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H.A. Webb M.A. & John R. Airey M.A. D.Sc.

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VIII. *The Practical Importance of the Confluent Hypergeometric Function.* *By* H. A. WEBB, *M.A.*, *and* JOHN R. AIREY, *M.A.*, *D.Sc.**

[Plate VI.]

§ 1. *Introduction.*

IT is well-known that many physical and engineering problems depend for solution on differential equations of the type

$$\frac{d^2y}{dx^2}+f(x)\cdot\frac{dy}{dx}+\phi(x)\cdot y=0, \quad . \quad . \quad . \quad (1)$$

where $f(x)$ and $\phi(x)$ are given functions of x. For example, the investigation of the periods of lateral vibration of a flexible non-uniform rope or chain †, or the periods of vibration of a circular disk ‡, leads to an equation of this type. Again, the whirling speed of a non-cylindrical shaft, or the period of lateral vibration of a non-cylindrical bar, such as an air-screw blade, can be found, with two-figure accuracy, by the solution of such an equation§; and in fact many vibration problems in various branches of physics lead to such equations. To take another illustration, the crippling end-load of a tapered aeroplane strut, whatever law of taper is adopted, could be found if we could solve equation (1); other problems of elastic instability lead to equations of this type, and may be brought into prominence in aeronautics by the urgency of saving weight.

In structures, such as aeroplanes or bridges, the liability to secondary failure (*i. e.* elastic instability) must be foreseen and estimated, as well as the liability to primary, or stress, failure. In running machinery it is important that the period of free vibrations shall be well above, or below, the given running speed, to avoid resonance; in instruments for producing sound, on the other hand, it is required that the period of free vibrations shall have a given value, to secure resonance.

In any of these cases, the problem presents itself to the designer somewhat as follows. The main outlines of the

* Communicated by the Authors.

† Airey, "The Oscillations of Chains," Phil. Mag. June 1911.

‡ Airey, "The Vibrations of Circular Plates," Proc. Phys. Soc. April 1911.

§ Webb, "The Whirling of Shafts," Engineering, November 1917.

design, including probably the over-all dimensions, are already settled by various considerations with which we are not now concerned. But we are allowed some latitude in detail design, which we are to use to avoid elastic failure, or to avoid, or to secure, resonance, as the case may be. We want therefore to be able to calculate, *roughly but quickly*, the effect on the crippling load, or the period, of various possible alterations. We want in fact to make several trials—the more the better—and choose the one we like best. Finally, when the design is complete, we wish to check it carefully by a more accurate calculation.

The functions $f(x)$ and $\phi(x)$ in equation (1) are to be considered, for a tentative design, to be defined by their graphs, which must be represented, for the range of values of x required, by empirical formulæ, the closeness of the representation giving some idea of the accuracy to be expected in the solution. These empirical formulæ should be of the simplest type, *e. g.* polynomials, or the ratios of linear or quadratic functions of x, otherwise time is wasted in constructing them. What is required therefore is a list of suitable equations of the type (1) that are soluble in terms of tabulated functions. The two important characteristics are that $f(x)$ and $\phi(x)$ should be of a simple type, and that they should contain several arbitrary constants; we can then hope to make them fit our graphs fairly well without much trouble.

When $f(x)$ and $\phi(x)$ are constants, the solution in terms of circular and exponential functions is well-known. A useful list of equations soluble by Bessel functions, with appropriate tables, has been given by Jahnke and Emde*. It is the object of this paper to show the value, from this point of view, of the confluent hypergeometric function, tables and graphs of which are given in § 4. For quick work graphs are more convenient than tables. A list of differential equations likely to be useful to designers, and soluble by means of these tables and graphs, is given in § 3. Some properties of the functions that were used in constructing the tables, and would be useful in extending them, are given in § 2.

It may perhaps be argued that few engineers have the mathematical ability for such scientific methods of design. But it should be remembered that many engineers acquire at their technical college or university a high degree of mathematical skill; and if they lose it afterwards, it is because

* *Funktionentafeln*, Teubner, 1909.

they find mathematical works of reference rather indigestible, and gradually cease to consult them. For example, an excellent summary, from a purely mathematical point of view, of the properties of the function we are going to consider is given in Whittaker and Watson's 'Modern Analysis'*; but it would be hard reading for engineers.

Or if it is objected that the engineer can hardly be expected to be familiar with the function theory of linear differential equations and may get into trouble over singularities, he might reply, if sufficiently well read, that the equation can't have singularities in the range of x considered, unless $f(x)$ or $\phi(x)$, or both, become infinite, and he would notice that from the graphs. Or he might say that he is not looking for a rule to which there are *no* exceptions. He wants a rule that *generally* works *quickly*, and he is prepared to risk an occasional failure, because he intends to refer the finished design for a final check to an expert mathematician. Divergent series have often been used by physicists in much the same spirit, and with few, if any, failures. Finally, many expert mathematicians have come into contact with engineering work recently under war conditions; they may have opportunity and inclination to assist in design on the lines we have indicated.

§ 2. *Properties of the confluent hypergeometric function.*

We define the function $M(\alpha, \gamma, x)$ as follows:—

$$M(\alpha, \gamma, x) = 1 + \frac{\alpha}{1 . \gamma} . x + \frac{\alpha(\alpha+1)}{1 . 2 . \gamma(\gamma+1)} . x^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{1.2.3 . \gamma(\gamma+1)(\gamma+2)} . x^3 + \ldots \text{ to infinity.} \quad . \quad . \quad . \quad (2)$$

The series is absolutely and uniformly convergent for all values of α, γ, and x, real or complex, except only when γ is zero or a negative integer; this case is supposed to be excluded.

The function $M(\alpha, \gamma, x)$ has been discussed under various notations by several writers †. The following is a list of such properties of the function as are of use for our purpose; most of them are easily verified from the definition (2).

* Second edition, 1915, Chapter XVI.

† For a list of references see Whittaker & Watson, *loc. cit.*

I. $y = M(\alpha, \gamma, x)$

satisfies the differential equation

$$x\frac{d^2y}{dx^2} + (\gamma - x)\frac{dy}{dx} - \alpha y = 0. \quad . \quad . \quad . \quad . \quad (3)$$

II. The complete solution of the differential equation (3) is

$$y = A \cdot M(\alpha, \gamma, x) + B \cdot x^{1-\gamma} \cdot M(\alpha - \gamma + 1, 2 - \gamma, x), \quad (4)$$

which we shall write for brevity

$$y = \overline{M}(\alpha, \gamma, x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where A and B are arbitrary constants of integration; *except only when γ is a positive integer*, in which case* the coefficient of B is either infinite or identical with the coefficient of A. In this case the complete solution of (3) may be written

$$y = [A + C\log x] \cdot M(\alpha, \gamma, x)$$
$$+ C\left[\frac{\alpha x}{\gamma}\left(\frac{1}{\alpha} - \frac{1}{\gamma} - 1\right) + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)} \cdot \frac{x^2}{1.2}\left(\frac{1}{\alpha} + \frac{1}{\alpha+1} - \frac{1}{\gamma} - \frac{1}{\gamma+1} - 1 - \frac{1}{2}\right)\right.$$
$$+ \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)} \cdot \frac{x^3}{1.2.3}\left(\frac{1}{\alpha} + \frac{1}{\alpha+1} + \frac{1}{\alpha+2} - \frac{1}{\gamma} - \frac{1}{\gamma+1} - \frac{1}{\gamma+2}\right.$$
$$\left. - 1 - \frac{1}{2} - \frac{1}{3}\right)$$
$$\left. + \ldots \text{ to infinity}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where A and C are arbitrary constants of integration.

III. $M(\alpha, \gamma, x) = e^x \cdot M(\gamma - \alpha, \gamma, -x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$

$x^{1-\gamma}M(\alpha - \gamma + 1, 2 - \gamma, x) = e^x \cdot x^{1-\gamma} \cdot M(1 - \alpha, 2 - \gamma, -x). \quad (8)$

From (7) and (8) it follows that tables will not be required for *negative* values of x, if the tables cover wide enough ranges of α and γ.

IV. The asymptotic expansion of $M(\alpha, \gamma, x)$ for large values of x is

* The situation is similar to that which arises with Bessel's equation when n is a positive integer, and a new function is required for the second solution.

$$
\begin{aligned}
M(\alpha, \gamma, x) &= \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)} \cdot (-x)^{-\alpha} \cdot \left\{ 1 - \frac{\alpha(\alpha-\gamma+1)}{1} \cdot \frac{1}{x} \right. \\
&\qquad \left. + \frac{\alpha(\alpha+1)(\alpha-\gamma+1)(\alpha-\gamma+2)}{1 \,.\, 2} \cdot \frac{1}{x^2} - \ldots \right\} \\
&+ \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \cdot e^x \cdot x^{\alpha-\gamma} \left\{ 1 + \frac{(1-\alpha)(\gamma-\alpha)}{1} \cdot \frac{1}{x} \right. \\
&\qquad \left. + \frac{(1-\alpha)(2-\alpha)(\gamma-\alpha)(\gamma-\alpha+1)}{1 \,.\, 2} \cdot \frac{1}{x^2} + \ldots \right\} . \quad (9)
\end{aligned}
$$

Both these series diverge for all values of x, but they have the property that the error involved in taking the sum to n terms to be the value of the series, is less than the nth term.

V. $$\frac{d}{dx} \cdot M(\alpha, \gamma, x) = \frac{\alpha}{\gamma} \cdot M(\alpha+1, \gamma+1, x). \quad . \quad . \quad . \quad (10)$$

$$(1-\alpha) \cdot \int_{x=0} M(\alpha, \gamma, x) \,.\, dx = (1-\gamma) \,.\, M(\alpha-1, \gamma-1, x) + (\gamma-1). \quad . \quad . \quad . \quad (11)$$

Hence the function can easily be differentiated or integrated.

VI. The following difference relations would be useful for extending the tables :—

$$
\left.
\begin{aligned}
&\frac{x}{\gamma} \cdot M(\alpha+1, \gamma+1, x) = M(\alpha+1, \gamma, x) - M(\alpha, \gamma, x), \\
&\alpha \,.\, M(\alpha+1, \gamma+1, x) = (\alpha-\gamma) \,.\, M(\alpha, \gamma+1, x) + \gamma \,.\, M(\alpha, \gamma, x), \\
&(\alpha+x) \,.\, M(\alpha+1, \gamma+1, x) = (\alpha-\gamma) \,.\, M(\alpha, \gamma+1, x) \\
&\qquad\qquad + \gamma \,.\, M(\alpha+1, \gamma, x), \\
&\alpha\gamma \,.\, M(\alpha+1, \gamma, x) = \gamma(\alpha+x) \,.\, M(\alpha, \gamma, x) \\
&\qquad\qquad - x(\gamma-\alpha) \,.\, M(\alpha, \gamma+1, x), \\
&\alpha \,.\, M(\alpha+1, \gamma, x) = (x+2\alpha-\gamma) \,.\, M(\alpha, \gamma, x) \\
&\qquad\qquad + (\gamma-\alpha) \,.\, M(\alpha-1, \gamma, x), \\
&\frac{\gamma-\alpha}{\gamma} \,.\, x \,.\, M(\alpha, \gamma+1, x) = (x+\gamma-1) \,.\, M(\alpha, \gamma, x) \\
&\qquad\qquad + (1-\gamma) \,.\, M(\alpha, \gamma-1, x).
\end{aligned}
\right\} (12)
$$

VII. If $\alpha=\frac{1}{2}\gamma$, $M(\alpha, \gamma, x)$ can be expressed in terms of a Bessel function. In fact

$$M(\tfrac{1}{2}\gamma, \gamma, x)=2^{\gamma-1} \cdot \Gamma\left(\frac{\gamma+1}{2}\right) \cdot e^{\frac{x}{2}} \cdot x^{\frac{1-\gamma}{2}} \cdot I_{\frac{\gamma-1}{2}}(\tfrac{1}{2}x) \quad . \quad (13)$$

VIII. The *Error Function* $=\phi(x)$

$$=\frac{2}{\sqrt{\pi}} \cdot \int_0^x e^{-x^2} \cdot dx$$

$$=\frac{2x}{\sqrt{\pi}} \cdot e^{-x^2} \cdot M(1, \tfrac{3}{2}, x^2). \quad . \quad . \quad . \quad . \quad (14)$$

The *Incomplete* γ *Function* $=\gamma(n, x)$

$$=\int_0^x e^{-t} \cdot t^{n-1} \cdot dt$$

$$=\frac{1}{n} \cdot e^{-x} \cdot x^n \cdot M(1, n+1, x). \quad . \quad . \quad . \quad (15)$$

Sonine's Polynomial $=T_m^{\ n}(x)$

$$=\frac{(-1)^n}{m!\, n!} \cdot M(-n, m+1, x). \quad . \quad . \quad . \quad (16)$$

The *Function of the Parabolic Cylinder*

$$=D_n(x)$$

$$=(-1)^n \cdot e^{\frac{1}{4}x^2} \cdot \frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2})$$

$=$ (if n is *even*)

$$\frac{(-2)^{\frac{n}{2}}}{\sqrt{\pi}} \cdot \Gamma\left(\frac{n+1}{2}\right) \cdot e^{-\frac{1}{4}x^2} \cdot M\left(-\frac{n}{2}, \tfrac{1}{2}, \tfrac{1}{2}x^2\right), \quad (17\,a)$$

and $=$ (if n is *odd*)

$$\frac{(-2)^{\frac{n-1}{2}}}{\sqrt{\pi}} \cdot \Gamma\left(\frac{n}{2}\right) \cdot e^{-\frac{1}{4}x^2} \cdot nx \cdot M\left(\frac{1-n}{2}, \tfrac{3}{2}, \tfrac{1}{2}x^2\right) \quad . \quad . \quad (17\,b)$$

We will also, following *Jahnke und Emde* *, define $Z_p(x)$ as

$$AJ_p(x)+BN_p(x),$$

where A and B are arbitrary constants, and $J_p(x)$, $N_p(x)$ are Bessel functions, in the usual notation. So that

$$y=Z_p(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

is the complete solution of the differential equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\cdot\frac{dy}{dx}+\left(1-\frac{p^2}{x^2}\right)y=0. \quad . \quad . \quad . \quad (19)$$

Reference should be made to *Jahnke und Emde's* tables and graphs of these functions †, which are presented in a form convenient for engineers.

§ 3. *Soluble differential equations.*

The following differential equations are soluble by means of Bessel functions or M functions, a, b, c, α, γ, l, m, n, p, q, r, s, t being any numerical constants whatever.

(A) $$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+ly=0.$$

(B) $$\frac{d^2y}{dx^2}+\frac{q}{x}\cdot\frac{dy}{dx}+\frac{m}{x}\cdot y=0.$$

(C) $$\frac{d^2y}{dx^2}+\frac{q}{x}\cdot\frac{dy}{dx}+\left(l+\frac{n}{x^2}\right)y=0.$$

(D) $$\frac{d^2y}{dx^2}+\frac{q}{x}\cdot\frac{dy}{dx}+\frac{1}{x^2}(lx^{2r}+n)y=0.$$

(E) $$\frac{d^2y}{dx^2}+(px+q)\frac{dy}{dx}+(mx+n)y=0.$$

(F) $$\frac{d^2y}{dx^2}+\left(p+\frac{q}{x}\right)\frac{dy}{dx}+\left(l+\frac{m}{x}\right)y=0.$$

(G) $$\frac{d^2y}{dx^2}+(px+q)\frac{dy}{dx}+(lx^2+mx+n)y=0.$$

(H) $$\frac{d^2y}{dx^2}+\left(p+\frac{q}{x}\right)\frac{dy}{dx}+\left(l+\frac{m}{x}+\frac{n}{x^2}\right)y=0.$$

* *Loc. cit.* p. 165.
† *Loc. cit.* pp. 106–168.

(K) $$\frac{d^2y}{dx^2}+\left(px+q+\frac{s}{x}\right)\frac{dy}{dx}+\tfrac{1}{2}q\left(lx^2+px+n+\frac{s}{x}\right)y=0.$$

(L) $$\frac{d^2y}{dx^2}+\frac{1}{x}(px^r+q)\frac{dy}{dx}+\frac{1}{x^2}(lx^{2r}+mx^r+n)y=0.$$

The solutions are as follows:—

(A) $$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+(n^2+\tfrac{1}{4}p^2)y=0$$

$$y=e^{-\frac{1}{2}px}(\mathrm{A}\cos nx+\mathrm{B}\sin nx).$$

(B) $$\frac{d^2y}{dx^2}+\frac{p+1}{x}\frac{dy}{dx}+\frac{m}{x}\cdot y=0.$$

$$y=x^{-\frac{1}{2}p}\cdot \mathrm{Z}_p(2\sqrt{mx}).$$

(C) $$\frac{d^2y}{dx^2}+\frac{1-2\alpha}{x}\cdot\frac{dy}{dx}+\left(\gamma^2+\frac{\alpha^2-p^2}{x^2}\right)y=0$$

$$y=x^{\alpha}\,.\,\mathrm{Z}_p(\gamma x).$$

(D) $$\frac{d^2y}{dx^2}+\frac{1-2\alpha}{x}\cdot\frac{dy}{dx}+\frac{1}{x^2}\cdot(\gamma^2r^2x^{2r}+\alpha^2-p^2r^2)y=0$$

$$y=x^{\alpha}\,.\,\mathrm{Z}_p(\gamma x^r).$$

(E) $$\frac{d^2y}{dx^2}+2(p+qx)\frac{dy}{dx}+y\,[4\alpha q+p^2-q^2m^2+2qx(p+qm)]=0$$

$$y=e^{-(p+qm)x}\,.\,\overline{\mathrm{M}}[\alpha,\tfrac{1}{2},-q(x-m)^2].$$

(F) $$\frac{d^2y}{dx^2}+\left(2p+\frac{\gamma}{x}\right)\frac{dy}{dx}+y[p^2-t^2+\frac{1}{x}\cdot(\gamma p+\gamma t-2\alpha t)]=0$$

$$y=e^{-(p+t)x}\,.\,\overline{\mathrm{M}}(\alpha,\gamma,2t\,x).$$

(G) $$\frac{d^2y}{dx^2}+2(p+qx)\frac{dy}{dx}+y[q+c(1-4\alpha)+(p+qx)^2-c^2(x-m)^2]=0$$

$$y=e^{-px-\frac{1}{2}qx^2-\frac{1}{2}c(x-m)^2}\,.\,\overline{\mathrm{M}}[\alpha,\tfrac{1}{2},c(x-m)^2].$$

(H) $$\frac{d^2y}{dx^2}+\left(2p+\frac{q}{x}\right)\frac{dy}{dx}+y[p^2-t^2+\frac{1}{x}(pq+\gamma t-2\alpha t)+\frac{1}{4x^2}(\gamma-q)(2-q-\gamma)]=0$$

$$y=e^{-(p+t)x}\,.\,x^{\frac{\gamma-q}{2}}\,.\,\overline{\mathrm{M}}(\alpha,\gamma,2tx).$$

(K) $$\frac{d^2y}{dx^2}+\left[\frac{2\gamma-1}{x}+2a+2(b-c)x\right]\frac{dy}{dx}$$

$$+y\left[\frac{a(2\gamma-1)}{x}+(a^2+2b\gamma-4\alpha c)+2a(b-c)x+b(b-2c)x^2\right]=0$$

$$y=e^{-ax-\frac{1}{2}bx^2}\,.\,\overline{\mathrm{M}}(\alpha,\gamma,cx^2).$$

(L) $$\frac{d^2y}{dx^2}+\frac{1}{x}(2px^r+qr-r+1)\frac{dy}{dx}$$

$$+\frac{y}{x^2}\left[(p^2-t^2)x^{2r}+r(pq+\gamma t-2\alpha t)x^r\right.$$

$$\left.+\tfrac{1}{4}r^2(\gamma-q)(2-q-\gamma)\right]=0$$

$$y=e^{\frac{-(p+t)}{r}.x^r}\,.\,x^{\frac{r}{2}(\gamma-q)}\,.\,\overline{\mathrm{M}}\left(\alpha,\gamma,\frac{2t}{r}\,.\,x^r\right).$$

§ 4. *Tables and Graphs of* M(α,γ,x).

The following tables of M(α,γ,x) were calculated, for small values of x, from the series in ascending powers of this argument, and for large values, from the asymptotic expansions. When α and γ are positive integers, two or three values of M, for a particular value of x, are required to give the other results by means of suitable recurrence formulæ. The last two formulæ of (12) were employed to find further values of M along vertical columns and horizontal rows; the first four, to "turn the corners" and fill in the results in the rectangle of values thus obtained. When α is a negative integer, the M function is a polynomial which is easily evaluated. A similar procedure was adopted in the case of α equal to half an odd positive or negative integer, only two preliminary calculations of M being required to give the remaining 47 for each value of the argument x.

Four significant figures are given in the tables. The numbers, however, must be multiplied by the power of ten indicated by the figure after the comma. Thus,

$$\mathrm{M}(4,1,4)=2603;\ \mathrm{M}(3,2,10)=132200;$$

and $$\mathrm{M}(-\tfrac{1}{2},4,10)=-3{\cdot}419.$$

	α	γ=1	2	3	4	5	6	7
x = 1.	4·0	1540, −2	5890, −3	3624, −3	2718, −3	2254, −3	1978, −3	1798, −3
	3·5	1220, −2	4920, −3	3146, −3	2423, −3	2048, −3	1823, −3	1675, −3
	3·0	9514, −3	4077, −3	2718, −3	2155, −3	1858, −3	1678, −3	1559, −3
	2·5	7279, −3	3348, −3	2338, −3	1911, −3	1683, −3	1544, −3	1450, −3
	2·0	5437, −3	2718, −3	2000, −3	1690, −3	1522, −3	1419, −3	1348, −3
	1·5	3932, −3	2179, −3	1701, −3	1490, −3	1375, −3	1302, −3	1253, −3
	1·0	2718, −3	1718, −3	1437, −3	1310, −3	1239, −3	1194, −3	1163, −3
	0·5	1753, −3	1328, −3	1204, −3	1147, −3	1114, −3	1093, −3	1079, −3
	−0·5	4252, −4	7262, −4	8218, −4	8682, −4	8955, −4	9135, −4	9262, −4
	−1·0	0000, 0	5000, −4	6667, −4	7500, −4	8000, −4	8333, −4	8571, −4
	−1·5	−3010, −4	3153, −4	5324, −4	6443, −4	7128, −4	7591, −4	7925, −4
	−2·0	−5000, −4	1667, −4	4167, −4	5500, −4	6333, −4	6905, −4	7321, −4
	−2·5	−6163, −4	4914, −5	3176, −4	4661, −4	5610, −4	6270, −4	6757, −4
	−3·0	−6667, −4	−4167, −5	2333, −4	3917, −4	4952, −4	5685, −4	6230, −4
x = 2.	4·0	1059, −1	2709, −2	1232, −2	7389, −3	5195, −3	4027, −3	3328, −3
	3·5	7495, −2	2019, −2	9595, −3	5973, −3	4326, −3	3434, −3	2892, −3
	3·0	5172, −2	1478, −2	7389, −3	4792, −3	3584, −3	2918, −3	2508, −3
	2·5	3458, −2	1059, −2	5613, −3	3811, −3	2952, −3	2470, −3	2168, −3
	2·0	2217, −2	7389, −3	4195, −3	3000, −3	2416, −3	2082, −3	1869, −3
	1·5	1340, −2	4978, −3	3073, −3	2335, −3	1964, −3	1747, −3	1607, −3
	1·0	7389, −3	3195, −3	2195, −3	1792, −3	1584, −3	1459, −3	1377, −3
	0·5	3442, −3	1905, −3	1516, −3	1353, −3	1265, −3	1212, −3	1176, −3
	−0·5	−3690, −4	3891, −4	6145, −4	7199, −4	7805, −4	8197, −4	8471, −4
	−1·0	−1000, −3	0000, 0	3333, −4	5000, −4	6000, −4	6667, −4	7143, −4
	−1·5	−1147, −3	−2254, −4	1345, −4	3297, −4	4526, −4	5373, −4	5993, −4
	−2·0	−1000, −3	−3333, −4	0000, 0	2000, −4	3333, −4	4286, −4	5000, −4
	−2·5	−6963, −4	−3600, −4	−8524, −5	1034, −4	2377, −4	3376, −4	4146, −4
	−3·0	−3333, −4	−3333, −4	−1333, −4	3333, −5	1619, −4	2619, −4	3413, −4

x	α	γ=1	2	3	4	5	6	7
x = 3.	4·0	5624, −1	1105, −1	4017, −2	2009, −2	1220, −2	8416, −3	6335, −3
	3·5	3662, −1	7542, −2	2867, −2	1492, −2	9386, −3	6670, −3	5149, −3
	3·0	2310, −1	5021, −2	2009, −2	1094, −2	7149, −3	5249, −3	4162, −3
	2·5	1399, −1	3241, −2	1375, −2	7885, −3	5383, −3	4096, −3	3345, −3
	2·0	8034, −2	2009, −2	9149, −3	5575, −3	4000, −3	3168, −3	2671, −3
	1·5	4272, −2	1178, −2	5866, −3	3847, −3	2926, −3	2424, −3	2117, −3
	1·0	2009, −2	6362, −3	3575, −3	2575, −3	2099, −3	1832, −3	1665, −3
	0·5	7380, −3	2981, −3	2019, −3	1653, −3	1471, −3	1365, −3	1297, −3
	−0·5	−1562, −3	−4756, −5	3657, −4	5496, −4	6520, −4	7169, −4	7615, −4
	−1·0	−2000, −3	−5000, −4	0000, 0	2500, −4	4000, −4	5000, −4	5714, −4
	−1·5	−1419, −3	−5961, −4	−1839, −4	6060, −5	2219, −4	3361, −4	4212, −4
	−2·0	−5000, −4	−5000, −4	−2500, −4	−5000, −5	1000, −4	2143, −4	3036, −4
	−2·5	3694, −4	−3203, −4	−2445, −4	−1058, −4	2022, −5	1255, −4	2125, −4
	−3·0	1000, −3	−1250, −4	−2000, −4	−1250, −4	−2857, −5	6250, −5	1429, −4
x = 4.	4·0	2603, 0	4186, −1	1274, −1	5460, −2	2910, −2	1800, −2	1237, −2
	3·5	1583, 0	2659, −1	8446, −2	3769, −2	2084, −2	1333, −2	9432, −3
	3·0	9282, −1	1638, −1	5460, −2	2550, −2	1470, −2	9750, −3	7124, −3
	2·5	5193, −1	9702, −2	3421, −2	1684, −2	1018, −2	7037, −3	5327, −3
	2·0	2730, −1	5460, −2	2060, −2	1080, −2	6900, −3	5000, −3	3938, −3
	1·5	1312, −1	2860, −2	1175, −2	6663, −3	4551, −3	3486, −3	2872, −3
	1·0	5460, −2	1340, −2	6200, −3	3900, −3	2900, −3	2375, −3	2062, −3
		1684, −2	5091, −3	2870, −3	2111, −3	1763, −3	1571, −3	1453, −3
	−0·5	−3519, −3	−6489, −4	5482, −5	3486, −4	5057, −4	6026, −4	6680, −4
	−1·0	−3000, −3	−1000, −3	−3333, −4	0000, 0	2000, −4	3333, −4	4286, −4
	−1·5	−9230, −4	−7586, −4	−4100, −4	−1571, −4	2365, −5	1572, −4	2594, −4
	−2·0	1000, −3	−3333, −4	−3333, −4	−2000, −4	−6667, −5	4762, −5	1429, −4
	−2·5	2111, −3	6138, −5	−2005, −4	−1808, −4	−1021, −4	−1568, −5	6522, −5
	−3·0	2333, −3	3333, −4	−6667, −5	−1333, −4	−1048, −4	−4762, −5	1587, −5

	α	$\gamma=1$	2	3	4	5	6	7
$x=5.$	4·0	1103, 1	1509, 0	3958, −1	1484, −1	7033, −2	3929, −2	2475, −2
	3·5	6331, 0	9013, −1	2461, −1	9598, −2	4721, −2	2730, −2	1775, −2
	3·0	3488, 0	5194, −1	1484, −1	6050, −2	3104, −2	1866, −2	1257, −2
	2·5	1824, 0	2861, −1	8614, −2	3697, −2	1990, −2	1251, −2	8761, −3
	2·0	8905, −1	1484, −1	4757, −2	2171, −2	1238, −2	8189, −3	6000, −3
	1·5	3938, −1	7074, −2	2453, −2	1209, −2	7397, −3	5207, −3	4022, −3
	1·0	1484 −1	2948, −2	1139, −2	6236, −3	4189, −3	3189, −3	2626, −3
	0·5	4008, −2	9419, −3	4382, −3	2841, −3	2190, −3	1855, −3	1658, −3
	−0·5	−7014, −3	−1537, −3	−3530, −4	1033, −4	3352, −4	4734, −4	5645, −4
	−1·0	−4000, −3	−1500, −3	−6667, −4	−2500, −4	0000, 0	1667, −4	2857, −4
	−1·5	6693, −4	−6543, −4	−5252, −4	−3157, −4	−1382, −4	2959, −6	1153, −4
	−2·0	3500, −3	1667, −4	−2500, −4	−2500, −4	−1667, −4	−7143, −5	1786, −5
	−2·5	3941, −3	6586, −4	9492, −7	−1430, −4	−1411, −4	−9309, −5	−3181, −5
	−3·0	2667, −3	7917, −4	1667, −4	−4167, −5	−9524, −5	−8631, −5	−5159, −5
$x=6.$	4·0	4397, 1	5245, 0	1210, 0	4034, −1	1718, −1	8731, −2	5061, −2
	3·5	2398, 1	2968, 0	7112, −1	2461, −1	1088, −1	5722, −2	3427, −2
	3·0	1251, 1	1614, 0	4034, −1	1457, −1	6707, −2	3671, −2	2280, −2
	2·5	6168, 0	8343, −1	2189, −1	8300, −2	4009, −2	2295, −2	1486, −2
	2·0	2824, 0	4034, −1	1121, −1	4505, −2	2302, −2	1390, −2	9451, −3
	1·5	1163, 0	1774, −1	5294, −2	2287, −2	1255, −2	8086, −3	5826, −3
	1·0	4034, −1	6707, −2	2202, −2	1051, −2	6341, −3	4451, −3	3451, −3
	0·5	9803, −2	1863, −2	7191, −3	4055, −3	2842, −3	2260, −3	1935, −3
	−0·5	1373, −2	−2946, −3	−9188, −4	−2082, −4	1307, −4	3242, −4	4482, −4
	−1·0	−5000, −3	−2000, −3	−1000, −3	−5000, −4	−2000, −4	0000, 0	1429, −4
	−1·5	3944, −3	−1900, −4	−5023, −4	−4043, −4	−2584, −4	−1239, −4	−9506, −6
	−2·0	7000, −3	1000, −3	0000, 0	−2000, −4	−2000, −4	−1429, −4	−7143, −5
	−2·5	5084, −3	1317, −3	3063, −4	−1672, −5	−1097, −4	−1144, −4	−8356, −5
	−3·0	1000, −3	1000, −3	4000, −4	1000, −4	−2857, −5	−7143, −5	−7143, −5

	α	$\gamma=1$	2	3	4	5	6	7
$x = 8.$	4·0	6151, 2	5863, 1	1093, 1	2981, 0	1054, 0	4513, −1	2240, −1
	3·5	3072, 2	3025, 1	5831, 0	1645, 0	6016, −1	2664, −1	1367, −1
	3·0	1461, 2	1490, 1	2981, 0	8733, −1	3318, −1	1526, −1	8122, −2
	2·5	6524, 1	6928, 0	1444, 0	4417, −1	1753, −1	8422, −2	4676, −2
	2·0	2683, 1	2981, 0	6521, −1	2097, −1	8763, −2	4432, −2	2587, −2
	1·5	9816, 0	1150, 0	2664, −1	9110, −2	4058, −2	2187, −2	1358, −2
	1·0	2981, 0	3725, −1	9287, −2	3445, −2	1673, −2	9829, −3	6622, −3
	0·5	6171, −1	8422, −2	2348, −2	9954, −3	5579, −3	3769, −3	2877, −3
	−0·5	−5666, −2	−9700, −3	−3065, −3	−1205, −3	−4510, −4	−6737, −5	1591, −4
	−1·0	−7000, −3	−3000, −3	−1667, −3	−1000, −3	−6000, −4	−3333, −4	−1429, −4
	−1·5	2095, −2	2558, −3	1482, −4	−3028, −4	−3432, −4	−2796, −4	−1918, −4
	−2·0	1700, −2	3667, −3	1000, −3	2000, −4	−6667, −5	−1429, −4	−1429, −4
	−2·5	4828, −4	1965, −3	9556, −4	3836, −4	1041, −4	−2380, −5	−7322, −5
	−3·0	−1233, −2	−3333, −4	4667, −4	3333, −4	1619, −4	4762, −5	−1587, −4
$x = 10.$	4·0	7658, 3	6094, 2	9545, 1	2203, 1	6643, 0	2449, 0	1057, 0
	3·5	3554, 3	2911, 2	4697, 1	1117, 1	3475, 0	1322, 0	5888, −1
	3·0	1564, 3	1322, 2	2203, 1	5419, 0	1744, 0	6871, −1	3169, −1
	2·5	6434, 2	5628, 1	9724, 0	2484, 0	8314, −1	3407, −1	1636, −1
	2·0	2423, 2	2203, 1	3965, 0	1057, 0	3702, −1	1590, −1	7998, −2
	1·5	8059, 1	7654, 0	1445, 0	4055, −1	1499, −1	6814, −2	3634, −2
	1·0	2203, 1	2203, 0	4403, −1	1318, −1	5232, −2	2566, −2	1480, −2
	0·5	4043, 0	4310, −1	9314, −2	3067, −2	1364, −2	7580, −3	4966, −3
	−0·5	2675, −1	−3466, −2	−9101, −3	−3419, −3	−1524, −3	−6964, −4	−2609, −4
	−1·0	−9000, −3	−4000, −3	−2333, −3	−1500, −3	−1000, −3	−6667, −4	−4286, −4
	−1·5	7910, −2	1084, −2	2296, −3	3911, −4	−1312, −4	−2617, −4	−2615, −4
	−2·0	3100, −2	7667, −3	2667, −3	1000, −3	3333, −4	4762, −5	−7143, −5
	−2·5	−2934, −2	−6370, −4	9926, −4	7192, −4	3921, −4	1742, −4	4603, −5
	−3·0	−4567, −2	−5667, −3	−6667, −4	1667, −4	2381, −4	1667, −4	8730, −5

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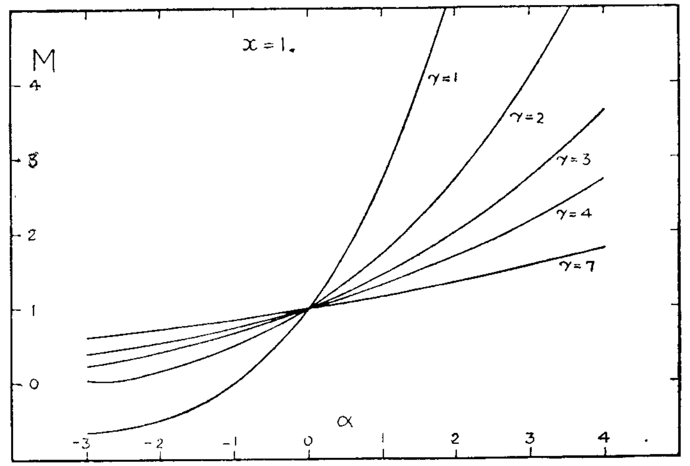

Fig. 1.

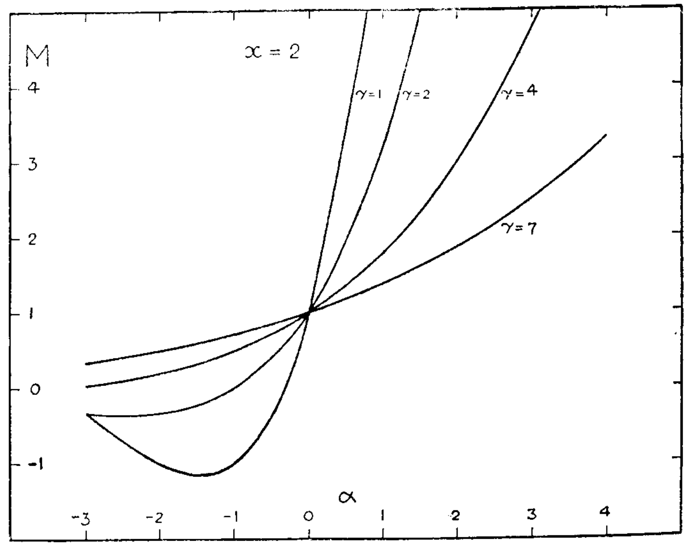

Fig. 2.

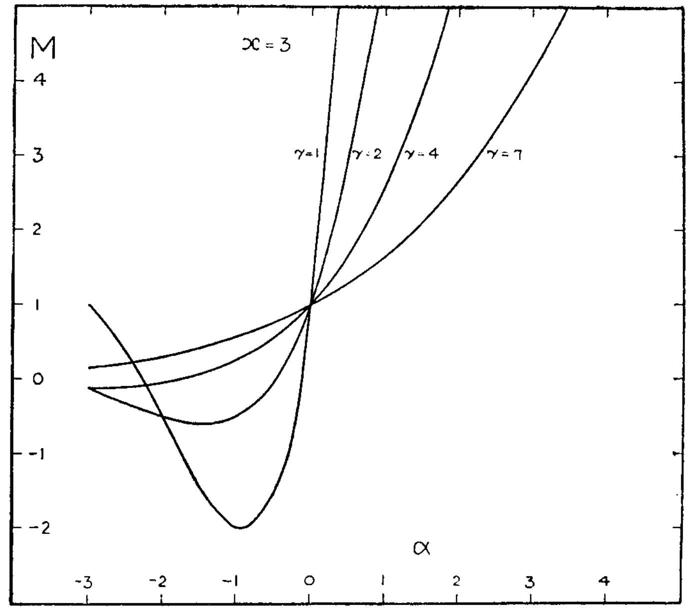

Fig. 3.

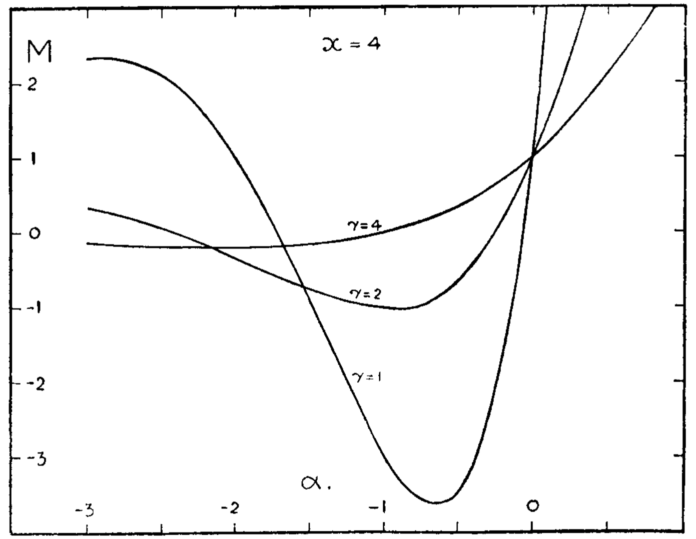

Fig. 4.

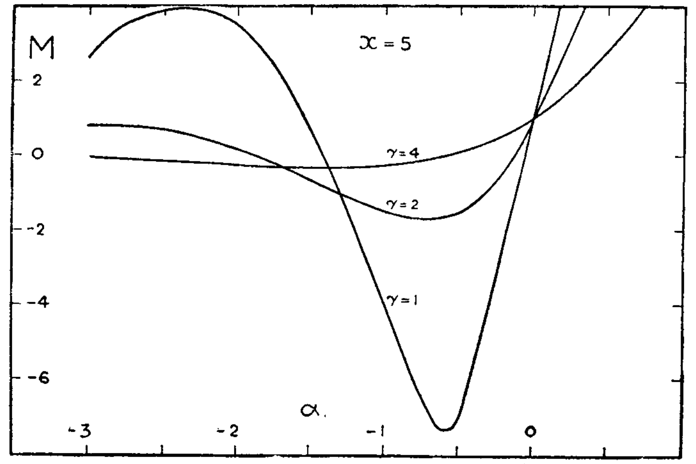

Fig. 5.

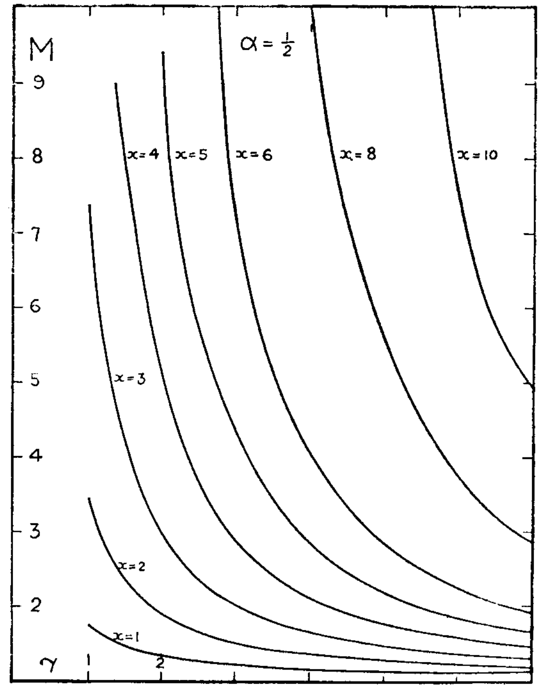

Fig. 6.

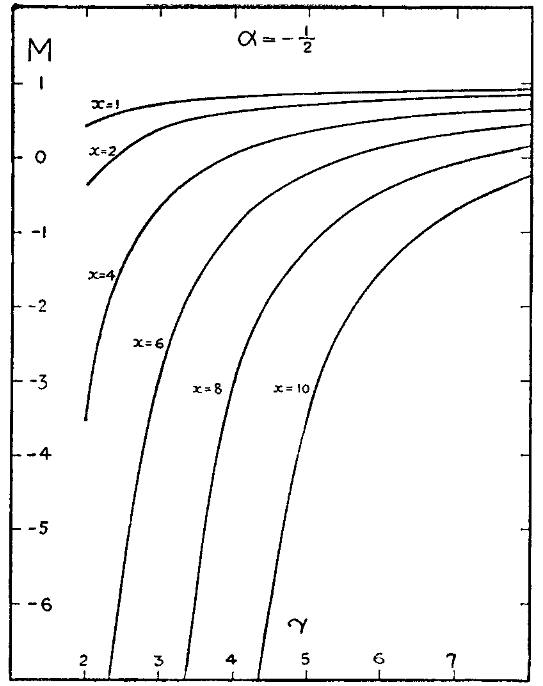

Fig. 7.

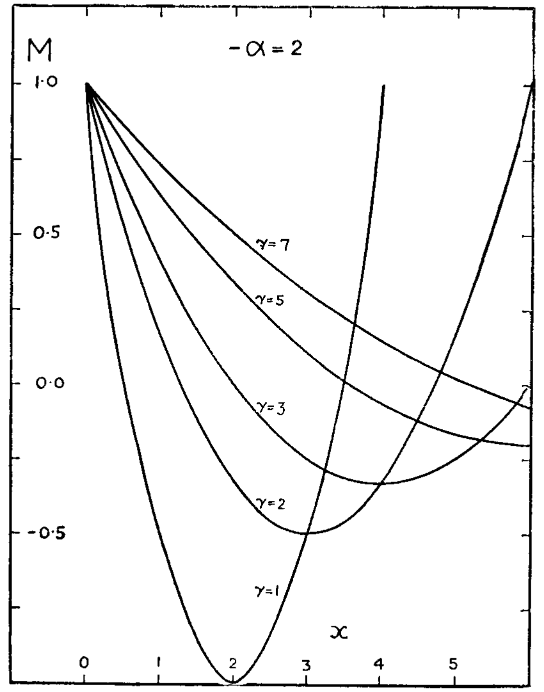

Fig. 8.

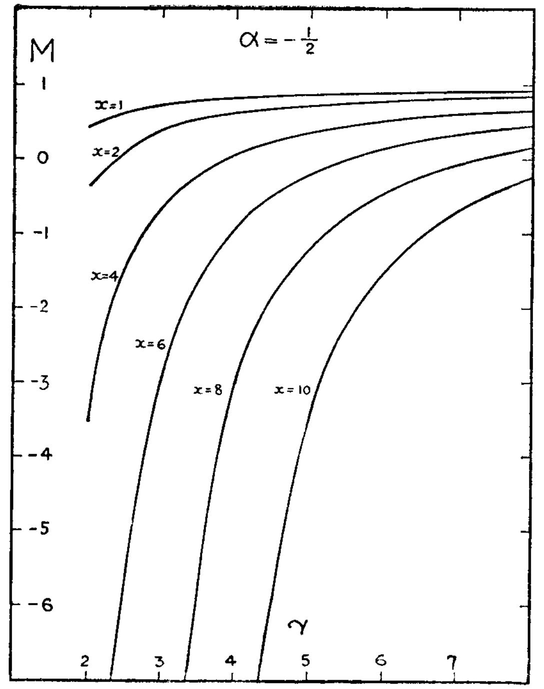

Fig. 7.

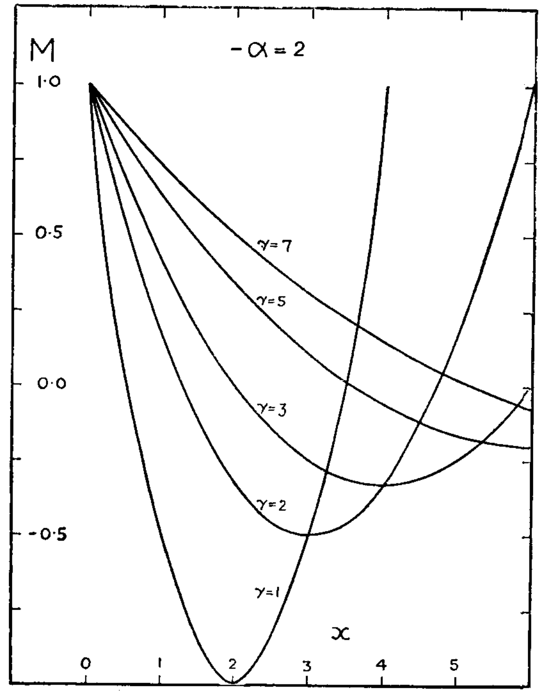

Fig. 8.

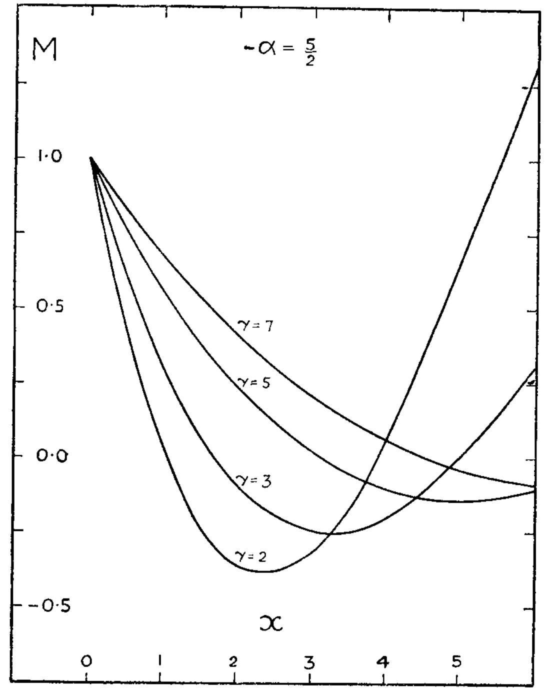

Fig. 9.

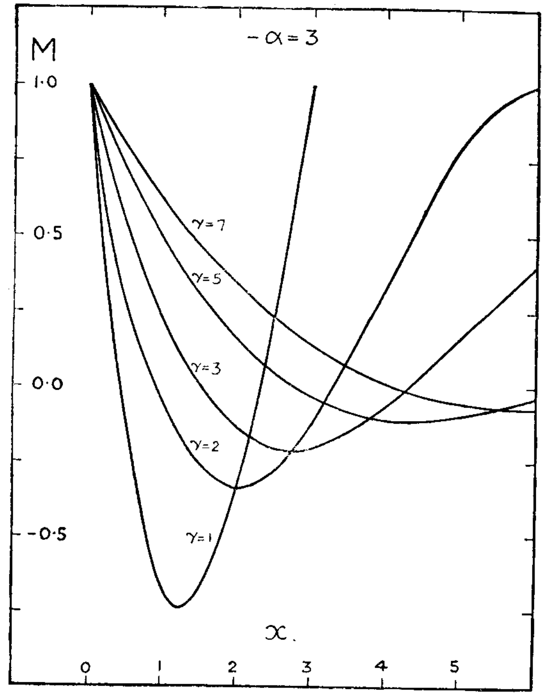

Fig. 10.