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November 25th, 1869.

Prof. HIRST, and subsequently Prof. SYLVESTER, V.P., in the Chair.

The Auditor (Mr. Merrifield) certified that he had examined the Treasurer's accounts, and found them perfectly correct. The Treasurer then suggested that a statement of the financial position of the Society should be drawn up and circulated amongst the Members. This proposal was unanimously agreed to by the Meeting.

The Rev. James White was elected a Member, and admitted into the Society. The Rev. Percival Frost, M.A., was proposed for election.

Dr. Henrici exhibited a model of the cubic surface

$$xyz-(\tfrac{3}{7})^3(x+y+z-1)^3=0,$$

which has three biplanar nodes (to a scale of $2\frac{1}{2}$ inches as unit). A sufficient number (eleven) of sections, $x+y+z-1=$ a constant, cut out in cardboard, are connected in a horizontal position, and kept at their proper distance by three vertical sections $y=z$, $z=x$, $x=y$, with regard to which the surface is symmetrical. The model contains the central part of the surface with the three nodes, and is bounded by a sphere of eight inches' radius, with its centre at the origin, large enough to show the position of the three straight lines in the surface (each counting for nine), and to give an idea how the surface extends to infinity. The interstices between the cardboard are intended to be filled up with plaster of Paris, so as to form a solid model.

Mr. W. K. Clifford then read the following paper:—

On Syzygetic Relations among the Powers of Linear Quantics.

In his *Géométrie de Direction* (Paris, 1869), M. Paul Serret makes very beautiful use of a principle which he states nearly as follows (p. 138):—

"*In order that a system of points (in a plane) may be so related that every curve of order m passing through all but one of them must pass through the remaining one, it is necessary and sufficient that the* m^{th} *powers of their distances from an arbitrary line should satisfy a linear homogeneous relation* $\lambda_1 P_1^m + \lambda P_2^m + \lambda_3 P_3 + \ldots\ldots \equiv 0.$" *

There is, of course, an analogous theorem for surfaces, and in fact M. Serret combines the two enunciations into one; he states also the correlative theorems concerning a system of lines or planes such that

* In the *Bulletin des Sciences Mathématiques et Astronomiques*, January, 1870, M. Darboux observes that this theorem, for the special case $m=2$, had been given by Hesse, *Vier Vorlesungen aus der analytischen Geometrie*, Leipzig, 1866.

every curve or surface touching all but one of them, touches also the remaining one. For the sake of clearness I have here stated in full only one of these four theorems.

By the use of Professor Sylvester's method of Contravariant Differentiation I have arrived at certain extensions of these theorems, which I now proceed to explain:—

Theorem I. *In order that a system of* N *points in a plane should all lie on a curve of order n, it is sufficient that the p^{th} powers of their distances from an arbitrary line should satisfy a linear homogeneous relation; the number* N *being given by the formula*

$$N = \frac{a}{2} n(n+3) + \tfrac{1}{2}(\beta+1)(\beta+2),$$

where a is the quotient and β the remainder of the division of p by n, so that $p = an+\beta$, and $\beta < n$.

Theorem II. *In order that a system of* N *points in space should all lie on a surface of order n, it is sufficient that the p^{th} powers of their distances from an arbitrary plane should satisfy a linear homogeneous relation; the number* N *being given by the formula*

$$N = \frac{a}{6} . n(n^2+6n+11) + \tfrac{1}{6}(\beta+1)(\beta+2)(\beta+3),$$

where as before $p = an+\beta$, $\beta < n$.

To render the nature of these theorems somewhat more clear, I add the following tables of the values of N for given values of p and n:—

TABLE A.—CURVES.

Values of p.	2	3	4	5	6	7	8	9	10	11	12
Line	5	7	9	11	13	15	17	19	21	23	25
Conic	6	8	11	13	16	18	21	23	26	28	31
Cubic		10	12	15	19	21	24	28	30	33	37
Quartic			15	17	20	24	29	31	34	38	43
Quintic				21	23	26	30	35	41	43	46
Sextic					28	30	33	37	42	48	55
Septic						36	38	41	45	50	56
Octavic							45	47	50	54	59

TABLE B.—SURFACES.

Values of p.	2	3	4	5	6	7	8	9	10	11	12
Plane	7	10	13	16	19	22	25	28	31	34	37
Quadric	10	13	19	22	28	31	37	40	46	49	55
Cubic		20	23	29	39	42	48	58	61	67	77
Quartic			35	38	44	54	69	72	78	88	103
Quintic				56	59	65	75	90	111	114	120
Sextic					84	87	93	103	118	139	167
Septic						120	123	129	139	154	175
Octavic							165	168	174	184	199

Here, for example, in the first table opposite the word Cubic and under the power 5 we find the number 15; the theorem corresponding to this is—

If 15 points are such that every quintic through 14 of them passes through the remaining one, all these points must lie on a cubic curve.

Now if we take 15 points arbitrarily on a cubic curve, it is not in general true that the fifth powers of their distances from an arbitrary line satisfy a linear homogeneous relation. That this may be the case, the 15 points must be intersections of the cubic with a quintic; and these are not arbitrary points, but 14 of them being given, the 15th is determined, by a theorem of Jacobi and Plücker. The theorem immediately derived from the table, then, must be completed by this statement; the points are not only all on a cubic, but they are intersections of a cubic and a quintic.

It is to be understood also that if we take a number N of points lying between any two adjacent numbers in the same vertical column of the table, then the same theorem is true about N that is true about the greater of these numbers. Thus we are informed by the first table that a syzygy among the 4th powers of the distances of 12 points makes them lie on a cubic, and that a similar syzygy for 15 points makes them lie on a quartic; this latter theorem is true for the intermediate numbers 13 and 14. It is not however *all* that is true in either of these cases; the 14 points are points of intersection of two quartics, and the 13 points are (I believe) points on a cubic such that no twelve of them are intersections of the cubic with a quartic. I wish particularly to draw attention to these intermediate cases, where it appears that more is true than can be proved by the method to be presently explained.

Method of Demonstration. Let the tangential equation of a point be

$$0 = \alpha\xi + \beta\eta + \gamma\zeta \; (\equiv p, \text{ say})$$

and let the equation of a curve of the n^{th} order be

$$0 = (\,*\,\between x, y, z)^n \; (\equiv \mathrm{B}_n, \text{ say})$$

then I say that

$$(\,*\,\between \tfrac{\delta}{\delta\xi}, \tfrac{\delta}{\delta\eta}, \tfrac{\delta}{\delta\zeta})^n . (\alpha\xi+\beta\eta+\gamma\zeta)^n = (\,*\,\between \alpha, \beta, \gamma)^n \, \underline{|n} \, ;$$

that is to say, *if we operate with* B_n *on the* n^{th} *power of* p, *we shall obtain the result of substituting the coordinates of* p *for* x, y, z *in* B_n. If, then, this result vanishes, the point p is on the curve B_n.

I will now prove that if the 12th powers of the *nil-facta* in the tangential equations of 43 points are connected by a linear syzygy, the 43 points are on a quartic curve. We can draw a quartic B_4 through 14 of the points; operate with B_4 on the syzygy, then these 14 points are cleared away, and there remains a syzygy between the 8th powers of the remaining 29 points. We have therefore now to prove that these 29 points are on a quartic. Draw a curve C_4 through 14 of them, and operate on the new syzygy with C_4. This clears away 14 more points, and we are left with a syzygy among the 4th powers of 15 points. But then by Serret's theorem these lie on a quartic. Hence, *any* 15 of the original 43 points are on the same quartic; therefore all the 43 are on the same quartic.

To prove that if the cubes of 13 points in space are connected by a syzygy, they lie on a quadric surface, operate with the plane through three of them; we are then left with a syzygy among the squares of 10 points, and Serret's theorem again applies.

The application of this method to the remaining cases will now be easy.

Mr. S. Roberts made some remarks on Invariants.

December 9th, 1869.

Prof. CAYLEY, President, in the Chair.

The Rev. Percival Frost, M.A., was elected a Member.

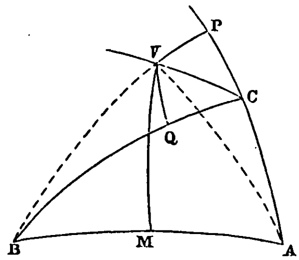
A Paper by Prof. H. J. S. Smith, "On the Focal Properties of Correlative Figures," was an Appendix to a former paper by the same author, "On the Focal Properties of Homographic Figures." By the term "Focal Properties" are intended those properties which arise from considering the imaginary circular points at an infinite distance in either figure, and the points corresponding to them in the other figure. These properties appear to be much less varied in their character in the case of two correlative figures than in the case of two homographic figures; and the two following theorems (of which the first is well known) will suffice to give an idea of the general nature of the results.

I. In two correlative figures in space, there are always two corresponding tetrahedra, such that three adjacent edges of each are rectangular; the three edges opposite to these being at an infinite distance, and the edges at a finite distance in either figure corresponding to the edges at an infinite distance in the other.

II. If we consider any point in either figure, and its correlative plane in the other, we have two definite planes passing through the point, and two corresponding points upon the plane, which may be called respectively the cyclic planes of the point, and the foci of the plane. If we take any third point in the plane, the angles which its focal radii vectores make with the line joining the foci are equal to the angles which the traces of the corresponding planes upon the cyclic planes make with the line of intersection of those two planes.

These theorems suppose only that in the two correlative figures the plane at an infinite distance in either figure answers to a point at a finite distance in the other.

Mr. Tucker read a proof, due to Mr. M. W. Crofton, F.R.S., of Gauss's Theorems and Napier's Analogies.



Bisect the side c at right angles by the arc MV, which meets the exterior bisector of the vertical angle C in V.

Draw the perpendiculars VP, VQ on the sides b, a of the triangle. Since VP=VQ, and VA=VB, and the angles at P and Q are right angles, the triangles AVP, BVQ are equal in all respects. Hence it will be seen that

$$\angle \mathrm{VAP} = \angle \mathrm{VBQ} = \frac{\mathrm{A}-\mathrm{B}}{2},$$

$$\angle \mathrm{VAM} = \angle \mathrm{VBM} = \frac{\mathrm{A}+\mathrm{B}}{2};$$

also, since CP=CQ,

$$\mathrm{BQ} = \mathrm{AP} = \frac{a+b}{2}, \quad \mathrm{CQ} = \mathrm{CP} = \frac{a-b}{2}.$$

Now the $\angle$ AVP = $\angle$ BVQ; adding AVQ to both, QVP = BVA, these two angles being bisected by VC and VM. Hence, in the right-angled triangles VMA, VPC,

$$\cos \mathrm{AM} \sin \mathrm{VAM} = \cos \mathrm{MVA} = \cos \mathrm{CVP} = \cos \mathrm{CP} \sin \mathrm{VCP},$$

that is, $\cos \frac{1}{2}c \sin \frac{1}{2}(\mathrm{A}+\mathrm{B}) = \cos \frac{1}{2}\mathrm{C} \cos \frac{1}{2}(a-b)$,

one of Gauss's theorems.

Again, considering the triangles AVP, CVP,

$$\sin \frac{a+b}{2} = \tan \mathrm{VP} \cot \frac{\mathrm{A}-\mathrm{B}}{2},$$

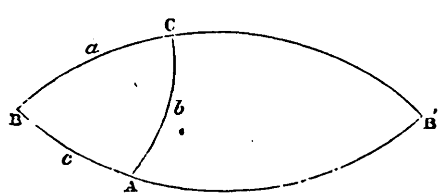
$$\sin \frac{a-b}{2} = \tan \mathrm{VP} \cot \tfrac{1}{2}(\pi-\mathrm{C});$$

dividing,

$$\frac{\sin \frac{a-b}{2}}{\sin \frac{a+b}{2}} = \tan \frac{\mathrm{C}}{2} \tan \frac{\mathrm{A}-\mathrm{B}}{2},$$

giving one of Napier's analogies.

The remaining formulas can be derived from the same construction, or an analogous one obtained by bisecting the angle C *internally*. But if one formula be proved, all the others follow. Thus, by applying the above formula of Gauss to the triangle AB′C, formed by producing a and c to meet, we have at once



$$\sin\frac{c}{2}\cos\frac{A-B}{2}=\sin\frac{C}{2}\sin\frac{a+b}{2};$$

and the remaining two follow from the polar triangle.

Mr. Crofton remarks, that "Prof. Gudermann, of Cleves, in his work on Spherics, gives a construction like mine, but deduces the formulas in a much more complicated form."

Mr. S. Roberts gave an account of his paper

On the Order of the Discriminant of a Ternary Form.

I propose to give a short analytical proof of two results obtained geometrically by Professor Cremona, relative to the influence of multiple points common to the curves of an involution. A few remarks are added on some limiting cases. The formula referred to will be found at pp. 261, 262 of the German translation of Professor Cremona's Introduction to the Geometrical Theory of Plane Curves (1865).

1. Let U, V be ternary forms of the n^{th} degree, so that $U+kV=0$ represents an involution of curves. In the general case, we obtain the number of possible double points of the involution, or the order relative to k of the discriminant, by means of the system of determinants

$$\left|\begin{matrix}\frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz}\\ \frac{dv}{dx} & \frac{dv}{dy} & \frac{dv}{dz}\end{matrix}\right|=0.$$

Taking the two equations

$$\frac{du}{dx}\frac{dv}{dz}-\frac{du}{dz}\frac{dv}{dx}=0\;......\;(1),\qquad \frac{du}{dy}\frac{dv}{dz}-\frac{du}{dz}\frac{dv}{dz}=0\;......\;(2),$$

we get the gross order $4(n-1)^2$, from which must be deducted the extraneous order of

$$\frac{du}{dz}=0\;......\;(3),\qquad \frac{dv}{dz}=0\;......\;(4),$$

or $(n-1)^2$, leaving $3(n-1)^2$.

2. Suppose, however, that all the curves of the involution have a common multiple point at $x=0$, $y=0$ (say $\overline{xy}$) of the order of multiplicity p; and suppose, further, that the tangents at the point are common, and that s of them coincide with $y=0$. We may write

$$U=y^s az^{n-p}+bz^{n-p-1}+\&c.,$$
$$V=y^s az^{n-p}+\beta z^{n-p-1}+\&c.$$

The highest power of z in (1), (2) is $2n-2p-2$, and its coefficient contains in each case y^{s-1}. These equations, therefore, represent curves having $\overline{xy}$ for a multiple point of the order $2p$, and there are $s-1$ common tangents; also (3), (4) represent curves having in common