The Reciprocators of Two Conics discussed Geometrically. By J. W. Russell, M.A.

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General Construction.

1. If two conics α and β have a common pole and polar, we can construct a conic Γ with respect to which they are reciprocal in the following way:—

Let U be the common pole; and let the common polar u cut α in A, A', and β in B, B'. Take X, X' the double points of the involution $(AB, A'B')$. Let any tangent q of α cut AA' in N. Let the line joining U to the fourth harmonic N' of N for XX' cut β in Q. Let QX cut q in M. Take R so that (QM, XR) is harmonic. Now take Γ as the conic touching UX at X, UX' at X', and passing through R.

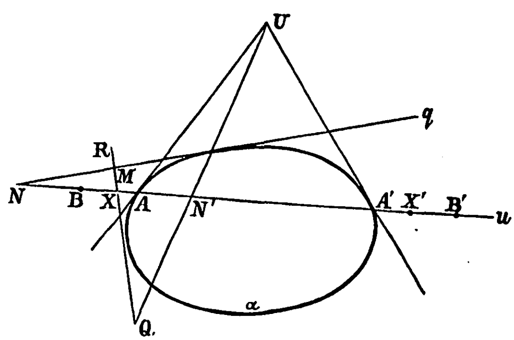

Fig. 1.

Then AA' is the reciprocal of U. Hence the reciprocal of B passes through U; and also through A, since (BA, XX') is harmonic. Hence UA is the reciprocal of B; so UB of A, UA' of B' and UB' of A'. Hence the reciprocal of α is a conic which touches UB at B, and UB' at B'. Now, since (NN', XX') is harmonic, the reciprocal of N is UQ; hence the reciprocal of Q passes through N; and also through M since (QM, XR) is harmonic. Hence q is the reciprocal of Q. Hence the reciprocal of α also passes through Q, and is therefore β.

2. It is convenient to call Γ a reciprocator of α and β.

Since we might have taken X, X' as the double points of the involution $(AB', A'B)$, and also have taken Q as the other point in which UN' meets β, the above construction gives four solutions of the problem if A, B, A', B' are distinct.

Notice that in all cases the reciprocators Γ constructed as above will have a self-conjugate triangle in common with α and β. For let V, W be the double points of the involution (AA', BB'). Now X, X' are the double points of the involution $(AB, A'B')$. Hence (AA', BB', XX') is an involution. Hence V, W, being the double points of this involution, are harmonic with X, X'. Hence V, W are conjugate for α, β, and Γ.

Particular Cases.

3. If the two conics intersect in four distinct points, we may take U at any vertex of the common self-conjugate triangle. Then A, A', B, B' will be distinct, and we get four solutions.

If the two conics touch α at A and have two distinct intersections D, E, there is one position of U, viz., the intersection of α with DE; and the common polar u of U is the line joining A to the intersection

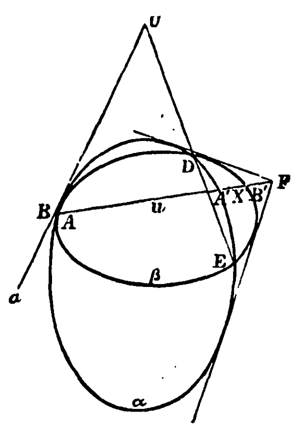

FIG. 2.

of the distinct common tangents. Here A, B coincide. If we take the involution $(AB, A'B')$, X coincides with A and B, and X' is such that $(AX', A'B')$ is harmonic. We cannot take the involution $(AB', A'B)$; for no two points can be found which are harmonic both with A, A' and with A, B'. Hence there will be two solutions.

If the two conics have double contact, we may take U at the common pole. Then A, B coincide, and so do A', B'. If we take the involution (AB, $A'B'$), X is at A and X' at A'; and Γ has double contact with α and β. This gives two solutions. If we take the involu-

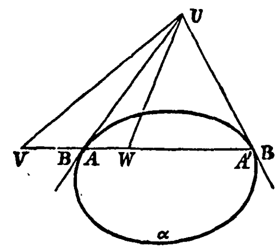

FIG. 3.

tion (AB', $A'B$), X and X' are any two points which divide AA' harmonically; and we get a doubly infinite set of solutions.

If the two conics have four-point contact, we may take U at any point on the common tangent; for any such point has the same polar

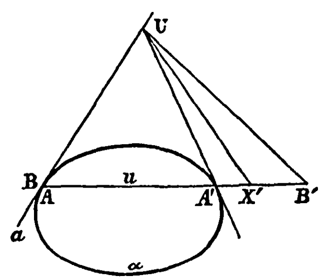

FIG. 4.

for both conics. Hence, as in the previous case, there is apparently a doubly infinite set of solutions (but see § 12).

Exceptional Construction.

4. If the conics have three-point contact and a distinct intersection, the above method fails; for then the conics have no common pole and polar. Let C be the point of contact, and K the distinct intersection. Let the common tangent k meet the tangent c at C in D, meet CK in L, and touch the conics α and β in A and B. Let the tangent b at K to α cut k in M. Take X, the fourth harmonic of C for K, L. Let MX cut KB in N, and take P such that (MN, XP) is harmonic.

Now take as Γ the conic which touches c at C, touches DX at X, and passes through P.

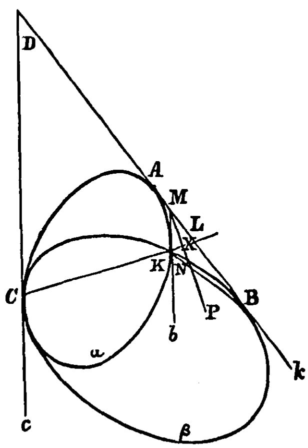

FIG. 5.

Then the reciprocal of D is CX. Hence the reciprocal of K is k for (CX, KL) is harmonic. But K is on a and k touches a; hence k touches and K is on a', the reciprocal of a for Γ. Now M is on k. Hence its reciprocal passes through K; and also through N, since (MN, XP) is harmonic. Hence KN is the reciprocal of M. Hence B is the reciprocal of MK which touches a. Hence B is on a'. Also since a and Γ touch at C, a' touches both at C. Hence a' and β coincide; for each touches c at C, touches k at B, and passes through K.

Notice that in this case Γ has three-point contact with a and β at C. For, if a and Γ have not three-point contact, they have a common pole and polar; hence a and β have a common pole and polar; which is not true.

The above Solution is Exhaustive.

5. We have now shown that in every case there is at least one solution of the problem. Let us next consider whether there are any solutions other than those already found.

Take first the exceptional case in which the conics have three-point contact. Here the conics a and β touch at C only. Hence Γ must touch both a and β at C, for coincident common points must reciprocate into coincident common tangents. Again, the unique intersection K must reciprocate into the unique common tangent k Hence X is known, and then P. Hence there is no other solution.

6. Take the case in which the conics α and β intersect in four distinct points. Then α and Γ must also intersect in four distinct points, for, if α and Γ touch, α and β touch. Hence α and Γ have one and only one common self-conjugate triangle UVW; and this reciprocates into itself and into a triangle which is self-conjugate for β and Γ. Hence UVW must be the common self-conjugate triangle of α and β. Hence α, β, Γ have the same common self-conjugate triangle. Hence the polar of U for Γ must be VW, *i.e.*, we must take the polar of U for α and β as the reciprocal of U. Hence the positions of X, X' are known. Also the reciprocal of q which passes through N and touches α must lie on the reciprocal of N (viz., UN'), and also on β. Hence Q is known, and then R. Hence there are four and only four reciprocators.

Take the case in which α and β touch at A, and intersect in two distinct points D, E. Then the line DE must reciprocate into F, the intersection of distinct common tangents. Also A must reciprocate into a. Hence U must reciprocate into AF.

Take the case in which α and β touch at A and A'. Then A must reciprocate into UA or UA', and A' into UA' or UA. In either case U reciprocates into AA'.

Take, lastly, the case in which the conics have four-point contact at A. Then A must reciprocate into a, the tangent at A. Hence the reciprocal of any point U on a must reciprocate into some line u through A, say $AA'B'$. Then the reciprocal of A' on a and u is the tangent to β from U, *i.e.*, is UB'. Hence u is the polar of U for β, and therefore for α also.

Hence in all cases we have obtained every reciprocator.

Self-reciprocal Conics.

7. Let us next inquire for what conics a given conic is its own reciprocal.

Let the point A on the conic α be the reciprocal of the tangent a of the conic α. Then the tangent a' at A is the reciprocal of the point of contact A' of a. Hence $(AA'BB'\ldots)$ of poles $= (aa'bb'\ldots)$ of polars $= (A'AB'B\ldots)$ of points of contact. Hence $(AA', BB', \ldots)$ is an involution. Hence AA', BB', ... meet in a fixed point, O say. Now let A be one, E, of the points of contact E, F of tangents from O. Then the reciprocal of E is the tangent at E. Hence the required reciprocator Γ touches OE at E. Similarly, Γ touches OF at F. Hence α and Γ have double contact.

Let AE cut $A'F$ in P, and let AF cut $A'E$ in P'. Then P and P' are on Γ. For, if a cut AF in L and EF in G,

$$(AL, FP') = A'(AL, FP') = (HG, FE),$$

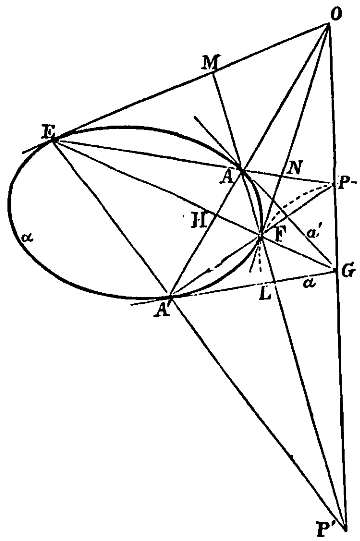

FIG. 6.

which is harmonic, for G is the pole of AA'. Now a is the reciprocal of A, and F is on Γ. Hence P' is on Γ, and similarly P.

We have still to verify that such a conic exists. Describe a conic Γ' touching OE at E, OF at F, and passing through P'. Then the reciprocal of a for this conic touches OE at E, and OF at F. Now, since (AL, FP') is harmonic, the reciprocal of A passes through L. Also AA' is the reciprocal of G, since (GH, FE) is harmonic. Hence the reciprocal of A passes through G. Hence a is the reciprocal of A. Hence the reciprocal of a touches OE at E, OF at F, and also touches a; and hence is α. Hence the reciprocal of α for Γ' is α.

Also the reciprocal of Γ' for α is Γ'. First notice that P is on the conic Γ' by the first part of the above proof. Also that PP' passes through O (and also through G) since PP', EF and AA' are the diagonals of the same quadrilateral. Now Γ' is generated from α by the intersection of EA' and AF, and α is generated from Γ' by the intersection of EP' and PF. Hence Γ' is self-reciprocal for α.

8. Such conics may be called a pair of self-reciprocal conics, each being its own reciprocal for the other. The fundamental property of self-reciprocal conics is that they are in harmonic homology, taking E as pole and OF as axis, or F as pole and OE as axis. For

$$(EN, AP) = O\,(EN, AP) = (EF, HG);$$

so for F. The simplest definition is that they have double contact, and have, at each point of contact, equal and opposite curvatures. For, since (MF, AP') is harmonic, $AF : FP' :: MA : MP' :: OF : OF$, when A and P' coincide with F. Hence $AF = FP'$ ultimately, and similarly $A'F = FP$. And the angles AFA' and $P'FP$ are equal. Hence the triangles AFA' and $P'FP$ are ultimately equal in all respects. Hence the curvatures of α and Γ at F are equal and opposite; and similarly at E. The above proof does not apply if F is at infinity, but in this case the proposition respecting curvature is nugatory.

Connexion between the Reciprocators.

9. Let us now study the connexion between the various conics Γ for which the same two conics α and β are reciprocal.

We notice that the reciprocators always occur in pairs which touch at two points X and X', say, and that another reciprocator cannot touch both of these conics at X, X'. We shall now prove that the two conics Γ_1, Γ_2 forming such a pair are self-reciprocal. For reciprocate for Γ_1. Then α reciprocates into β and β into α. Γ_1 reciprocates into itself. Also Γ_2 reciprocates into a reciprocator; for the proposition that α and β are reciprocal for Γ_2 reciprocates into the proposition that β and α are reciprocal for the reciprocal Γ_2' of Γ_2. Hence Γ_2' is a reciprocator having double contact with Γ_1 at X, X'. And the reciprocal for Γ_1 of no conic except Γ_1 can coincide with Γ_1. Hence Γ_2' coincides with Γ_2, *i.e.*, Γ_1 and Γ_2 are self-reciprocal.

10. If the two conics intersect in four distinct points, then each of the four conics Γ_1, Γ_2, Γ_3, Γ_4 is self-reciprocal for every other. For let X_1, X_1', X_2, X_2' be the double points of the above construction on VW; Y_1, Y_1', Y_2, Y_2' those on WU; and Z_1, Z_1', Z_2, Z_2' those on UV, where UVW is the common self-conjugate triangle of α and β. Let Γ_1, Γ_2 touch UX_1, UX_1' at X_1, X_1', and let Γ_3, Γ_4 touch UX_2, UX_2' at X_2, X_2'. Then Γ_1, Γ_2 are self-reciprocal, and so are Γ_3, Γ_4. Now two of the four touch VY_1, VY_1' at Y_1, Y_1'; but Γ_1, Γ_2 cannot touch again, nor can Γ_3, Γ_4. Hence Γ_1 must touch Γ_3, or Γ_2 touch Γ_4. Then Γ_2 touches Γ_4, or Γ_1 touches Γ_3 at Y_2, Y_2'. Hence Γ_1, Γ_3 are self-reciprocal,

and so are Γ_2, Γ_4. Similarly, Γ_1, Γ_4 touch on UV, and so do Γ_2, Γ_3. Hence Γ_1, Γ_4 are self-reciprocal, and so are Γ_2, Γ_3. Hence every one of the conics Γ_1, Γ_2, Γ_3, Γ_4 is self-reciprocal for every other.

If the two conics touch and intersect in two points, the two conics Γ_1, Γ_2 are self-reciprocal; for they have double contact.

11. If the two conics α and β have double contact at A and A', the two reciprocators Γ_1 and Γ_2 which have double contact with them at A and A' are, as before, self-reciprocal. Let Γ_3 and Γ_4 be the reciprocators touching at X, X', where X, X' are harmonic with A, A'; these also are, as before, self-reciprocal. Take V, W the double points of the involution (AA', XX'). Then V, W are conjugate for α, β, Γ_1, Γ_2, Γ_3, Γ_4. Hence UVW is a self-conjugate triangle for these conics. Also no other reciprocator has UVW as a self-conjugate triangle. For suppose the reciprocators γ_3', γ_4' touching at x, x' have UVW as a self-conjugate triangle. Then V, W are harmonic with x, x', which are harmonic with A, A'. Hence x, x' coincide with X, X'; for these are also harmonic with both A, A' and V, W. Now the four reciprocators obtained by using V or W as pole have also UVW as a self-conjugate triangle (see § 2, end). Hence these conics are the same as Γ_1, Γ_2, Γ_3, Γ_4. Hence, as in § 10, they are self-reciprocal in pairs. Hence we conclude that each of the variable conics Γ_3, Γ_4 has double contact with the fixed conics Γ_1, Γ_2 both on UV and on UW.

12. Finally, if the conics α and β have four-point contact, the reciprocators consist of one conic Γ' and a system of conics self-reciprocal for Γ'. For (see Fig. 4), the conic Γ' having four-point contact with both α and β at A, and passing through X' is a reciprocator. For the tangent at X' is UX', since the tangents at A' and X' to α and Γ' meet on AU; hence AA' is the reciprocal of U. Hence the reciprocal of UA' is B'. Hence the reciprocal of α for Γ' is a conic having four-point contact with β at A, and passing through B', *i.e.*, is β. Also Γ' is one of the reciprocators belonging to AA', for it is a reciprocator and touches UX' at X' and UA at A. And the other is self-reciprocal for Γ'.

13. We can extend the above construction to two quadrics. For instance, take the case in which the two quadrics have a common pole and polar, U and u. Let the quadrics be ϕ and ψ, and let their sections by the plane u be α and β. Take any reciprocator γ of the conics α and β. Let any tangent plane q of ϕ cut u in the line n.

Let the line joining U to the pole N' of n for γ cut ψ in the point Q. Let the line joining Q to any point X on γ meet q in M, and take the point R so that (QM, XR) is harmonic. Then a reciprocator of the quadrics ϕ and ψ is the quadric Γ which passes through R and touches the cone joining U to γ along γ.

For let ϕ' be the reciprocal of ϕ for Γ. Then, since ϕ touches the cone joining U to α along α, ϕ' touches the cone joining U to β along β; for U is the reciprocal of u. Now Q lies on UN'. Also N' is the reciprocal of Un. Hence the reciprocal of Q passes through the intersection of u and Un, *i.e.*, through n; and it passes through M, since (QM, XR) is harmonic; hence it is q. But q touches ϕ; hence Q is on ϕ'. Hence ϕ' is ψ.

Notice that, since Q may have two positions, we get twice as many reciprocators of the quadrics as there are of the conics α and β.

By A. E. JOLLIFFE. Received May 8th, 1895. Communicated May 9th, 1895.

The following analytical investigation was undertaken with the object of confirming Mr. Russell's statements as to the number of the conics Γ in the different cases, concerning which some doubts had been expressed. For this reason the result in § 4 is obtained by rigorous analysis instead of being assumed as geometrically obvious. The result of § 2 and the existence of the infinite systems in the cases of double and four-point contact were first obtained as below by means of analysis. It is hoped that the additional matter contained in § 7 and onwards may be not without interest.

1. In the case of four distinct intersections, the equations of the two conics may be written

$$S \equiv x^2+y^2+z^2 = 0,$$

$$S' \equiv px^2+qy^2+rz^2 = 0.$$

Take the equation of Γ in the most general form possible, viz.,

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0.$$