § 60.})$$

Now

$$\sigma^2 = \frac{z^2 + (h-r)^2}{h^2}$$

$$= \frac{z^2 + \eta^2}{h^2} \quad (h \text{ and } r \text{ large})$$

$$= \frac{\rho^2}{h^2}, \text{ say.}$$

Hence

$$\sigma\xi = \frac{(a+b)\xi\rho}{\gamma R}$$

$$= \xi\rho. \quad (12)$$

11. The presence of $1/\gamma$ outside the integral makes the expression for U infinite, but the charge R is now also infinite. If we reduce the charge to a finite amount R' , we must multiply the expression for U by R'/R , and thus, since

$$R\gamma = a+b,$$

finally obtain for U the value

$$R' \int_0^\infty \frac{N' J_0(\rho\xi)}{\sinh(a+b)\xi} d\xi.$$

This represents the potential at a point whose coordinates are X, ρ , due to the electricity induced in the two planes by the charge $-R'$.

The particular case when $a = b$ is

$$U = R' \int_0^\infty \frac{\cosh X\xi e^{-a\xi} J_0(\rho\xi)}{\cosh a\xi} d\xi. \quad (13)$$

Potential at a Point outside the Inverted Surface.

12. Referring to the figure in § 3, we invert with regard to the point O , the radius of inversion being R .

The intersection of the planes OA, OB will invert into a circle through C and O in a plane perpendicular to the plane of the paper; the planes themselves into segments of spheres cutting in this circle at an external angle γ . The new conductor will be at potential unity. If we denote the potential at any point Q outside of it by \bar{U} , we shall have, by the ordinary rules for inversion,

$$\bar{U} = \frac{R}{(OQ^2 CQ^2 - R^2)^{\frac{1}{2}}} \int_0^\infty \frac{\bar{N}}{\cosh \pi\xi \sinh \gamma\xi} k_\pi(\cosh \sigma) d\xi, \quad (14)$$

in which \bar{N} denotes the same expression as in § 7 or § 8, the angle θ now being the angle between the segment corresponding to OA and the segment passing through Q . The angle θ is measured from the former segment to the latter in the opposite direction from that in which it is measured in § 7 or § 8.

The quantity $\cosh \bar{\sigma}$ denotes

$$\frac{UQ^2 + OQ^2}{2\sqrt{OQ^2 \cdot OQ^2 - R^2 z^2}},$$

where z is the perpendicular from Q to the plane of the paper.

Since \bar{U} , as may be easily shown from the above expression, and is otherwise obvious, is a symmetrical function about the axis of the inverted conductor, we may put $z = 0$, and consider only points Q in the plane of the paper.

13. For points on the axis of the conductor

$$\bar{U} = \frac{R}{UQ} \int_0^\infty \frac{N}{\cosh \pi \xi \sinh \gamma \xi} d\xi.$$

As a particular case, let $\gamma = 2\pi$; then, § 7,

$$\bar{U} = \frac{R}{UQ} \int_0^\infty \frac{\cosh(\pi - \theta) \xi \cosh(\pi - \alpha) \xi}{\cosh^2 \pi \xi} d\xi.$$

This integral can be evaluated by expanding $\operatorname{sech}^2 \pi \xi$ in powers of $e^{-2\xi}$. The result of integrating can then be arranged so as to exhibit its equality to

$$\frac{R}{2\pi UQ} \left\{ \frac{\pi - \frac{1}{2}(\theta + \alpha)}{\sin \frac{1}{2}(\theta + \alpha)} + \frac{\frac{1}{2}(\theta - \alpha)}{\sin \frac{1}{2}(\theta - \alpha)} \right\}.$$

This gives the potential at any point of the axis of a spherical bowl charged to potential unity.

Capacity.

14. The capacity of the inverted conductor may be determined by means of a result given in my paper on "Ellipsoidal Harmonics" (*Phil. Trans.*, Vol. CLXXXII., 1891) to the following effect:—If P represent the potential at the point of inversion due to the electricity induced in any conductor A by a charge equal to $-R$, where R is the radius of inversion, then the capacity of the conductor inverted from A is PR .

Thus, in the general case above, since for the point of inversion $r = R$, $z = 0$, and therefore $\sigma = 0$, $k = 1$, if we put $\theta = 2\pi - \alpha$ in § 7, and $\theta = \alpha$ in § 8, we find that, whether the angle γ is greater or less than π , the capacity is given by

$$R \int_0^\infty \frac{\sinh \pi \xi \cosh(\gamma - 2\alpha) \xi + \sinh(\gamma - \pi) \xi}{\cosh \pi \xi \sinh \gamma \xi} d\xi. \quad (17)$$

Difference of Charges on the Two Segments.

15. It is but a variation of the proof of the theorem quoted in § 14 to extend it to separate parts of the inverted conductor, provided we know the potentials due to the corresponding distributions before inversion. Thus, in the case on hand, the charge on the surface into which OA inverts is given, § 7, by

$$\begin{aligned} E_a &= Rf(O, R, 2\pi - \alpha) \\ &= Rf(O, R, \alpha), \end{aligned} \quad (18)$$

and similarly $E_b = Rg(O, R, \gamma - \alpha). \quad (19)$

From (18) and (19), on substituting the values of f and g , and observing that

$$\cosh(\pi - \alpha) \xi \sinh(\gamma - \alpha) \xi - \cosh(\gamma - \pi - \alpha) \xi \sinh \alpha \xi$$

reduces to $\cosh \pi \xi \sinh(\gamma - 2\alpha) \xi,$

we have
$$\begin{aligned} E_a - E_b &= R \int_0^\infty \frac{\sinh(\gamma - 2\alpha) \xi}{\sinh \gamma \xi} d\xi \\ &= R \cdot \frac{1}{2} \frac{\pi}{\gamma} \cot \frac{\alpha}{\gamma} \pi. \end{aligned} \quad (20)$$

This theorem I formerly proved for γ less than π . It is now shown to be true for any value of γ from 0 to 2π , the extreme cases corresponding to these angles being two spheres in contact and a spherical bowl.

*Note on Mr. W. D. Niven's paper on the Electric Capacity of a Conductor formed by Two Intersecting Spheres. By H. M. MACDONALD. Communicated December 10th, 1896.**

In his paper "On the Electrical Capacity of a Conductor bounded by Two Spherical Surfaces cutting at any angle," *Proc. Lond. Math. Soc.*, xii., 1880, and also in a further "Note on the Electric Capacity of a Conductor in the form of Two Intersecting Spheres," communicated

* It was ordered by the Council that Mr. Macdonald's note should follow Mr. Niven's paper.

to the Society on November 12th, 1896, Mr. W. D. Niven obtains the potential due to the distribution induced on a wedge-shaped conductor bounded by two intersecting planes by an electrified point, its value being expressed at any point on the circle having its plane perpendicular to the two intersecting planes, its centre on their line of intersection, and passing through the electrified point. The expression there given is $f(\theta) + g(\theta - 2\pi + \gamma)$, which is equal to

$$\frac{1}{\gamma} \int_0^\infty \frac{\cosh \frac{\pi - \theta}{\gamma} \xi \sinh \frac{\gamma - \alpha}{\gamma} \xi + \cosh \frac{3\pi - \theta - \gamma}{\gamma} \xi \sinh \frac{\alpha}{\gamma} \xi}{\cosh \frac{\pi \xi}{\gamma} \sinh \xi} d\xi.$$

The potential at the point θ due to the inducing charge is

$$-\frac{1}{2} \operatorname{cosec} \frac{1}{2} (\theta + \alpha) = -\frac{1}{\gamma} \int_0^\infty \frac{\cosh \frac{\theta + \alpha - \pi}{\gamma} \xi}{\cosh \frac{\pi \xi}{\gamma}} d\xi;$$

therefore the potential V at the point θ on the circle due to the whole system is given by

$$V = \frac{1}{\gamma} \int_0^\infty \frac{d\xi}{\cosh \frac{\pi \xi}{\gamma} \sinh \xi} \left\{ \cosh \frac{\pi - \theta}{\gamma} \xi \sinh \frac{\gamma - \alpha}{\gamma} \xi + \cosh \frac{3\pi - \theta - \gamma}{\gamma} \xi \sinh \frac{\alpha}{\gamma} \xi - \cosh \frac{\theta + \alpha - \pi}{\gamma} \xi \sinh \xi \right\},$$

that is,
$$V = \frac{2}{\gamma} \int_0^\infty \frac{\sinh \frac{\alpha \xi}{\gamma}}{\sinh \xi} \sinh \frac{2\pi - \theta - \gamma}{\gamma} \xi \tanh \frac{\pi \xi}{\gamma} d\xi.$$

Now
$$\tanh \frac{\pi \xi}{\gamma} = 2 \int_0^\infty \frac{\sin \frac{2\pi \xi \zeta}{\gamma}}{\sinh \pi \zeta} d\zeta$$

for all values of ξ ; therefore

$$V = \frac{4}{\gamma} \int_0^\infty \int_0^\infty \frac{\sinh \frac{\alpha \xi}{\gamma} \sinh \frac{2\pi - \theta - \gamma}{\gamma} \xi \sin \frac{2\pi \xi \zeta}{\gamma} d\xi d\zeta}{\sinh \xi \sinh \pi \zeta},$$

that is,
$$V = \frac{2}{\gamma} \int_0^\infty \int_0^\infty \frac{\sin \frac{2\pi \xi \zeta}{\gamma} d\xi d\zeta}{\sinh \zeta \sinh \pi \xi} \times \left\{ \cosh \frac{2\pi - \theta - \gamma + \alpha}{\gamma} \pi \xi - \cosh \frac{2\pi - \theta - \gamma - \alpha}{\gamma} \pi \xi \right\}.$$

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Again, writing

$$I = \int_0^x \frac{\sin \kappa \pi \xi \cosh \pi \lambda \xi}{\sinh \pi \xi} d\xi = \frac{1}{2\pi i} \int_0^1 \frac{(x^{-\kappa} - x^{\kappa})(x^{\lambda} + x^{-\lambda})}{x^{-1} - x} \frac{dx}{x},$$

that is
$$I = \frac{1}{2\pi i} \int_0^1 \left\{ \frac{x^{\lambda+\kappa} - x^{-\lambda-\kappa}}{x - x^{-1}} - \frac{x^{\lambda-\kappa} - x^{-\lambda+\kappa}}{x - x^{-1}} \right\} \frac{dx}{x},$$

whence
$$I = \frac{1}{4i} \left[\tan(\lambda + \kappa i) \frac{\pi}{2} - \tan(\lambda - \kappa i) \frac{\pi}{2} \right], \quad 1 > \lambda > 0;$$

we have
$$\int_0^{\infty} \frac{\sin \kappa \pi \xi \cosh \pi \lambda \xi}{\sinh \pi \xi} d\xi = \frac{1}{2} \frac{\sinh \kappa \pi}{\cosh \kappa \pi + \cos \lambda \pi}, \quad 1 > \lambda > 0.$$

It follows that

$$2 \int_0^{\infty} \frac{\sin \frac{2\pi \xi}{\gamma} \cosh \frac{2\pi - \theta - \gamma + \alpha}{\gamma} \pi \xi}{\sinh \pi \xi} d\xi = \frac{\sinh \frac{2\pi \xi}{\gamma}}{\cosh \frac{2\pi \xi}{\gamma} + \cos \frac{2\pi - \theta - \gamma + \alpha}{\gamma} \pi},$$

provided
$$1 > \frac{2\pi - \theta - \gamma + \alpha}{\gamma} > 0,$$

or
$$1 > \frac{\theta + \gamma - 2\pi - \alpha}{\gamma} > 0;$$

hence

$$V = \frac{1}{\gamma} \int_0^{\infty} \frac{d\xi}{\sinh \xi} \left\{ \frac{\sinh \frac{2\pi \xi}{\gamma}}{\cosh \frac{2\pi \xi}{\gamma} - \cos \frac{2\pi - \theta + \alpha}{\gamma} \pi} - \frac{\sinh \frac{2\pi \xi}{\gamma}}{\cosh \frac{2\pi \xi}{\gamma} - \cos \frac{2\pi - \theta - \alpha}{\gamma} \pi} \right\}$$

for all values of θ in the dielectric. This expression agrees with the general value of V given in my paper Vol. XXII, p. 160; and I therefore desire to withdraw my previous conclusion that it was incorrect. Putting in my expression for the potential $z = 0$, $\gamma = \gamma'$, and therefore $\eta = 0$, q being $-\gamma'$, it becomes

$$V = \frac{1}{2\alpha} \int_0^{\infty} \frac{d\xi}{\sinh \frac{\xi}{2}} \left\{ \frac{\sinh \frac{\pi \xi}{\alpha}}{\cosh \frac{\pi \xi}{\alpha} - \cos \frac{\pi(\theta + \theta')}{\alpha}} - \frac{\sinh \frac{\pi \xi}{\alpha}}{\cosh \frac{\pi \xi}{\alpha} - \cos \frac{\pi(\theta - \theta')}{\alpha}} \right\},$$

the same as the above, remembering that α in the last expression is γ in the former, θ is $\theta - 2\pi + \gamma$, and θ' is $\gamma - \alpha$.