On a Twofold Generalization of Stieltjes' Theorem. By HENRY TABER, Worcester, Mass.

Received June 8th, 1896. Read June 11th, 1896.

In a paper "Sur une propriété des déterminants symétriques gauche" which appeared in Volume XVII., Second Series (1892), of the *Mémoires de la Société Royale des Sciences de Liège*, M. François Deruyts gave the following very interesting theorem :—

If the minors of order $2k$ of a skew symmetric determinant are all zero, the minors of order $2k-1$ are all zero also.

As an immediate consequence of this theorem follow certain theorems of some interest relating to orthogonal substitutions, among which is included a two-fold generalization of Stieltjes' theorem.

Let the transformation A defined by the system of equations

$$x'_r = a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rn}x_n \quad (r = 1, 2, \ldots n)$$

be any orthogonal substitution in n variables. Let $A_{(+1)}$ denote the linear transformation defined by the system of equations

$$x'_r = a_{r1}x_1 + \ldots + a_{r,r-1}x_{r-1} + (a_{rr} - \rho)\, x_r + a_{r,r+1}x_{r+1} + \ldots + a_{rn}x_n \quad (r = 1, 2, \ldots n),$$

for $\rho = +1$, and let $A_{(-1)}$ denote the linear transformation defined by this system of equations for $\rho = -1$.

Further, let Det. $\left[A^m_{(+1)}\right]$, m being any positive integer, denote the determinant of the transformation $A^m_{(+1)}$, the m^{th} power of $A_{(+1)}$, obtained by m applications (*i.e.*, $m-1$ repetitions) of the transformation $A_{(+1)}$. Similarly, let Det. $\left[A^m_{(-1)}\right]$ denote the determinant of $A^m_{(-1)}$, the m^{th} power of $A_{(-1)}$. Then we have at once the following theorems :—

I. *If the orthogonal substitution A is proper, and if the minors of order $2k$ of the determinant Det.* $\left[A^{2m+1}_{(+1)}\right]$ *are all zero, the minors of order $2k-1$ are all zero also.*

II. *If the determinant of the orthogonal substitution A is equal to -1,* *and the minors of order* $2k+1$ *of Det.* $\left[A^{2m+1}_{(+1)}\right]$ *are all zero, the minors of order* $2k$ *are all zero also.*

III. *If the determinant of the orthogonal substitution* A *is equal to* $+1$, *and if the* $(2\kappa)^{\text{th}}$* *minors of Det.* $\left[A^{2m+1}_{(-1)}\right]$ *are all zero, the* $(2\kappa+1)^{\text{th}}$ *minors are all zero also.*

IV. *If the determinant of the orthogonal substitution* A *is equal to* -1, *and if the* $(2\kappa-1)^{\text{th}}$ *minors of Det.* $\left[A^{2m+1}_{(-1)}\right]$ *are all zero, the* $(2\kappa)^{\text{th}}$ *minors are all zero also.*

Theorem III. is the two-fold generalization of Stieltjes' theorem above referred to. For $m=0$, we have

$$\text{Det.}\left[A_{(-1)}\right] = \begin{vmatrix} a_{11}+1, & a_{12}, & \dots & a_{1n} \\ a_{21}, & a_{22}+1, & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1}, & a_{n2}, & \dots & a_{nn}+1 \end{vmatrix};$$

and the theorem is that, if the $(2\kappa)^{\text{th}}$ minors of this determinant are all zero, the $(2\kappa+1)^{\text{th}}$ minors are all zero also. For $\kappa=0$, this is Stieltjes' theorem.†

Deruyts' theorem gives at once the conditions that must be satisfied by the numbers belonging to the roots ± 1 of the characteristic equation of an orthogonal substitution. Thus let $+1$ be a root of multiplicity p of the characteristic equation of the orthogonal substitution A. Then the nullity of the transformation $A_{(+1)}$ is at least one,‡ and the nullity of $A^2_{(+1)}$, &c., the successive powers of $A_{(+1)}$, in-

---

* That is, the minors of Det. $\left[A^{2m+1}_{(-1)}\right]$ of order $n-2\kappa$.

† See Netto, *Acta Mathematica*, Vol. IX., p. 295.

‡ The nullity of the linear transformation defined by the system of equations

$$x'_r = a_{r1}x_1 + a_{r2}x_2 + \dots + a_{rn}x_n \quad (r = 1, 2, \dots n)$$

is p if all the $(p-1)^{\text{th}}$ minors of the matrix

$$\begin{matrix} a_{11} a_{12} & \dots & a_{1n} \\ a_{21} a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots \\ a_{n1} a_{n2} & \dots & a_{nn} \end{matrix}$$

are zero, but not all the p^{th} minors.

creases until a power of exponent μ is attained whose nullity is equal to p. The nullity of the $(\mu+1)^{\text{th}}$ and of higher powers of $A_{(+1)}$ is then also p. Let

$$p_1,\ p_2,\ \dots\ p_{\mu-1},\ p_\mu = p,$$

designate respectively the nullities of

$$A_{(+1)},\ A^2_{(+1)},\ \dots\ A^{\mu-1}_{(+1)},\ A^{\mu}_{(+1)};$$

I term the numbers p_1, p_2, &c., the numbers *belonging* to the root $+1$ of the characteristic equation of A. For any linear transformation A, we must have

$$p_1 \geqq p_2 - p_1 \geqq \dots \geqq p_\mu - p_{\mu-1} \geqq 1;^*$$

and, since A is orthogonal, it follows from Deruyts' theorem that p_1, p_3, &c., the numbers with odd suffixes belonging to the root $+1$ of the characteristic equation of A, are all even or all odd according as $p = p_\mu$ is even or odd.

Similarly, if q_1, q_2, ... q_ν are the numbers belonging to the root -1 of the characteristic equation of A,

$$q_1 \geqq q_2 - q_1 \geqq \dots \geqq q_\nu - q_{\nu-1} \geqq 1,$$

and q_1, q_3, &c., the numbers with odd suffixes, are all even or all odd according as $q = q_\nu$ is even or odd.†

These theorems may be proved as follows:—If $+1$ is a root of multiplicity p, and -1 a root of multiplicity q of the characteristic equation of the orthogonal substitution A, an orthogonal substitution E can always be found, such that

$$A = EBE^{-1},$$

where B is an orthogonal substitution in n variables defined by the

---

* In Volume XXVI. of these *Proceedings*, page 368, line 19, the conditions for the numbers belonging to a root other than ± 1, should read $m_1 \geqq m_2 - m_1 \geqq$ &c., as there given.

† The conditions given above are the only conditions for the numbers belonging to the root $+1$ or -1.

three orthogonal substitutions

$$X'_r = b^{(1)}_{r1} X_1 + b^{(1)}_{r2} X_2 + \dots + b^{(1)}_{rp} X_p \qquad (r = 1, 2, \dots p),$$

$$X'_{p+r} = b^{(2)}_{r1} X_{p+1} + b^{(2)}_{r2} X_{p+2} + \dots + b^{(2)}_{rq} X_{p+q} \qquad (r = 1, 2, \dots q),$$

$$X'_{p+q+r} = b^{(3)}_{r1} X_{p+q+1} + b^{(3)}_{r2} X_{p+q+2} + \dots + b^{(3)}_{r, n-p-q} X_n \qquad (r = 1, 2, \dots \overline{n-p-q}),$$

$+1$ being a root of multiplicity p of the characteristic equation of the transformation B_1 defined by the first p equation, -1 being a root of multiplicity q of the characteristic equation of the transformation B_2 defined by the second set of equations (q in number), while the roots of the characteristic equation of B_3, the transformation defined by the third system of equations, are the roots other than ± 1 of the characteristic equation of A.

Let the transformation I_1 be defined by the system of equations

$$X'_r = X_r \qquad (r = 1, 2, \dots p);$$

let I_2 be defined by the equations

$$X'_{p+r} = X_{p+r} \quad (r = 1, 2, \dots q);$$

and let I be defined by the equations

$$X'_r = X_r \qquad (r = 1, 2, \dots n).$$

Then I_1 is the identical transformation for the p-way extension $(X_1, X_2, \dots X_p)$; I_2 is the identical transformation for the q-way extension $(X_{p+1}, X_{p+2}, \dots X_{p+q})$; and I is the identical transformation for the n-way extension $(X_1, X_2, \dots X_n)$.

Since -1 is not a root of the characteristic equation of the linear transformation B_1 in p variables, the determinant of $I_1 + B_1$* is not zero; and so we may put

$$C_1 = (I_1 - B_1)(I_1 + B_1)^{-1}$$

---

* Following Cayley, I regard the operations of addition and subtraction as capable of extension to linear transformations; if the linear transformations A and B are defined respectively by the two systems of equations

$$X'_r = a_{r1} X_1 + a_{r2} X_2 + \dots + a_{rn} X_n \quad (r = 1, 2, \dots n),$$

$$X'_r = b_{r1} X_1 + b_{r2} X_2 + \dots + b_{rn} X_n \quad (r = 1, 2, \dots n),$$

the transformation $A \pm B$ is defined by the system of equations

$$X'_r = (a_{r1} \pm b_{r1}) X_1 + (a_{r2} \pm b_{r2}) X_2 + \dots + (a_{rn} \pm b_{rn}) X_n \quad (r = 1, 2, \dots n).$$

Whence, denoting by $\breve{C}_1$ the transverse or conjugate of C_1,* we have

$$\begin{aligned}\breve{C}_1 &= (\breve{I}_1+\breve{B}_1)^{-1}(\breve{I}_1-\breve{B}_1)\\ &= (I_1+B_1^{-1})^{-1}(I_1-B_1^{-1})\\ &= -(I_1+B_1)^{-1}(I_1-B_1) = -C_1,\end{aligned}$$

that is, C_1 is skew symmetric. Moreover,

$$(I_1+C_1)(I_1+B_1) = 2I_1;$$

therefore the determinant of I_1+C_1 is not zero; and, consequently,

$$B_1 = (I_1-C_1)(I_1+C_1)^{-1}.$$

Whence we obtain $\quad B_1-I_1 = -2C_1(I_1+C_1)^{-1}$,

$$(B_1-I_1)^{2m+1} = (-2)^{2m+1}C_1^{2m+1}(I_1+C_1)^{-2m-1}.$$

Since the determinant of I_1+C_1 is not zero, the nullity of $(B_1-I_1)^{2m+1}$ is equal to the nullity of C_1^{2m+1}; therefore, if the minors of order r of the determinant of $(B_1-I_1)^{2m+1}$ are all zero, the minors of order r of the determinant of C_1^{2m+1} are all zero, and conversely. But C_1^{2m+1} is skew symmetric, being an odd power of a skew symmetric linear transformation. Therefore, by Deruyts' theorem, if the minors of order $2k$ of the determinant of $(B_1-I_1)^{2m+1}$ are all zero, the minors of order $2k-1$ are all zero also. That is, if the nullity of $(B_1-I_1)^{2m+1}$ is as great as $p-2k+1$, it is as great as $p-2k+2$.

Furthermore, since $+1$ is not a root of the characteristic equation of B_2, nor of the characteristic equation of B_3, the nullity of $(B-I)^{2m+1}$ is equal to the nullity of $(B_1-I_1)^{2m+1}$. Again, the nullity of

$$A_{(+1)}^{2m+1} = (A-I)^{2m+1} = (EBE^{-1}-I)^{2m+1} = E(B-I)^{2m+1}E^{-1}$$

is equal to the nullity of $(B-I)^{2m+1}$. Therefore, if the nullity of $A_{(+1)}^{2m+1}$ is as great as $p-2k+1$, it is as great as $p-2k+2$.

---

* If the transformation C is defined by the equations

$$X_r' = C_{r1}X_1+C_{r2}X_2+\ldots+C_{rn}X_r \quad (r = 1, 2, \ldots n),$$

the transformation $\breve{C}$, the transverse of C, is defined by the equations

$$X_r' = C_{1r}X_1+C_{2r}X_2+\ldots+C_{nr}X_r \quad (r = 1, 2, \ldots n).$$

We have $\quad \breve{CC'} = \breve{C'}\breve{C}, \quad (\breve{C^{-1}}) = (\breve{C})^{-1}.$

Similarly, since $+1$ is not a root of the characteristic equation of B_2, we may put

$$C_2 = (I_2+B_2)(I_2-B_2)^{-1}.$$

Whence we derive, as before,

$$\begin{aligned}\breve{C}_2 &= (\breve{I}_2-\breve{B}_2)^{-1}(\breve{I}_2+\breve{B}_2)\\ &= (I_2-B_2^{-1})^{-1}(I_2+B_2^{-1})\\ &= -(I_2-B_2)^{-1}(I_2+B_2) = -C_2.\end{aligned}$$

We also have $$(I_2+C_2)(I_2-B_2) = 2I_2;$$

consequently, the determinant of I_2+C_2 is not zero, and therefore we have

$$B_2 = -(I_2-C_2)(I_2+C_2)^{-1}.$$

Whence we obtain

$$(B_2+I_2)^{2m+1} = 2^{2m+1}C_2^{2m+1}(I_2+C_2)^{-2m-1}.$$

By the same reasoning as that employed above, since the determinant of I_2+C_2 is not zero, and since C_2^{2m+1} is skew symmetric, it follows from Deruyts' theorem that, if the nullity of $(B_2+I_2)^{2m+1}$ is as great as $q-2k+1$, it is as great as $q-2k+2$. But, since -1 is not a root of the characteristic equation of either B_1 or B_3, the nullity of $(B_2+I_2)^{2m+1}$ is equal to the nullity of $(B+I)^{2m+1}$. Again, the nullity of

$$A_{(-1)}^{2m+1} = (A+I)^{2m+1} = (EBE^{-1}+I)^{2m+1} = E(B+I)^{2m+1}E^{-1}$$

is equal to the nullity of $(B+I)^{2m+1}$. Therefore, if the nullity of $A_{(-1)}^{2m+1}$ is as great as $q-2k+1$, it is as great as $q-2k+2$.

If the determinant of A is equal to $+1$, p, the multiplicity of the root $+1$ of the characteristic equation of A, is even or odd according as n is even or odd. Therefore, *if A is a proper orthogonal substitution, and if n is $\frac{\text{even}}{\text{odd}}$, the nullity of $A_{(+1)}^{2m+1}$ is $\frac{\text{even}}{\text{odd}}$.*

On the other hand, if A is improper, p is odd if n is even, and is even if n is odd. Therefore in this case, *if n is $\frac{\text{even}}{\text{odd}}$, the nullity of $A_{(+1)}^{2m+1}$ is $\frac{\text{odd}}{\text{even}}$.* These two theorems are equivalent respectively to I. and II.

Further, irrespective of the value of n, q is even if A is proper, and is odd if A is improper. Therefore, *if A is $\frac{proper}{improper}$, the nullity of $A_{(-1)}^{2m+1}$ is $\frac{even}{odd}$.* This theorem is equivalent to III. and IV.

If $p_1, p_2, \ldots p_\mu$ are the numbers belonging to the root $+1$ of multiplicity p of the characteristic equation of A, then, since the nullity of $A_{(+1)}^{2m+1}$ is even or odd according as p is even or odd, it follows that p_1, p_3, &c., the numbers with odd suffices, are all even if $p = p_\mu$ is even, or all odd if p is odd.

Similarly, if $q_1, q_2, \ldots q_\nu$ are the numbers belonging to the root -1 of multiplicity q of the characteristic equation of A, then, since the nullity of $A_{(-1)}^{2m+1}$ is even or odd according as q is even or odd, it follows that q_1, q_3, &c., are all even or all odd according as $q = q_\nu$ is even or odd.

[*Postscript, December* 25*th*, 1896.—Employing the notation of the preceding paper, Stieltjes' theorem, as originally enunciated, is

If A and B are two proper orthogonal substitutions, the determinant of $A+B$ vanishes only if the first minors of this determinant all vanish.

The theorem was given by Stieltjes for n (the number of variables) equal to two, or equal to three. Stieltjes stated that he believed the theorem to hold for $n = 4$; and he suggested the inquiry whether it held for any value of n.*

If B is the identical substitution, this theorem becomes the theorem given in the preceding paper as Stieltjes' theorem, namely:

If A is a proper orthogonal substitution, the determinant of $A+I$ (where I is the identical substitution) vanishes only if the first minors of this determinant all vanish.

This theorem is formally included in Stieltjes' theorem, and is apparently only a special case of Stieltjes' theorem. But the latter also follows from this theorem. For, if A and B are any two proper orthogonal substitutions, AB^{-1} (where B^{-1} denotes the reciprocal of B, which, since B is orthogonal, is thus the transverse or conjugate of B) is also a proper orthogonal substitution; and, since the determinant of B is not zero, the nullity of $A+B = (AB^{-1}+I)\,B$ is equal to the nullity of $AB^{-1}+I$. By the preceding theorem, the determinant of $AB^{-1}+I$ vanishes only if the first minors of this determinant all vanish. That is, if the nullity of $AB_{-1}+I$ is as

* *Acta Mathematica*, Vol. VI., p. 319.

great as 1, it is as great as 2. Therefore, if the nullity of $A+B$ is as great as 1, it is as great as 2, which is Stieltjes' theorem.

Since the content of both theorems is the same, the latter may also be termed Stieltjes' theorem.

I have recently learned that the most interesting case of Theorem III., also of Theorem IV., of the preceding paper, namely, for $m = 1$, were given by A. Voss, in the *Mathematische Annalen*, Vol. XIII., p. 330. For $m = 1$, Theorems III. and IV. are:

If the substitution

$$X_r = a_{r1}y_1 + a_{r2}y_2 + \ldots + a_{rn}y_n \qquad (r = 1, 2, \ldots n)$$

is orthogonal, and the r^{th} minors of

$$\begin{vmatrix} a_{11}+1, & a_{12}, & \ldots \\ a_{21}, & a_{22}+1, & \ldots \\ \ldots & \ldots & \ldots \end{vmatrix}$$

are all zero, but not all the $(r+1)^{th}$ minors, r is an odd number if the substitution is proper, and is even if the substitution is improper.

Voss states that this theorem is true of the determinant

$$\text{Det.}\left[A_{(+1)}\right] = \begin{vmatrix} a_{11}-1, & a_{12}, & \ldots \\ a_{21}, & a_{22}-1, & \ldots \\ \ldots & \ldots & \ldots \end{vmatrix}.$$

This is the case if n is even, but not otherwise. Thus, for $n = 3$, if $a_{11} = +1$, $a_{22} = a_{33} = -1$, and $a_{rs} = 0$ for $r \neq s$, the substitution is proper, and Det. $\left[A_{(+1)}\right]$ vanishes, but not its first minors. The proper statement, for n either odd or even, of the theorem relating to the Det. $\left[A_{(+1)}\right]$ corresponding to Voss' theorem, just given, regarding the Det. $\left[A_{(-1)}\right]$, is obtained from Theorems I. and II., by putting $m = 1$. The theorem thus obtained may be shown to follow immediately from Voss' investigations. But I do not know that it has anywhere been given.

To Voss is due, I believe, the first complete proof of Stieltjes' theorem. Netto's proof does not extend to every case.

I have also found that Theorems I., II., III., and IV. may be derived from theorems relating to the elementary divisors $(\rho \pm 1)^{e_r}$ of the characteristic function of an orthogonal substitution, given by Frobenius in *Crelle's Journal*, Vol. LXXXIV.

Let ρ be a variable parameter not contained in the coefficients of the linear substitution A; and let ρ_0 be a root of multiplicity l of the characteristic equation of A, viz.,

$$\text{Det.}\left[A-\rho I\right]=0,$$

where I is the identical substitution. Then $(\rho_0-\rho)^l$ is a divisor of the left-hand member of this equation, the characteristic function of A.

The minors of all orders of this determinant are polynomials in ρ, and the minors of order $n-r$ may all contain $\rho_0-\rho$ if $l>r$. Let l_r be the highest power of $\rho_0-\rho$ contained in all the minors of order $n-r$. If l_ω is the last of the series l_1, l_2, &c., that is not zero,

$$l>l_1>l_2\ldots>l_\omega>0;$$

and, if $\quad e=l-l_1,\quad e_1=l_1-l_2,\ \ldots,\ e_{\omega-1}=l_{\omega-1}-l_\omega,\quad e_\omega=l_\omega,$

then $\quad e\geqq e_1\geqq e_2\geqq\ldots\geqq e_\nu\geqq 1,$

$$(\rho_0-\rho)^l=(\rho_0-\rho)^e(\rho_0-\rho)^{e_1}(\rho_0-\rho)^{e_2}\ldots(\rho_0-\rho)^{e_\omega}.$$

The several divisors $(\rho_0-\rho)^e$, $(\rho_0-\rho)^{e_1}$, &c., of $(\rho_0-\rho)^l$ are elementary divisors (*elementar theiler*) of the characteristic function of A. These divisors all vanish for $\rho=\rho_0$. Corresponding to each root of the characteristic equation of A is a system of elementary divisors that vanish if ρ be equal to the root in question.

Let now m_1, m_2, ... $m_\mu=l$ be the numbers belonging to the root ρ_0. Form a diagram corresponding to ρ_0 by arranging l dots in rows and columns, so that there shall be μ rows respectively of

$$m_1,\ m_2-m_1,\ \ldots\ m_\mu-m_{\mu-1}$$

equidistant dots, and so that the first dot in each row falls in the same (left-hand) column. The number of columns will then be m_1, which will be found to be equal to $\omega+1$; and the number of dots in the successive columns (counting from the left) will be equal severally and respectively, to e, e_1, e_2, ... e_ω.

For an orthogonal substitution Frobenius has shown that the elementary divisors $(\rho\pm 1)^{2k}$ occur in pairs with equal exponent. It will be found that this is equivalent to the theorem given above, that the numbers with odd subscripts belonging to the root ± 1 of the characteristic equation of an orthogonal substitution are all even or all odd according as the multiplicity of the roots ± 1 is even or odd. From this theorem, Theorems I., II., III., and IV. may be derived.]