On the Intersections of Two Cubics. *By* H. M. Taylor. Received January 6th, 1898. Read January 13th, 1898. Received, in amended form, May 5th, 1898.

It has long been a well-known theorem that every cubic drawn through eight given points passes through a ninth fixed point. Geometrical methods have been given for finding this ninth point by means of conics, and also by means of straight lines only.*

In this paper expressions will be found for the coordinates of the ninth point in terms of the other eight. It will appear that (I.) the coordinates of the ninth point are functions of the eighth degree in the coordinates of each of the other points; (II.) that when seven of the points are given, if the eighth point lies upon a straight line, the locus of the ninth point is an octavic curve having a triple point at each of the seven given points: and that, if the straight line on which the eighth point lies passes through one or two of the seven points, the locus of the ninth point degenerates and contains respectively one or two nodal cubics, having nodes at those points through which the straight line passes; and also, if the eighth point lies upon a conic, the locus of the ninth point is a curve of the sixteenth degree having a sextuple point at each of the seven given points: and that, if the conic on which the eighth point lies passes through one, two, three, four, or five of the seven given points, the locus of the ninth point degenerates and contains respectively one, two, three, four, or five nodal cubics having nodes at the points through which the conic passes.

The general equation of a cubic may be written

$$\phi(x, y, z) \equiv ax^3+by^3+cz^3+fx^2y+gy^2z+hz^2x+lx^2z+my^2x+nz^2y+pxyz = 0.$$

If this passes through nine points (x_r, y_r, z_r), then

$$\phi(x_r, y_r, z_r) = 0,$$

for $$r = 1, 2, \ldots, 9.$$

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* Weddle, *Cambridge and Dublin Mathematical Journal*, Vol. VI., p. 83, and Hart, *Cambridge and Dublin Mathematical Journal*, Vol. VI., p. 181; and Cayley, *Quarterly Journal*, Vol. V., p. 222, *Collected Works*, Vol. IV., p. 495.

When these nine equations are independent, they determine the ratios of the constant coefficients in ϕ; but, if

$$x_q^\rho y_q^\sigma z_q^\tau = \sum_{i=1}^{8} \lambda_i x_i^\rho y_i^\sigma z_i^\tau$$

for all combinations of ρ, σ, τ such that

$$\rho+\sigma+\tau = 3,$$

then the nine equations $\phi(\quad) = 0$ are equivalent to only eight, and any cubic curve through eight of the points will pass through the ninth. In this case we see that, by elimination of the constants λ from nine of the last ten equations, we obtain ten relations between the coordinates of the nine points, each relation being symmetrical with respect to the points, and only two of them being independent.

If we represent the determinants formed by leaving out the first, second, ..., tenth columns in the array

$$\| x_i^3,\ y_i^3,\ z_i^3,\ x_i^2 y_i,\ y_i^2 z_i,\ z_i^2 x_i,\ x_i^2 z_i,\ y_i^2 x_i,\ z_i^2 y_i,\ x_i y_i z_i \|,$$

for $i = 0, 1, 2, \ldots, 8$, where $x_0, y_0, z_0 = x, y, z$, by A', B', C', F', G', H', L', M', N', P', then each of the equations $A' = 0$, $B' = 0$, ... represents the fact that the point 9 lies on a cubic curve passing through the points 1, 2, ..., 8.

Every pair of these ten cubics intersect in the eight given points and a fixed ninth point. We shall proceed to find the coordinates of this ninth point in terms of the coordinates of the eight given points.

If these equations were written in the form

$$A' \equiv A_b y^3 + A_c z^3 + A_f x^2y + A_g y^2z + A_h z^2x + A_l x^2z + A_m y^2x + A_n z^2y + A_p xyz = 0,$$

$$B' \equiv B_a x^3 + B_c z^3 + B_f x^2y + B_g y^2z + B_h z^2x + B_l x^2z + B_m y^2x + B_n z^2y + B_p xyz = 0,$$

$$C' \equiv C_a x^3 + C_b y^3 \quad C_f x^2y + C_g y^2z + C_h z^2x + C\, x^2z + C_m y^2x + C_n z^2y + C_p xyz = 0,$$

$$F' \equiv F_a x^3 + F_b y^3 + F_c z^3 + F_g y^2z + F_h z^2x + F_l x^2z + F_m y^2x + F_n z^2y + F_p xyz = 0,$$

$$G' \equiv G_a x^3 + G_b y^3 + G_c z^3 + G_f x^2y + G_h z^2x + G_l x^2z + G_m y^2x + G_n z^2y + G_p xyz = 0,$$

$$H' \equiv H_a x^3 + H_b y^3 + H_c z^3 + H_f x^2y + H_g y^2z + H_l x^2z + H_m y^2x + H_n z^2y + H_p xyz = 0,$$

$$L' \equiv L_a x^3 + L_b y^3 + L_c z^3 + L_f x^2y + L_g y^2z + L_h z^2x + L_m y^2x + L_n z^2y + L_p xyz = 0,$$

$$M' \equiv M_a x^3 + M_b y^3 + M_c z^3 + M_f x^2y + M_g y^2z + M_h z^2x + M_l x^2z + M_n z^2y + M_p xyz = 0,$$

$$N' \equiv N_a x^3 + N_b y^3 + N_c z^3 + N_f x^2y + N_g y^2z + N_h z^2x + N_l x^2z + N_m y^2x + N_p xyz = 0,$$

$$P' \equiv P_a x^3 + P_b y^3 + P_c z^3 + P_f x^2y + P_g y^2z + P_h z^2x + P_l x^2z + P_m y^2x + P_n z^2y = 0,$$

then $\quad A_b = B_a, \quad A_c = -C_a, \quad A_f = F_a, \quad A_g = -G_a, \ldots,$

and, since from any pair of these equations we can obtain each of the other eight equations, we deduce relations among the coefficients of the form

$$\frac{A_aB_f - A_fB_a}{F_a} \equiv -B_a$$

$$= \frac{A_bB_f - A_fB_b}{F_b} = \frac{A_cB_f - A_fB_c}{F_c} = \text{\&c.}$$

Consider two cubics $\phi(x, y, z) = 0$

and $\Phi(x, y, z) \equiv Ax^3 + By^3 + Cz^3 + Fx^2y + Gy^2z + Hz^2x$

$$+ Lx^2z + My^2x + Nz^2y + Pxyz = 0;$$

we have, at once, for their nine points of intersection

$$\frac{x_1x_2 \dots x_9}{U} = \frac{y_1y_2 \dots y_9}{V} = \frac{z_1z_2 \dots z_9}{W},$$

where U denotes the eliminant of

$$by^3 + cz^3 + gy^2z + nz^2y \quad \text{and} \quad By^3 + Cz^3 + Gy^2z + Nz^2y,$$

V denotes the eliminant of

$$ax^3 + cz^3 + hz^2x + lx^2z \quad \text{and} \quad Ax^3 + Cz^3 + Hz^2x + Lx^2z,$$

and W denotes the eliminant of

$$ax^3 + by^3 + fx^2y + my^2x \quad \text{and} \quad Ax^3 + By^3 + Fx^2y + My^2x;$$

for, if x be eliminated between $\phi = 0$ and $\Phi = 0$, the result is

$$Wy^9 + \dots + Vz^9 = 0,$$

and likewise for the elimination of y and of z.

Now, suppose that the two cubics $\phi = 0$, $\Phi = 0$ are identical with the two cubics $A' = 0$, $B' = 0$. Then, after some reductions in the determinantal forms of the eliminants, we find

$$U = A_b^3 \{C_nB_gN_g + C_g^2B_g - C_bC_gB_n + C_nB_n^2 - 2C_bC_nB_g + C_b^2B_c\},$$

$$V = A_b^3 \{C_hL_hA_l + \dots \qquad \},$$

$$W = A_b^3 \{A_fM_fB_n + \dots \qquad \}.$$

As the relation $x_1 = 0$ does not necessitate either of the relations

$$y_1y_2y_3y_4y_5y_6y_7y_8y_9 = 0, \quad z_1z_2z_3z_4z_5z_6z_7z_8z_9 = 0,$$

it follows that x_1 is a factor of the expression

$$C_nB_gN_g + \text{\&c.}$$

Similarly, x_2, x_3, x_4, x_5, x_6, x_7, x_8 must be factors of the expression.

Similarly the products $y_1 y_2 y_3 y_4 y_5 y_6 y_7 y_8$ and $z_1 z_2 z_3 z_4 z_5 z_6 z_7 z_8$ must be factors of the expressions

$$C_h L_h A_l + \&c.,$$
$$A_f M_f B_m + \&c.$$

Hence we may write $$\frac{x_9}{O_a} = \frac{y_9}{O_b} = \frac{z_9}{O_c},$$

where O_a, O_b, O_c are homogeneous functions of the eighth degree in the coordinates of each of the points 1, 2, 3, 4, 5, 6, 7, 8.

We might infer from these equations that, if seven of the points were given, and an eighth lay upon a given straight line, then the ninth must lie on an octavic curve having a triple point at each of the seven given points; but, for convenience, we will choose points 1, 2, 3 as coincident with the angular points A, B, C of the triangle of reference.

In this case the equation of the cubic takes the form

$$fx^2y + gy^2z + hz^2x + lx^2z + my^2x + nz^2y + pxyz = 0,$$

and, since $$x_1 = y_2 = z_3 = 1$$

and $$y_1 = z_1 = z_2 = x_2 = x_3 = y_3 = 0,$$

each of the factors B_g, C_n, C_h, A_l, &c., is of the first degree in the coordinates which vanish, but N_g, L_h, M_f do not vanish with these coordinates; hence, if we denote by f, g, h, l, m, n, p the determinants formed by leaving out the first, second, ..., seventh columns in the scheme

$$\| x_i^2 y_i,\ y_i^2 z_i,\ z_i^2 x_i,\ x_i^2 z_i,\ y^2 x_i,\ z_i^2 y_i,\ x_i y_i z_i \|,$$

for $i = 0, 4, 5, 6, 7, 8$, where

$$x_0, y_0, z_0 = x, y, z,$$

and if further we denote the coefficients in the equations $f = 0$, $g = 0$, ..., $p = 0$ by a notation similar to one used already and explained by the following scheme:—

$$\begin{aligned}
f &\equiv \qquad\quad f_g\, y^2z + f_h\, z^2x + f_l\, x^2z + f_m\, y^2x + f_n\, z^2y + f_p\, xyz = 0,\\
g &\equiv g_f\, x^2y \qquad\quad + g_h\, z^2x + g_l\, x^2z + g_m\, y^2x + g_n\, z^2y + g_p\, xyz = 0,\\
h &\equiv h_f\, x^2y + h_g\, y^2z \qquad\quad + h_l\, x^2z + h_m\, y^2x + h_n\, z^2y + h_p\, xyz = 0,\\
l &\equiv l_f\, x^2y + l_g\, y^2z + l_h\, z^2x \qquad\quad + l_m\, y^2x + l_n\, z^2y + l_p\, xyz = 0,\\
m &\equiv m_f x^2y + m_g y^2z + m_h z^2x + m_l x^2z \qquad\quad + m_n z^2y + m_p xyz = 0,\\
n &\equiv n_f\, x^2y + n_g\, y^2z + n_h\, z^2x + n_l\, x^2z + n_m\, y^2x \qquad\quad + n_p\, xyz = 0,\\
p &\equiv p_f\, x^2y + p_g\, y^2z + p_h\, z^2x + p_l\, x^2z + p_m\, y^2x + p_n\, z^2y \qquad\quad = 0,
\end{aligned}$$

we may write the equations in the form

$$\frac{x_1x_2x_3x_4x_5x_6x_7x_8x_9}{x_2m_gx_3h_nn_g+\&c.} = \frac{y_1y_2y_3y_4y_5y_6y_7y_8y_9}{y_3n_hy_1f_il_h+\&c.} = \frac{z_1z_2z_3z_4z_5z_6z_7z_8z_9}{z_1l_fz_2g_mm_f+\&c.}.$$

Hence
$$\frac{x_4\ldots x_9}{m_gh_nn_g} = \frac{y_4\ldots y_9}{n_hf_il_h} = \frac{z_4\ldots z_9}{l_fg_mm_f},$$

or
$$\frac{x_4\ldots x_9}{\dfrac{n_g}{l_f}} = \frac{y_4\ldots y_9}{\dfrac{l_h}{m_g}} = \frac{z_4\ldots z_9}{\dfrac{m_f}{n_h}}.$$

Now,
$$n_g = x_4x_5x_6x_7x_8S_1,$$

where S_1 is the determinant

$$|\ x_ry_r,\ x_rz_r,\ y_r^2,\ z_r^2,\ y_rz_r\ |,$$

for $r = 4, 5, 6, 7, 8$.

Similarly,
$$l_h = y_4y_5y_6y_7y_8S_2,$$

and
$$m_f = z_4z_5z_6z_7z_8S_3,$$

with corresponding values for S_2 and S_3. Therefore we may write

$$\frac{x_9}{\dfrac{S_1}{l_f}} = \frac{y_9}{\dfrac{S_2}{m_g}} = \frac{z_9}{\dfrac{S_3}{n_h}}.$$

If the points 4, 5, 6, 7 be assumed given, the equation $S_1 = 0$ implies that the point 8 must lie on a conic through the points A, 4, 5, 6, 7. Similarly, $S_2 = 0$ is the conic through B, 4, 5, 6, 7; $S_3 = 0$ is the conic through C, 4, 5, 6, 7.

Similarly, the equation

$$l_f \equiv |\ y_r^2z_r,\ y_r^2x_r,\ z_r^2x_r,\ z_r^2y_r,\ x_ry_rz_r\ | = 0,$$

where $r = 4, 5, 6, 7, 8$, implies that the point 8 lies on a cubic which has a double point at A, and passes through the points B, C, 4, 5, 6, 7.

Similarly, $m_g = 0$ is a cubic which has a double point at B, and passes through the points A, C, 4, 5, 6, 7; and $n_h = 0$ is a cubic which has a double point at C and passes through the points A, B, 4, 5, 6, 7.

We conclude that, if the point 9 lies on a given straight line

$$\lambda x + \mu y + \nu z = 0,$$

then the point 8 must lie on an octavic curve

$$\lambda \frac{S_1}{l_f} + \mu \frac{S_2}{m_g} + \nu \frac{S_3}{n_h} = 0,$$

having a triple point at each of the points A, B, C, 4, 5, 6, 7.

Every such octavic contains only two arbitrary constants corresponding to the two constants determining the straight line on which the point 9 lies.

Two such octavics intersect nine times at each of the points A, B, C, 4, 5, 6, 7; and therefore have but one other intersection.

If the point 9 lies on a straight line through the point 1,

$$\mu y + \nu z = 0,$$

the octavic locus of the point 8 degenerates into the cubic $l_f = 0$ and a quintic curve

$$u \frac{S_2}{l_f} + \nu \frac{S_3}{n_h} = 0,$$

passing through A and having double points at B, C, 4, 5, 6, 7.

Again, if the point 9 lies on the straight line BC joining the points 2, 3, then the octavic degenerates into the two cubics $m_g = 0$, $n_h = 0$ and the conic $S_1 = 0$.

If the point 9 were upon the straight line joining two of the other points, 6, 7, then the octavic locus of the point 8 would degenerate into two cubics $C_6 = 0$, $C_7 = 0$ and a conic $S' = 0$. $C_6 = 0$ has a double point at 6 and passes through A, B, C, 4, 5, 7, and $C_7 = 0$ has a double point at 7 and passes through A, B, C, 4, 5, 6, and $S' = 0$ passes through A, B, C, 4, 5. Hence we see that two of the cubics, $m_g = 0$ and $C_6 = 0$, intersect twice at each of the points B, 6 and once at each of the points A, C, 4, 5, 7; and therefore at no other point.

Similarly, one of the cubics and one of the conics, say $n_h = 0$ and $S' = 0$, intersect twice at C and once at each of the points A, B, 4, 5, 6; and therefore at no other point.

It appears therefore that in this case the point 8 is the fourth intersection of the two conics $S_1 = 0$ and $S' = 0$, both being conics circumscribing the triangle ABC—a result which might have been anticipated.

If the point 9 lies upon a conic

$$\alpha x^2 + \beta y^2 + \gamma z^2 + \lambda yz + \mu zx + \nu xy = 0,$$

the point 8 must lie on a curve of the sixteenth degree

$$\alpha\frac{S_1^2}{l_f^2}+\beta\frac{S_2^2}{m_g^2}+\gamma\frac{S_3^2}{n_h^2}+\lambda\frac{S_2S_3}{m_gn_h}+\mu\frac{S_3S_1}{n_hl_f}+\nu\frac{S_1S_2}{l_fm_g}=0,$$

having sextuple points at the points A, B, C, 4, 5, 6, 7. Two such sedecimics will intersect thirty-six times at each of the seven given points, and will have four other intersections corresponding to the four intersections of the two conics from which the sedecimics were obtained.

If the point 9 lies upon a conic

$$\beta y^2+\gamma z^2+\lambda yz+\mu zx+\nu xy=0,$$

passing through A, the sedecimic locus of 8 degenerates into the cubic $l_f=0$ and a tredecimic

$$\beta\frac{S_2^2}{m_g^2}+\gamma\frac{S_3^2}{n_h^2}+\lambda\frac{S_2S_3}{m_gn_h}+\mu\frac{S_3S_1}{n_hl_f}+\nu\frac{S_1S_2}{l_fm_g}=0,$$

having a quadruple point at A and quintuple points at B, C, 4, 5, 6, 7.

If the point 9 lies on a conic

$$\gamma z^2+\lambda yz+\mu zx+\nu xy=0,$$

passing through two of the points A, B, the locus of the point 8 degenerates into the two cubics $l_f=0$ and $m_g=0$ and a decimic curve

$$\gamma\frac{S_3^2}{n_h^2}+\lambda\frac{S_2S_3}{m_gn_h}+\mu\frac{S_3S_1}{n_hl_f}+\nu\frac{S_1S_2}{l_fm_g}=0,$$

having triple points at A, B and quadruple points at C, 4, 5, 6, 7.

If the point 9 lies on a conic

$$\lambda yz+\mu zx+\nu xy=0,$$

passing through three of the points A, B, C, then the locus of the point 8 degenerates into the cubics $l_f=0$, $m_g=0$, $n_h=0$ and a septimic

$$\lambda\frac{S_2S_3}{m_gn_h}+\mu\frac{S_3S_1}{n_hl_f}+\nu\frac{S_1S_2}{l_fm_g}=0,$$

having double points at each of the points A, B, C and triple points at each of the points 4, 5, 6, 7.

It has been seen three times in succession that, if the conic locus of the point 9 pass through one of the given points, the locus of the point 8 degenerated and contained a cubic curve. It can be shown that this degeneration still continues when the conic locus of 9 passes through other of the fixed points.

If the conic locus of the point 9 pass through the point 4, then

$$\frac{\lambda}{x_4}+\frac{\mu}{y_4}+\frac{\nu}{z_4}=0,$$

and the septimic may be written

$$\frac{\lambda}{z_4}l_f S_2 S_3+\frac{\mu}{z_4}m_g S_3 S_1-\left(\frac{\lambda}{x_4}+\frac{\mu}{y_4}\right)n_h S_1 S_2=0,$$

or
$$\lambda S_2\left(\frac{l_f}{z_4}S_3-\frac{n_h}{x_4}S_1\right)+\mu S_1\left(\frac{m_g}{z_4}-\frac{n_n}{y_4}S_2\right)=0.$$

Now, after reductions, we find

$$\frac{S_3}{z_4}l_f-\frac{S_1}{x_4}n_p=\frac{Yy_4}{P_5\Delta_5}\Sigma_4,$$

$$\frac{S_3}{z_4}m_g-\frac{S_2}{y^4}n_h=\frac{Xx_4}{P_5\Delta_5}\Sigma_4,$$

where P_5 is the determinant $(y_5z_5,\ z_6x_6,\ x_7y_7)$,

Δ_5 ,, ,, (x_5, y_6, z_7),

X ,, ,, $(x^2,\ y_5z_5,\ z_6x_6,\ x_7y_7)$,

Y ,, ,, $(y^2,\ y_5z_5,\ z_6x_6,\ x_7y_7)$,

and Z ,, ,, $(z^2,\ y_5z_5,\ z_6x_6,\ x_7y_7)$.

Hence the septimic may be written

$$\lambda S_2\frac{Yy_4}{P_5\Delta_5}\Sigma_4+\mu S_1\frac{Xx_4}{P_5\Delta_5}\Sigma_4=0,$$

and it therefore degenerates into the cubic $\Sigma_4=0$ and the quartic

$$\lambda S_2 Yy_4+\mu S_1 Xx_4=0.$$

Since $X=0$, $Y=0$, $Z=0$ represent the conics $BC567$, $CA567$, $AB567$ respectively, the cubic $\Sigma_4=0$ passes through the six points A, B, C, 5, 6, 7 and has a double point at the point 4, and the quartic passes through the points A, B, C, 4 and has double points at 5, 6, 7.

If the conic locus of the point 9 pass also through the point 5, then

$$\frac{\lambda}{x_5}+\frac{\mu}{y_5}+\frac{\nu}{z_5}=0,$$

and therefore $\lambda : \mu = x_4x_5\,(y_4z_5-y_5z_4) : y_4y_5\,(z_4x_5-z_5x_4)$.

Using these values, we find that the quartic $\lambda S_2 Y y_4+\mu S_1 X x_4=0$ degenerates into a line $D=(x,\,y_6,\,z_7)=0$ and into a cubic $\Sigma_5=0$, where we may write

$$\Sigma_5 = D_5 PL + P_5 DQ,$$

P denoting the determinant $(yz,\,z_6x_6,\,x_7y_7)$, Q denoting

$$x_5yz\begin{vmatrix} y_6, & z_6 \\ y_2, & z_7\end{vmatrix}\begin{vmatrix} P_4, & P_5 \\ x_4D_4, & x_5D_5\end{vmatrix} + \text{corresponding terms in } zx,\ xy,$$

and $L=0$ being a tangent to $Q=0$ at the point 5. Now this cubic Σ_5 passes through all the points A, B, C, 4, 5, 6, 7; manifestly it has a double point at the point 5, because $L=0$ is a tangent to the conic $Q=0$.

This proves that, in the case when the conic locus of 9 passes through the points A, B, C, 4, 5, the locus of the point 8 degenerates into the five cubics $l_f=0$, $m_g=0$, $n_h=0$, $\Sigma_4=0$, $\Sigma_5=0$, having double points at A, B, C, 4, 5 respectively, and the straight line $D=0$ joining the remaining pair of the seven fixed points 6, 7.

Thursday, March 10th, 1898.

Professor E. B. ELLIOTT, F.R.S., President, in the Chair.

Present, seventeen members and three visitors.

Mr. A. N. Whitehead was admitted into the Society, and then read a paper entitled "The Geodesic Geometry of Surfaces in non-Euclidean Space." Messrs. Berry, Macaulay, and Burnside joined in a discussion on the paper.

Prof. Burnside read a paper on "Linear Homogeneous Continuous Groups whose Operations are Permutable."

Mr. T. I. Dewar, in the absence of Prof. Greenhill, exhibited, with the aid of stereoscopes, some stereoscopic diagrams of Pseudo-Elliptic Catenaries and Geodesics. A unanimous vote of thanks was passed to Mr. Dewar.

Lt.-Col. Cunningham gave a short supplement to his paper "On Aurifeuillians."

The President briefly communicated a paper by Mr. W. F. Sheppard, "On the Calculation of the most Probable Values of Frequency-Constants, for Data arranged according to Equidistant Division of a Scale";* and then (Lt.-Col. Cunningham in the Chair) read a paper "On the Transformation of Linear Partial Differential Operators by extended Linear Continuous Groups."

The following presents were made to the Library:—

"Queen's College, Galway, Calendar for 1897-98," 8vo; Dublin, 1898.

"Proceedings of the Royal Society," Vol. LXII., Nos. 384-5.

"Year-Book of the Royal Society," 1897-8.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXII., St. 2; Leipzig, 1898.

"Nyt Tidsskrift for Matematik," Aargang 8, A, Nr. 6, 1898; B, Nr. 4, 1897; Copenhagen.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," Jahrgang 42, 1897, Heft 3-4.

"Zeitschrift für Mathematik und Physik," Bd. XLIII., Heft 1; Leipzig, 1898.

"The Nautical Almanac for 1901," 8vo; London, 1898. From the Lords Commissioners of the Admiralty.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. IV., Fasc. 2; Napoli, 1898.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. IV., No. 5; New York, 1898.

Lorenz, L.—"Œuvres Scientifiques," revues et annotées, par H. Valentiner, Tome I., Fasc. 2, 8vo; Copenhague, 1898.

"Jahrbuch über die Fortschritte der Mathematik," Bd. XXVI., Heft 3; Berlin, 1898.

"Bulletin des Sciences Mathématiques," Tome XXII., Fév., 1898; Paris.

Bruno, Peter.—"Beobachtungen am sechszölligen Repsoldschen Heliometer der Leipziger Sternwarte," Abh. II., No. 3, 8vo; Leipzig, 1898.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. XLII., Pt. 1; 1897-98.

"Sitzungsberichte der K. Preuss. Akademie der Wissenschaften zu Berlin," Nos. XL.-LIII.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. VII., Fasc. 2, 3; Roma, 1898.

Ball, Sir R.—"The Twelfth and Concluding Memoir on the Theory of Screws," 4to; Dublin, 1898.

"Annales de la Faculté des Sciences de Toulouse," Tome XI.; Paris, 1897.

"Educational Times," March, 1898.

"Annals of Mathematics," Vol. XI., No. 6; Virginia, 1898.

Cayley, A.—"The Collected Mathematical Papers, Supplementary Volume containing Titles of Papers and Index," 4to; Cambridge, 1898.

"Indian Engineering," Vol. XXIII., Nos. 4-7, Jan. 22-Feb. 5, 1898.

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