An Essay on the Geometrical Calculus. By E. Lasker.

Received July 15th, 1896. Communicated November 12th, 1896, by Mr. Tucker.

Introduction.

In 1844 a remarkable book was published, entitled *Die Ausdehnungslehre*, by H. Grassmann. It deals with a new species of magnitudes—such as have a number of extensions, and which are composed of a number of independent units—and teaches how to interpret their addition and multiplication.

The book did not excite much interest, probably because its author omitted to demonstrate that his calculus, although quite different from ordinary algebra, was nevertheless, if properly interpreted, nothing but a way of writing algebraical identities. Had he done so, he might have considerably reduced the volume of his book, enlarged its sphere, and would probably have found the keystone to modern algebra.

In 1847 an essay of his, on the geometrical calculus of Leibnitz, took the prize of the Jablonowsky Society at Leipzig. And, in 1862, prompted by some of the few mathematicians of renown who had studied his writings, he brought out a second edition of his *Ausdehnungslehre*. This second edition was far superior to the former, and gained a small, but select, circle of friends. Since that time the literature of the subject has grown to a considerable extent, and it is yet continually on the increase.

It is to be regretted that Grassmann made very little practical use of his calculus. It is surprising that, though he had been for many years in the possession of a new and powerful calculus, he failed to contribute in any way to the development of the mathematical ideas which interested his contemporaries. Yet the time was full of mathematical life and vigour, under the influence of such men as Hesse, Clebsch, Riemann, Sylvester, Cayley, not to mention the older generation of which Gauss and Cauchy were still alive.

The author's essay attempts to demonstrate that Grassmann's *Ausdehnungslehre* is a shape into which projective geometry or modern algebra may be thrown; that it is coextensive with these two branches of mathematics, and that its symbolism embodies probably the shortest and clearest and most suggestive manner of expressing the truths of these sciences. The manner of deduction is purely geometrical, based on a few assumptions concerning the nature of plane spaces of any manifoldness.

1. We shall assume in the following an abstract universe, a space S which shall contain any kind of space whose definition is faultless. S itself will have no property to distinguish it in any way. The Euclidean properties of our space of three manifoldness are consistent with each other. S will therefore contain the straight line joining any two of its points, the plane determined by any three, and the space of three dimensions determined by any four of its points.

If, now, any point P outside of a space of three dimensions S_3 is joined by straight lines with all the points of the S_3, then a space will be originated which we shall call a S_4, a plane space of four dimensions, or of four manifoldness. Let A, B be any two points of the S_3, and A', B' any other two points upon the lines PA, PB. Then the line $A'B'$ will be contained by the plane PAB; which, in accordance with its generation, must itself be contained by the S_4. The S_4 will therefore contain the line joining any two of its points, the plane joining any three and the S_3 joining any four of the points of the S_4. A S_4 is determined by any S_3 contained by it, and any point P besides, not belonging to the S_3. It follows that it is uniquely determined by any five of its points not situated in one S_3. Continuing the process by which the possibility of the geometrical existence of a S_4 was evolved, we demonstrate in the same way the (abstract) existence of a S_n, a plane space of n manifoldness. Its fundamental properties will be or are assumed to be—

(1) That it is uniquely determined by any $n+1$ of its points not belonging to any S_k where k is smaller than n;

(2) That it contains any S_h with which it can be shown to have $h+1$ points in common not situated in a S_l where l is smaller than h;

(3) That it is continuous everywhere;

(4) That no part of it is distinguished by itself from any other part of it which is identically constructed; and

(5) That any S_{n-1} cuts a S_n into two compartments A, B which are identically constructed, and such that any point moving in a con-

tinuous line (for instance, a straight one) from A into B must of necessity pass the S_{n-1}.

Let A, B, C, D, E be any five points fixing the position of a S_4. Join AB and construct all the lines passing through A and perpendicular to AB. Then the totality of these lines form a plane space S_3, which will be said to be perpendicular to AB. In S_3 $ABCD$ construct the plane e, containing A, perpendicular to AB; and similarly the plane c in the S_3 $ABDE$ and d in the S_3 $ABEC$. $ABCD$ and $ABDE$ have the plane ABD in common; consequently e and c the perpendicular δ on AB in A contained by ABD; similarly c and d the perpendicular ϵ in ABE; and d and e the perpendicular γ in ABC. δ and ϵ determine, in accordance with Euclid's axioms in space S_3, the plane c uniquely; and so ϵ and γ the plane d; γ and δ the plane e. c, d, e are therefore contained in a space Σ_3 of three manifoldness, fixed by A and one point each of γ, δ, ϵ. c, d, e divide Σ_3 into altogether four compartments connected in A, in such wise that any point moving in the Σ_3 would have to pass one of the planes c, d, e, in order to move from one compartment into another. Let us only consider two of these compartments L, M which are opposite to each other, but are otherwise identically constructed. Move one of the bordering planes c, d, e, say e, through one of the border lines, say δ, in L (and M) continuously. In its initial position e was perpendicular to AB. If the angle that e forms with AB should vary, if, for instance, the angle eAB should be larger than R, as far as L was concerned, and therefore smaller than R if its value is measured in M, until e comes in the course of its movement to coincide with a plane e' in L, M—when again the angle $e'AB$ may be R—then the spaces L', M' described by the moving plane will again be opposite corners, like L, M, and identically constructed.

But any plane through δ in L', M' will make with AB an angle larger than R in L', smaller than R in M'. This would be equivalent to a permanent property distinguishing L' from M' and must therefore be rejected. It follows, then, that all lines through A contained by the Σ_3 must be perpendicular to AB. Let us assume any line l through A perpendicular to AB, but not belonging to Σ_3. Any one of the points of l not coinciding with A may be called P. The S_4 in which we operate may be regarded as generated by the Σ_3 and point B in the manner originally described. B and P will therefore be collinear with some point Q of the Σ_3. AP, AQ, AB being all contained in the plane ABP, the two lines AP and AQ perpendicular to AB must coincide. The proposition to be demonstrated is therefore verified. In exactly the same manner we may prove by induction the more general theorem that the aggregate of lines l through A perpendicular to some line AB in space S_n constitute a plane space S_{n-1}, called perpendicular to AB.

Spaces perpendicular to the same line l are called parallel. They cannot have any point in common. For, were P any such point, A and B the points which the two spaces have in common with l, then PAB would be a triangle with two right angles A and B.

Any plane space may be moved into coincidence with any other plane space of the same manifoldness. This is a direct consequence of the fundamental properties (1), (2), (3), (4) of plane spaces.

Through any point P belonging to a S_k, itself contained in a S_{k+1}, a line l may be drawn in that S_{k+1}, perpendicular to the S_k. And only *one* such line is possible. To find l, draw any line l' in S_{k+1} anywhere; and at any one of its points P' erect the perpendicular S'_k. After that move the figure conceived to be rigid, so that P' coincides with P, S'_k with S_k. The position which l' assumes will indicate l.

From any point P a perpendicular can be let fall on any space Σ_k outside of it. To construct it erect the perpendicular at any point Q of Σ_k, in space $\Sigma_k P$; and draw in the plane of P and this perpendicular the parallel through P to it. Only one such perpendicular is possible, according to Euclid's parallel axiom.

$n+1$ points not situated in a S_h, where h is smaller than n, form a figure which will here be called a pyramid of n manifoldness. The points $A_1, \dots A_{n+1}$ will be called the corner-points, the lines $A_1A_2, A_1A_3, \dots A_cA_g, \dots$ the edges, the planes $A_1A_2A_3, \dots A_cA_gA_k \dots$ the border-planes, the spaces S_3 $A_1A_2A_3A_4$, $A_1A_2A_3A_5, \dots$ the border S_3 of the pyramid; and so on generally. A pyramid of n manifoldness will thus have $n+1$ corner-points, $\frac{n+1 \,.\, n}{2}$ edges, $\frac{n+1 \,.\, n \,.\, n-1}{1\,.\,2\,.\,3}$ border-planes ... $(n+1)_{k+1}$ border S_k.

Of any limited part of the space S_n we shall assume that it possesses magnitude. And we shall further assume that our ordinary conceptions concerning the addition and the measurement of geometrical magnitudes in space S_3 are applicable to them. One of our assumptions is therefore that magnitude in space S_n is expressible by a number multiplied by the n^{th} power e^n of the unit of length.

Let H and K be two spaces of manifoldness $n-1$ contained in a S_n; both perpendicular to some line l and therefore parallel to each other. In H draw any figure L, and by lines parallel to l project it on to K into the position M. Then the figure bordered by L, M, and the lines parallel to l is measured by the length l of the perpendicular bordered by the two spaces H, K multiplied by the measure of the magnitude $L (= M)$. This may be proved by a most elementary process of integration, in conformity with the assumption that our conception of the addition of parts of spaces should be applicable to parts of spaces in the S_n.

If P be any point, A and B any two parallel S_{n-1} in a S_n, and we draw from P a continuous aggregate of lines, which cut out of A (supposed to be nearer to P than B) a figure A', out of B a figure B', then the geometrical body of n manifoldness bordered by A', B', and the lines emanating from P is larger than $A'.h$ and smaller than $B'.h$; h being the distance of A' from B'.

It might be difficult to prove this when the foot of the perpendicular let fall from P on A and B falls outside the figures A' and B'. It is, however, sufficient for the present purpose to suppose that these points fall inside these two figures. Then, indeed, it is easily seen that the perpendicular projection of the figure A' on B' is totally enclosed by B', and that the projection of B' on A' encloses the figure A' on all sides. So, then, in conjunction with our last proposition, the truth of the theorem is at once seen to follow from this.

We are now enabled to prove the fundamental theorem: If Δ be the magnitude of any pyramid of n manifoldness and D be the magnitude of any one of its border pyramids of $n-1$ manifoldness, h the distance of that border S_{n-1} from the opposite corner, then

$$\Delta = \frac{D.H}{n}.$$

It will not detract from the generality of the theorem, if we prove it to be true only when the foot of the perpendicular from the vertex on the S_{n-1} is enclosed by the figure whose magnitude is D. For, if it be true under this restriction, it is easily seen that it must be generally true. We make therefore that assumption.

Cut h into any number m of equal parts, the points of intersection being A (the corner-point or vertex) and $H_1, H_2, \ldots H_m$ (which last is the foot of the perpendicular let fall from A on the S_{n-1}). Through each H_i draw the space of $n-1$ manifoldness $S_{n-1}^{(i)}$ perpendicular to h, therefore parallel to the S_{n-1}. Then the whole pyramid is divided into m parts; each contained between two parallel spaces of $n-1$ manifoldness $S_{n-1}^{(i)}$ and $S_{n-1}^{(i+1)}$.

Now the pyramid of $n-1$ manifoldness in the $S_{n-1}^{(i)}$ is in all proportions similar to the S_{n-1} (as can be easily proved from the corresponding figure *in plano*). If the unit of length in the S_{n-1} is e, then the unit of length in the $S_{n-1}^{(i)}$ will be $e\,\frac{i}{m}$. If therefore D is the measure of the pyramid in the S_{n-1}, then $D\left(\frac{i}{m}\right)^{n-1}$ is the measure of the pyramid in the $S_{n-1}^{(i)}$ (applying supposition 4). It follows that, Δ being equal to the sum of the slices between $S_{n-1}^{(i)}$ and S_{n-1}^{i+1}, Δ is larger than $D\,\frac{h}{m}\,\Sigma\left(\frac{i}{m}\right)^{n-1}$, the i ranging from 0 up to $m-1$, and Δ is smaller than $D\,\frac{h}{m}\,\Sigma\left(\frac{i}{m}\right)^{n-1}$, the i ranging from 1 up to m.

If m now becomes indefinitely large, this becomes equivalent to the equation to be demonstrated, $\Delta = \frac{D.h}{n}$.

2. Let A, C, D, ... L be the corner-points of a pyramid; and let B be any point collinear with and intermediate between A and C. Then, from our conception of geometrical magnitude, the pyramid $ABD \ldots L$ + the pyramid $BCD \ldots L$ is equal to the pyramid $ACD \ldots L$. This equation we shall agree to write in an abbreviated form, thus,

$$AB + BC = CA,$$

implying the above.

The equation written down may now be understood to be quite generally true, whether B be intermediate between A and C or not, as long as B is collinear with A and C. This implies that the magnitude of a pyramid must also be capable of assuming negative values; that, to be definite,

$$AB = -BA,$$

and $$AA = 0$$

In words: If two corner-points of a pyramid are transposed, its absolute value does not change; but in as far as the above theorem of addition is general it must change its sign. And, if two corner-points of a pyramid coincide, its value is $= 0$.

For the objects that we are attempting to attain the true values of the volumes of pyramids are not so much wanted as their proportions. We shall therefore, for the sake of brevity, designate by the words— "volume of the pyramid $A_1 A_2 \ldots A_{n+1}$" and denote by $[A_1 \ldots A_{n+1}]$ the

$n!$ fold of its true volume. We then obtain the following equation:—

$[A_1 \dots A_{n+1}] = [A_1 \dots A_n]$ multiplied by the distance from A_{n-1} to the space $A_1 \dots A_n$.

$[AB \dots L]$ may be calculated by multiplying $[AB]$, the perpendicular from C on line AB, the perpendicular from D on plane ABC, &c. by each other. The perpendicular from a point upon a plane space, in which it is itself situated, is of course 0. We see therefore that

$$[AB \dots L] = 0$$

whenever any one of the corner-points of the pyramid is situated in the space of the others. If, conversely,

$$[ABC \dots KL] = 0,$$

then either

$$[ABC \dots K] = 0,$$

or else L must belong to the space $ABC \dots K$. From this it could be shown by induction that, whenever

$$[ABC \dots KL] = 0,$$

any one of the corner-points of the pyramid must be situated in the space of the others. If therefore $n+1$ is the number of the points $A, \dots L$,

$$[A \dots L] = 0$$

is the necessary and sufficient condition that the $A \dots L$ may be situated in plane space of less than n manifoldness.

If ξ, η are two border-spaces of the pyramid $A \dots L$ that contain all of its points together but have between themselves no corner-points in common, and if we denote by $[\xi]$ and $[\eta]$ the pyramids of the corner-points situated in ξ and η respectively, then, from the manner of forming the magnitude $[AB \dots L]$, it is clear that

$$[AB \dots L] = [\xi].[\eta].[\xi\eta],$$

where $[\xi\eta]$ is some factor that is only dependent on the relative situation of the two spaces ξ and η to each other and not on the special position that the corner-points occupy within ξ and η. In the same manner, if $\xi_1, \dots \xi_k$ are k border-spaces of $A \dots L$ comprising all of these points, but of which none has any corner-point in common with any other, then

$$[AB \dots L] = [\xi_1] \dots [\xi_k] \cdot [\xi_1 \xi_2 \dots \xi_k],$$

where $[\xi_i]$ denotes the volume of the pyramid of the corner-points situated in ξ_i and $[\xi_1 \xi_2 \dots \xi_k]$ a certain number determined only by the position of the *spaces* ξ_i relatively to each other.

If $[A \dots L]$ is different from zero, then also $[\xi\eta]$ is different from zero; and ξ and η will have no point in common with each other. If the point P belonged to both ξ and η, we might move the corner-points in ξ and η, so that one of them in either space ξ and η coincides with P, while the value of $[\xi]$ and $[\eta]$ remains unchanged. The value of the pyramid would not be altered by this process; but, two of its corner-points coinciding (in P), it must be zero. The necessary and sufficient condition that any two spaces ξ and η have any one point in common is consequently

$$[\xi\eta] = 0.$$

If $[\xi\eta]$ is different from zero, the space composed by the two (*i.e.*, the space of the $A \dots L$) will be denoted by $\xi\eta$. In the same manner, $\xi_1 \xi_2 \dots \xi_k$ denotes the space composed by $\xi_1, \xi_2, \dots \xi_k$, *i.e.*, the space containing all the points of $\xi_1, \xi_2, \dots \xi_k$. We may easily verify that $[\xi_1 \dots \xi_k]$ is obtained by multiplying $[\xi_1 \xi_2]$ into the magnitude that ξ_3 forms with space $\xi_1 \xi_2$, this again into the magnitude that ξ_4 forms with space $\xi_1 \xi_2 \xi_3$, &c.; and that the necessary and sufficient condition that any ξ_i should have any point in common with the space composed by the other ξ is

$$[\xi_1 \xi_2 \dots \xi_k] = 0.$$

To define our terms: The space $\xi_1 \dots \xi_k$ will be called the composed space, the ξ_i the different components. If $[\xi] = 1$, ξ will be said to be in its normal form. If $[\xi] = c$, different from 1, c will be called the *weight* of ξ. And $[\xi_1 \dots \xi_k]$ will be called the factor of the composition. We thus may say: The weight of the composed space is equal to the product of the weights of the different components multiplied by the factor of the composition.

3. If we fix a point O on any straight line L and lay down a unit of length, all the points of the line may be determined by their distance from O. If, then, a is the measure of OA, c that of OB, then $[AB] = c - a$. Positive and negative sign are distinguished in accordance with the law

$$[AB] + [BC] = [AC].$$

From the algebraical identity

$$(b-a)(d-c)+(c-b)(d-a)+(a-c)(d-b) = 0,$$

we conclude

$$[AB].[CD]+[BC].[AD]+[CA].[BD] = 0,$$

where also $$[AB]+[BC]+[CA] = 0.$$

Let us denote by k, l, m the values of

$$[AB], \quad [BC], \quad [CA];$$

then $$k[CD]+l[AD]+m[BD] = 0.$$

D denotes here any point of L whatever. It is therefore natural to write the equation between $[CD]$, $[AD]$, $[BD]$ as an equation between the points C, A, B themselves, leaving D to be determined until the moment when the requirements of the work make it convenient.

The equation may be translated into words as follows:—Between any three points of a straight line a linear equation must exist. Or else any point C of a line may be expressed as a linear form of any two others A and B,

$$C = \alpha A+\beta B,$$

so that $$\alpha+\beta = 1$$

(on account of $k+l+m = 0$).

By such an equation C is uniquely determined. For it must imply, according to the meaning of such an equation, that

$$[CD] = \alpha[AD]+\beta[BD],$$

which becomes, when D coincides with C,

$$0 = \alpha[AC]+\beta[BC].$$

The ratio in which C divides the segment AB, and therefore C itself, is known when α and β are given.

Let us now agree to use equations of this kind

$$aA+bB+\ldots+lL = 0,$$

even when the points A, B, ... L are not situated upon one line. If the space S of these points is of n manifoldness, then we define the meaning of such an equation by laying down that, if valid at all, it must be equivalent to

$$a[AX]+b[BX]+\ldots+l[LX]=0,$$

where X may be any arbitrary S_{n-1} within the S_n of the points $A, \ldots L$.

We shall now successively prove (1), if an equation like the above holds good in space of n manifoldness S_n, it will not cease to be valid in any space containing that S_n.

(2) Between any $n+2$ points of a space of n manifoldness which are not situated in space of less than n manifoldness, there exists exactly one linear equation

$$aA+bB+\ldots+lL=0,$$

where

$$a+b+\ldots+l=0,$$

and (3) if $A \ldots L$ is a pyramid of non-vanishing magnitude in space of n manifoldness, and any point P of that space is given as a linear form of the corner-points of that pyramid

$$P=aA+\ldots+\lambda L,$$

then P is uniquely determined by this equation.

To prove (1) let

$$a[AX]+b[BX]+\ldots+l[LX]=0,$$

where X is any S_{n-1} in the S_n containing all the points $A, B, \ldots L$. Let further Σ be a S_{n+1} containing the S_n; and let Y be any space of n manifoldness contained by Σ.

Y has with any straight line of Σ a point in common. This follows from the original definition of plane spaces (as well as by $[Yl]=0$ where l is any line of Σ, which is easy to prove). With a plane contained by Σ it will therefore have in common at least two points, consequently a straight line; and similarly, with the S_n of the points $A, \ldots L$, a S_{n-1} which may be called X. Let P be any point of Y not contained by X. Then

Y is composed by X and P;

therefore

$$[AY]=[AX].[(AX)P]$$

$$=[AX].[S_nP].$$

Similarly,

$$[BY]=[BX].[S_nP],$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$[LY]=[LX].[S_nP].$$

It follows that

$$a[AY]+b[BY]+\ldots+l[LY]=0.$$

From this the proposition (1) is verified by induction.

The truth of (2) is also demonstrated by induction. Assume it to be true when $n=k$. Then it will continue to hold good when $n=k+1$. $A, B, C, \ldots L$ being a group of $k+3$ points in space of manifoldness $k+1$, the line AB will have one point P in common with the space S_{n-1} of the remaining points $C, \ldots L$. Now, we know that between A, B, P there exists some equation such as

$$\alpha A+\beta B-P=0, \quad \alpha+\beta=1,$$

and according to assumption there exists an equation such as

$$\gamma C+\delta D+\ldots+\lambda L-P=0, \quad \gamma+\delta+\ldots+\lambda=1.$$

The space S of the points $A \ldots L$ comprises the spaces where these equations are valid; they are therefore both valid within it. It follows then that the two expressions

$$\alpha A+\beta B \quad \text{and} \quad \gamma C+\delta D+\ldots+\lambda L$$

must be the same; and it is incidentally noticed that

$$\alpha+\beta=\gamma+\delta+\ldots+\lambda.$$

One equation of the form

$$aA+bB+cC+\ldots+lL=0, \quad a+b+\ldots+l=0,$$

is therefore sure to exist. Assume the existence of another

$$a'A+b'B+c'C+\ldots+l'L=0.$$

Compose X by $C \ldots L$. Then

$$[CX]=[DX]=\ldots=[LX]=0,$$

and

$$a[AX]+C[BX]=0,$$

$$a'[AX]+C'[BX]=0,$$

showing that $a:b=a':b'$. Similarly

$$a:b:c:\ldots:l=a':b':c':\ldots:l'$$

proving (2) completely.

Let, finally,

$$P=\alpha A+\beta B+\ldots+\lambda L.$$

From what we know about points upon a straight line $\alpha A+\beta B$ is the $\alpha+\beta$-ple of some definite point A' upon the line AB,

$$\alpha A+\beta B+\gamma C=(\alpha+\beta)A'+\gamma C$$

is the $\alpha+\beta+\gamma$-ple of some definite point on the line $A'C$; *i.e.*, some definite point in the plane ABC. By induction, it is immediately clear that

$$\alpha A+\beta B+\ldots+\lambda L, \quad \text{where} \quad \alpha+\beta+\ldots+\lambda=1,$$

is some definite point in the space $AB \ldots L$.

The values $\alpha, \beta, \gamma, \ldots \lambda$ which fix the position of a point

$$P=\alpha A+\beta B+\gamma C+\ldots+\lambda L$$

will be called the "coordinates" of P, whenever $A, B, \ldots L$ is a fixed pyramid used all through in one piece of work.

From the theorems given many properties of plane spaces can at once be deduced. For instance, any S_k has with any S_h, both being contained in space S_n, where $k+h=n$, one point in common. For, assume any pyramid of $k+1$ points in the S_k $A_1 \ldots A_{k+1}$, and a similar pyramid in the S_h $B_1 \ldots B_{h+1}$. The total number of these points which are all situated in the S_n being $h+k+2=n+2$, one linear equation must exist between them (generally speaking)

$$a_1A_1+\ldots+a_{k+1}A_{k+1}+b_1B_1+\ldots+b_{h+1}B_{h+1}=0,$$

$$\Sigma a+\Sigma b=0.$$

If $\Sigma a=c$ then $a_1A_1+\ldots+a_{k+1}A_{k+1}$ is the c-ple of the point common to both S_k and S_h.

Similarly it may be shown that a S_k and a S_h both of which are contained in space S_n must have a plane space of manifoldness $k+h-n$ in common ($k+h$ supposed to be larger than n).

If S_h, S_k, S_l are any three spaces contained in a S_n, and

$$h+k+l=n-1,$$

one straight line (generally speaking) can be drawn to cut all three spaces. Assuming

$h+1$ points in the S_h, $A_1 \ldots A_{h+1}$,

$k+1$ points in the S_k, $B_1 \ldots B_{k+1}$,

$l+1$ points in the S_l, $C_1 \ldots C_{l+1}$, we have $h+k+l+3 = n+2$ points in the S_n, between which therefore a linear equation will exist, viz.,

$$a_1A_1 + \dots + a_{h+1}A_{h+1} + b_1B_1 + \dots + b_{k+1}B_{k+1} + c_1C_1 + \dots + c_{k+1}C_{k+1} = 0.$$

Consequently the three points A, B, C belonging respectively to S_h, S_k, S_l, defined by

$$\alpha A = a_1A_1 + \dots + a_{h+1}A_{k+1},$$

$$\beta B = b_1B_1 + \dots + b_{k+1}B_{k+1},$$

$$\gamma C = c_1C_1 + \dots + c_{l+1}C_{l+1},$$

are collinear.

(Some theorems based on the propositions of this section are given in two articles by the author in *Nature*, August 8th and October 17th, 1895.)

The coordinates of a point have a certain geometrical significance.

Let $$a_1A_1 + a_2A_2 + \dots + a_{n+2}A_{n+2} = 0$$

for $n+2$ points $A_1, \dots A_{n+2}$ in a S_n.

From this

$$a_1[A_1X] + a_2[A_2X] + \dots + a_{n+2}[A_{n+2}X] = 0.$$

Assume for X the space $A_3A_4 \dots A_{n+2}$; then

$$a_1[A_1A_3 \dots A_{n+2}] + a_2[A_2A_3 \dots A_{n+2}] = 0.$$

If, therefore, $$a_1 = \quad [A_2A_3 \dots A_{n+1}],$$

then $$a_2 = -[A_2A_3 \dots A_{n+1}],$$

and generally a_k is found by replacing, in $[A_2A_3 \dots A_{n+2}]$, a_k by a_1 and changing the sign of the whole.

In conformity with the rule of signs, we shall obtain

$$a_1 = \quad [A_2A_3 \dots A_{n+2}],$$

$$a_2 = -[A_1A_3 \dots A_{n+2}],$$

$$a_3 = \quad [A_1A_2A_4 \dots A_{n+2}],$$

$$a_4 = -[A_1A_2A_3A_5 \dots A_{n+2}],$$

&c.

4. Let $$A_{n+2} = a_1A_1 + \ldots + a_{n+1}A_{n+2}.$$

In accordance with the meaning of such equations, we obtain, by adding any combination $X = P_1P_2 \ldots P_n$ of n points in space $A_1 \ldots A_{n+1}$ to each member,

$$[P_1P_2 \ldots P_nA_{n+2}] = a_1[P_1P_2 \ldots P_nA_1] + \ldots + a_{n+1}[P_1P_2 \ldots P_nA_{n+1}].$$

Replacing A_{n+2} by its linear expression in the A_i, this assumes the form of an identity

$$[P_1P_2 \ldots P_n(a_1A_1 + \ldots + a_{n+1}A_{n+1}]$$
$$= a_1[P_1P_2 \ldots P_nA_1] + a_2[P_1P_2 \ldots P_nA_2] + \ldots .$$

Treating the P similarly, it is clearly shown that the values of pyramids whose points are linear forms of other points are found by treating them as if they were algebraical products.

Let A, B, P, Q be four points on one line, and

$$P = aA + bB,$$
$$Q = cA + dB;$$

then, according to the above,

$$[PQ] = ac[AA] + ad[AB] + bc[BA] + bd[BB].$$

If, now, the rule of signs is taken into consideration, this is seen to be

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} [AB].$$

Let, similarly, P, Q, R be three points in a plane, linearly expressed by means of three points A, B, C in the same plane whose triangle does not vanish

$$P = aA + bB + cC,$$
$$Q = a'A + b'B + c'C,$$
$$R = a''A + b''B + c''C.$$

Then it is easily seen that

$$[PQR] = a\begin{vmatrix} b' & c' \\ b'' & c'' \end{vmatrix}[ABC] + b\begin{vmatrix} c' & a' \\ c'' & a'' \end{vmatrix}[BCA] + c\begin{vmatrix} a' & b' \\ a'' & b'' \end{vmatrix}[CAB].$$

But, according to the rule of signs, $[ABC]$, $[BCA]$, $[CAB]$ all denote the same value; therefore

$$[PQR] = \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} [ABC].$$

It could, in exactly the same manner and in connection with some elementary properties of determinants, be shown that, *if* $n+1$ *points* $P_1, \ldots P_{n+1}$ *are expressed as linear forms of the* $n+1$ *corner-points of some pyramid* $A_1 \ldots A_{n+1}$ *in their space*

$$P_i = a_{i,1} A_1 + \ldots + a_{i,n+1} A_{n+1},$$

then

$$[P_1 P_2 \ldots P_{n+1}] = \begin{vmatrix} a_{1,1} & \ldots\ldots & a_{1,n+1} \\ \vdots & & \\ a_{n+1,1} & \ldots\ldots & a_{n+1,n+1} \end{vmatrix} [A_1 \ldots A_{n+1}].$$

Let $A_1 \ldots A_{n+1}$ be a fixed pyramid, P a variable point. Put

$$P = x_1 A_1 + \ldots + x_{n+1} A_{n+1};$$

then the totality of points for which the x satisfy a linear equation

$$\mathfrak{a}_1 x_1 + \ldots + \mathfrak{a}_{n+1} x_{n+1} = 0$$

form a certain S_{n+1}. Indeed, let $P_1, \ldots P_{n+1}$ be $n+1$ points whose coordinates $x_{i,j}$ satisfy the given linear equation. Then from elementary properties of determinants their determinant $(x_{i,j})$ vanishes; therefore also $[P_1 \ldots P_{n+1}]$, showing that P_{n+1} belongs to the space of the other n points.

If Q, R are any two points, and

$$a_1 x_1 + a_2 x_2 + \ldots + a_{n+1} x_{n+1} = u = 0$$

be the equation of any S_{n-1}, then

$$uQ : uR,$$

or the proportion of the values obtained by inserting in place of the running coordinates contained in u those of Q and R is equal to

$$[S_{n-1} Q] : [S_{n-1} R],$$

that is, equal to the proportion of the perpendiculars from the two points on the S_{n-1}. Indeed, let $P_1, \ldots P_n$ be any n points fixing the S_{n-1}, X a variable point

$$X = x_1 A_1 + \ldots + x_{n+1} A_{n+1};$$

then $[P_1 \ldots P_n X]$ must be a multiple of u, both being homogeneous and linear in the x_i and expressing by their vanishing the same circumstance.

5. Let $A, B, \ldots L$; $A', B', \ldots L'$ be any two pyramids in the same space ξ. Let η be any other space. Then, from the formula of composition,

$$[AB \ldots L\eta] = [AB \ldots L] \,.\, [\xi\eta],$$

$$[A'B' \ldots L'\eta] = [A'B' \ldots L'] \,.\, [\xi\eta].$$

Should therefore

$$[AB \ldots L] = k\,[A'B' \ldots L'],$$

then also $$[AB \ldots L\eta] = k\,[A'B' \ldots L'\eta].$$

This equation may again be written in an abbreviated form thus,

$$AB \ldots L = k \,.\, A'B' \ldots L'.$$

In accordance with this result we may lay down that by the words "space ξ" is meant any combination of points in that space whose pyramid is $= 1$; and by $k\xi$, k denoting any constant, any combination of points in that space whose pyramid is $= k$.

In order to express the fact that any two spaces S and T are identical, we write $S \equiv T$, saying S congruent to T. Thus the straight line L, joining two points A, B, is $L \equiv AB$. However, according to the above, L only $= AB$ when $[AB] = 1$. Generally $AB = [AB] \,.\, L$.

A linear equation between spaces of the same manifoldness $\xi_1, \xi_2, \ldots \xi_k$,

$$\text{(E)} \quad c_1\xi_1 + c_2\xi_2 + \ldots + c_k\xi_k = 0,$$

is defined as an abbreviation for the equation between numerical values

$$c_1\,[\xi_1\eta] + c_2\,[\xi_2\eta] + \ldots + c_k\,[\xi_k\eta] = 0,$$

η denoting an arbitrary space, such that

$$\xi_1\eta \equiv \xi_2\eta \equiv \ldots \equiv \xi_k\eta.$$

It is immediately seen that from (E) it follows that also

$$c_1\xi_1\vartheta + c_2\xi_2\vartheta + \ldots + c_k\xi_k\vartheta = 0,$$

ϑ denoting an arbitary space.

If a linear equation exists between two spaces, they must be congruent.

Let, indeed, ξ and η be any two spaces, P any point of η. From

$$c_1\xi = c_2\eta,$$

we conclude $$c_1\xi P = c_2\eta P = 0;$$

i.e., any point belonging to η also belongs to ξ; therefore $\xi \equiv \eta$.

If a linear equation exists between three spaces of k manifoldness ξ_1, ξ_2, ξ_3, they must have a S_{k-1} in common; and they will be contained in a S_{k+1}. And, conversely, if this is so, then a linear relation must exist between them.

Indeed, let
$$c_1\xi_1 + c_2\xi_2 + c_3\xi_3 = 0.$$
If P is a point common to ξ_2 and ξ_3, then
$$\xi_2 P = 0 \quad \text{and} \quad \xi_3 P = 0;$$
but
$$c_1\xi_1 P + c_2\xi_2 P + c_3\xi_3 P = 0;$$
consequently also
$$\xi_1 P = 0;$$
i.e., a point common to two spaces ξ belongs also to the third.

Let Σ be the space common to $\xi_1\xi_2\xi_3$, A_1 any point of ξ_1 not contained in Σ, A_2 any point of ξ_2 not contained in Σ. Join A_1, A_2. From
$$c_1\xi_1 A_1 A_2 + c_2\xi_2 A_1 A_2 + c_3\xi_3 A_1 A_2 = 0,$$
we conclude, since
$$\xi_1 A_1 = 0, \quad \xi_2 A_2 = 0,$$
that also
$$\xi_3 A_1 A_2 = 0;$$
i.e., that the line A_1A_2 has a point in common with ξ_3. Let this point be denoted by A_3.

Let, now, A_1' be a point, if such a point exists, belonging to ξ_1, but not to ΣA_1. Join A_1' with A_2. Any line cutting ξ_1 and ξ_2 will, as we have seen, also cut ξ_3. Let $A_1'A_2$ cut ξ_3 in A_3'. A_1A_1' and A_3A_3' will be contained in the plane $A_1A_2A_1'$; therefore have a point P in common. P belonging to both ξ_1 and ξ_3 will be contained in Σ, *i.e.*,
$$\Sigma A_1 A_1' = 0.$$
We see therefore that all points of ξ_1 are contained in ΣA_1; so we obtain
$$\xi_1 \equiv \Sigma A_1, \quad \xi_2 \equiv \Sigma A_2, \quad \xi_3 \equiv \Sigma A_3.$$
Σ is therefore of manifoldness $k-1$, and the space containing ξ_1, ξ_2, ξ_3 is $\Sigma A_1 A_2$, of manifoldness $k+1$.

Conversely, if ξ_1, ξ_2, ξ_3 are three spaces of manifoldness k, having a S_{k-1} (Σ) in common and contained in a S_{k+1}, they will be cut by any line of the S_{k+1} not belonging to the S_{k-1} in one point each, denoted respectively by A_1, A_2, A_3,
$$\xi_1 \equiv \Sigma A_1, \quad \xi_2 \equiv \Sigma A_2, \quad \xi_3 \equiv \Sigma A_3.$$
A_1, A_2, A_3 being collinear, a certain relation will exist between them,
$$c_1 A_1 + c_2 A_2 + c_3 A_3 = 0;$$

therefore also $$c_1 \Sigma A_1 + c_2 \Sigma A_2 + c_3 \Sigma A_3 = 0,$$

which may also be written

$$c_1 [\Sigma A_1] \xi_1 + c_2 [\Sigma A_2] \xi_2 + c_3 [\Sigma A_3] \xi_3 = 0.$$

The theorem proved may be expressed differently thus :—

(1) If two spaces ξ_1, ξ_2 of k manifoldness have a space S_{k-1} in common, then any linear form of them $c_1\xi_1 + c_2\xi_2$ is congruent to some space ξ_3, also containing the S_{k-1}.

(2) If two spaces ξ_1, ξ_2 of k manifoldness have not a space S_{k-1} in common, then any linear form of them $c_1\xi_1 + c_2\xi_2$ cannot be congruent to any space, but will have a symbolical significance only.

If, for instance, l_1, l_2 are two lines in space which have no point in common, then $c_1 l_1 + c_2 l_2$, or any expression equivalent to it, will not be represented by a line. But nature adds forces in the same manner as lines are added in the sense defined above, so that such expressions $c_1 l_1 + c_2 l_2$ may very well be employed to express the effect of a system of forces acting upon a rigid body, or the instantaneous movement of such a body.

Let $\xi_1, \xi_2, \ldots \xi_m$ be spaces of k manifoldness. Then any linear form of them

$$c_1\xi_1 + c_2\xi_2 + \ldots + c_m\xi_m + \ldots$$

will be called a form of manifoldness k.

Between the border S_k's of any pyramid no linear relation can possibly exist. But any space of k manifoldness in the space of the pyramid can be represented as a linear form of these border S_k's.

Let $A_1, \ldots A_{n+1}$ be the corner-points of any pyramid in space S_n. To prove the first part of the proposition assume any linear relation between the border S_k of the pyramid

$$c_1 A_1 \ldots A_{k+1} + \ldots + c_j \xi_j + \ldots = 0,$$

the c denoting constants, the ξ_j border S_k's. Add the combination

$$\eta = A_{k+2} A_{k+2} \ldots A_{n+1},$$

containing all corner-points but the $A_1 \ldots A_{k+1}$; then

$$[\xi_j \eta] = 0;$$

therefore $$c_1 [A_1 \ldots A_{k+1} A_{k+2} \ldots A_{n+1}] = 0.$$

$[A_1 \ldots A_{n+1}]$ is different from zero according to hypothesis. Therefore $c_1 = 0$, and generally $c_j = 0$.

In any space ξ of k manifoldness in the S_n, assume any $k+1$ points $P_1, P_2, \dots P_{k+1}$, whose pyramid is $=1$. Then, according to definitions

$$\xi = P_1 P_2 \dots P_{k+1}.$$

But, the P being situated in the S_n fixed by the pyramid $A_1 \dots A_{n+1}$, they must be expressible as linear forms of these points A_i,

$$\begin{array}{ll} P_1 & = c_{1,1} A_1 + c_{1,2} A_2 + \dots + c_{1,n+1} A_{n+1}, \\ \vdots & \quad \vdots \qquad \vdots \qquad\qquad \vdots \\ P_{k+1} & = c_{k+1,1} A_1 + c_{k+1,2} A_2 + \dots + c_{k+1,n+1} A_{n+1}. \end{array}$$

$P_1 \dots P_{k+1}$ is therefore obtained by multiplying the linear expressions with each other, under consideration of the rule of signs, and the law that two identical factors give zero as result. The expression thus obtained defines ξ uniquely, as no other expression of the same form can be equal to it. The coefficients of the border S_k's are expressible in the form of determinants of the matrix

$$\left| \begin{array}{ccc} c_{1,1} & \dots\dots & c_{1,n+1} \\ \vdots & & \vdots \\ c_{k+1,1} & \dots\dots & c_{k+1,n+k} \end{array} \right|,$$

according to what has been shown before.

The most general form of manifoldness k in space S_n is consequently a linear form of the border S_k's of any pyramid situated in the S_n. As such it has altogether $(n+1)_{k+1}$ independent coefficients (coordinates).

The forms of manifoldness 2 can be given a certain shape which seems simpler than any other.

If a and b are any two lines, and $[ab]=0$, then a and b will have a point in common, and any linear form $ha+kb$ will therefore be congruent to some line. If, however, $[ab]$ is different from zero, such a reduction will not be possible, as was shown before.

Let A, B, C, D be the four corners of a pyramid in a S_3. The general form of the second order in the S_3 is

$$X = c_1 AB + c_2 AC + c_3 AD + c_4 BC + c_5 BD + c_6 CD,$$

i.e.,

$$= A\,(c_1 B + c_2 C + c_3 D) + c_4 BC + c_5 BD + c_6 CD.$$

But $$c_1B+c_2C+c_3D \quad \text{is} \equiv \text{some point,}$$

$$c_4BC+c_5BD+c_6CD \text{ is} \equiv \text{some line,}$$

in the plane BCD; therefore

$$X = ha+kb,$$

where a passes through an arbitrary point A, and b is totally situated in an arbitrary plane BCD (not containing A).

This, then, is the reduced form of the second order in space.

Similarly, A_1, A_2, ... A_5 being the five corner-points of a pyramid in a S_4,

$$X = hA_1P+Y,$$

where X is the most general form of the second order in the S_4, and Y the most general form of manifoldness 2 in the S_3 $A_2A_3A_4A_5$, and P belongs to that S_3.

Put, then, $$Y = ka+lb,$$

so that a passes through P; then hA_1P+ka is again $\equiv$ some line c; and we finally obtain

$$X = mc+nb,$$

where m, n are some constants. The line of reasoning is thus indicated. We conclude: *The most general form of the second order in space S_n is*

$$X = c_1a_1+\ldots+c_\nu a_\nu,$$

where the c are constants and the a lines, and ν is $= \frac{1}{2}n$ or $= \frac{1}{2}(n+1)$ according as n is even or odd.

If the a_i are all situated in a space Σ, we may say that X belongs to Σ. What, then, is the condition that X belongs to a space of manifoldness h, and not to space of lower manifoldness?

If $$X = ha+kb,$$

$$XX = X^2 = hkab,$$

aa and bb being zero, and

$$X^3 = 0.$$

If $$X^2 = 0,$$

then $$h.k = 0 \quad \text{or} \quad [ab] = 0,$$

from which we conclude that X must then simply be a line.

Therefore the necessary and sufficient condition that X should be a line is

$$X^2 = 0,$$

and that X should belong to a S_3 is

$$X^3 = 0,$$

and quite generally, concluding in the same way as above: *The necessary and sufficient condition that X should belong to a space of $2n+1$ manifoldness is*

$$X^{n+2} = 0.$$

And the space to which X belongs is $\equiv X^{n+1}$.

In the language of determinants this leads to the following theorem. Let, first of all, four points A_1, A_2, A_3, A_4 be the corner-points of a pyramid in space S_3, and

$$P = c_1 A_1 + c_2 A_2 + c_3 A_3 + c_4 A_4,$$

$$Q = d_1 A_1 + d_2 A_2 + d_3 A_3 + d_4 A_4.$$

Then $$X = PQ = \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} A_1 A_2 + \begin{vmatrix} c_1 & c_3 \\ d_1 & d_3 \end{vmatrix} A_1 A_3 + \ldots$$

$$= \Delta_{1,2} A_1 A_2 + \Delta_{1,3} A_1 A_3 + \Delta_{1,4} A_1 A_4 + \Delta_{2,3} A_2 A_3$$

$$+ \Delta_{2,4} A_2 A_4 + \Delta_{3,4} A_3 A_4.$$

This not being the most general form of manifoldness 1 in S_3, the Δ must satisfy the relation expressed by

$$X^2 = 0.$$

Developed, this relation is seen to be

$$\text{(E)} \quad \Delta_{1,2} \Delta_{3,4} + \Delta_{1,3} \Delta_{2,4} + \Delta_{1,4} \Delta_{2,3} = 0.$$

Or, in the language of algebra, between the six determinants of the matrix

$$\begin{vmatrix} c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

the above relation (E) holds good, and, if (E) be satisfied, then the magnitudes $\Delta_{i,j}$ can be expressed as determinants of such a matrix.

Let, now, quite generally $A_1 \dots A_{n+1}$ be any pyramid in space S_n,

$$a_1 \text{ be any line } \equiv \Delta^{(1)}_{1,2} A_1 A_2 + \Delta^{(1)}_{1,3} A_1 A_3 + \dots + \Delta^{(1)}_{i,j} A_i A_j + \dots,$$

$$a_2 \quad ,, \quad ,, \quad \equiv \Delta^{(2)}_{1,2} A_1 A_2 + \dots\dots\dots + \Delta^{(2)}_{i,j} A_i A_j + \dots,$$

$$a_\nu \quad ,, \quad ,, \quad \equiv \Delta^{(\nu)}_{1,2} A_1 A_2 + \dots\dots\dots + \Delta^{(\nu)}_{i,j} A_i A_j + \dots,$$

$$X = h_1 a_1 + h_2 a_2 + \dots + h_\nu a_\nu = \Delta_{1,2} A_1 A_2 + \dots + \Delta_{i,j} A_i A_j + \dots.$$

If ν is smaller than $\frac{1}{2}(n+1)$ or $\frac{1}{2}n$, then X is not the most general form of the second order in space $A_1 \dots A_{n+1}$. The $\Delta_{i,j}$ must therefore satisfy a number of conditions. They are all expressed by the equation

$$\text{(E)} \quad X^{\nu+1} = 0,$$

when X^ν is supposed to be different from 0. In algebraic form

$$\text{(E}'\text{)} \quad \Sigma \Delta_{i_1, j_1} \cdot \Delta_{i_2, j_2} \cdot \Delta_{i_3, j_3} \dots \Delta_{i_{\nu+1}, j_{\nu+1}} = 0,$$

where the summation is to be extended over all indices

$$i_1, j_1, \dots, i_{\nu+1}, j_{\nu+1},$$

different from each other, but which belong to a circle of $2\nu+2$ integers; (E′) then is a series of necessary and sufficient conditions that magnitudes $\Delta_{i,j}$ shall be expressible as one and the same linear form

$$h_1 \Delta^{(1)}_{i,j} + h_2 \Delta^{(2)}_{i,j} + \dots + h_\nu \Delta^{(\nu)}_{i,j}$$

of the determinants $\Delta^{(k)}_{i,j}$ of matrices

$$\begin{vmatrix} c^{(k)}_1 & \dots\dots & c^{(k)}_{n+1} \\ d^{(k)}_1 & \dots\dots & d^{(k)}_{n+1} \end{vmatrix}.$$

We might put this again into determinant form, but we leave the matter here, as it lies too far apart from the object here pursued.

Forces acting on a rigid body have a certain line of action a, and a certain intensity h, so that, by identifying them with $h \cdot a$, they are perfectly defined.

Two forces in the same plane have the same effect as one force according to the parallelogram of forces. Let a, b be the two lines of the forces f_1, f_2, P the point of intersection of a and b. Further, determine A on a, and B on b, so that

$$h = [PA],$$

and

$$k = [PB],$$

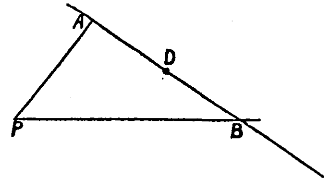
where

$$f_1 = h \cdot a = PA,$$

and

$$f_2 = k \cdot b = PB.$$

The resultant of f_1 and f_2 is double PD, where D is the centre of the finite line BA; therefore



$$D = \tfrac{1}{2}(A+B),$$

according to the parallelogram of forces. So, then, the resultant of f_1 and f_2 is

$$= 2PD = PA + PB = f_1 + f_2.$$

This is also true when a and b are parallel, and even when

$$h + k = 0$$

(the two forces then forming a couple); only that then the line of action of the force is a certain exceptional line which will afterwards be spoken of as the line infinity, and which would again be characterized by $f_1 + f_2$, in situation as well as in regard to a certain factor (the magnitude connected with the couple).

By applying the calculus to the formulæ of mechanics, the same result would be attained for forces acting on rigid bodies in space. The effect of a system of forces upon a rigid body would then be seen to be expressible by a form of manifoldness 1. The resultant of a system of forces would simply be their sum (in the sense I defined above), and the corresponding infinitesimal motion of the body would also be determined by the same form of manifoldness 1.

6. We have not in the preceding sections again mentioned the possibility that presented itself in the introduction, namely, that two spaces might be parallel. Projective geometry shows how to connect parallelism with the general theory. Parallel spaces *in plano*, or in space, are such as intersect in a certain line or plane, the line or plane at infinity. The same is true for spaces of any degree of manifoldness.

For three points on a line we had

$$[AB] + [BC] + [CA] = 0,$$

similarly for four points in a plane

$$[ABC] - [BCD] + [CDA] - [DAB] = 0,$$

and generally, $A_1, A_2, \ldots, A_{n+1}$, denoting any pyramid fixing a space

S_n, and P any point of that space

$$[A_2 \dots A_{n+1}P]-[A_1A_3 \dots A_{n+1}P]+[A_1A_2A_4 \dots A_{n+1}P]-\dots$$
$$\dots \pm [A_1A_2 \dots A_{n+1}] = 0.$$

This formula, indeed, is obtained by means of the last proposition of § 3, identifying A_{n+2} with P, and applying

$$a_1+a_2+\dots+a_{n+2}=0.$$

The equation may also be written, the space

$$A_2 \dots A_{n+1}-A_1A_3 \dots A_{n+1}+A_1A_2A_4 \dots A_{n+1}, \text{ \&c.},$$

being denoted by I,

$$[IP] = \mp [A_1A_2 \dots A_{n+1}].$$

I is defined as a certain form of manifoldness $n-1$ in space of manifoldness n. Two S_{n-1}'s contained in a S_n have a S_{n-2} in common. Therefore a linear form of such S_{n-1}'s is again some S_{n-1}; and consequently I must be some S_{n-1}. The above equation, however, shows that all points P form with I one and the same magnitude. This is an apparent contradiction which has to be explained.

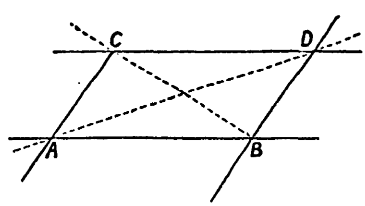
Let A, B, C, D be the four corners of a parallelogram. The diagonals bisecting each other, their point of intersection must be

$\frac{1}{2}(A+D)$ and also $\frac{1}{2}(B+C)$

showing that

$$A+D = B+C,$$

or $$A-B = C-D.$$



We found that $\alpha A+\beta B$ expressed the $\alpha+\beta$-ple of a point on the line AB. Here we see that, when $\alpha+\beta=0$, the point in question also belongs to any line parallel to AB. The significance of $\alpha+\beta=0$ is shown by letting $\alpha+\beta$ assume continuously varying values, with zero as limit. Let, for instance, α be stable $=1$, β approach -1 as limit. The point $P=\alpha A+\beta B$ will then travel away from the points A, B; the more nearly β approaches -1, the farther away P will move, and the more nearly the value of $[PA]:[PB]$ will approach unity. In the limit $\beta \doteq -1$, P will be at infinity; *$A-B$ is therefore $\equiv$ the point at infinity of the line AB.*

The equation

$$[IP] = \mp [A_1 \dots A_{n-1}]$$

may now be interpreted as meaning

$$[IP] = [IQ],$$

where Q is any other point of the S_n; or else

$$[I(P-Q)] = 0,$$

showing that I contains all the points at infinity, belonging to any line PQ in space S_n. I is accordingly called the S_{n-1} infinity.

We shall define I by the form

$$I = A_2 A_3 \dots A_{n+1} - A_1 A_3 \dots A_{n+1} + \dots$$

with the condition, however, that the value of $\mp [A_1 \dots A_{n+1}]$ giving $[IP]$ must $= 1$. Hence $[IP] = 1$, where P is any finite point;

therefore $\quad A_2 A_3 \dots A_{n+1} - A_1 A_3 + \dots A_{n+1} + \dots = c \,.\, I,$

where c is the value of $\mp [A_1 \dots A_{n+1}]$.

Two S_{n-1} in the S_n, say S and T, which are parallel, intersect in I.

Indeed let AB be any line in S. From A let fall a perpendicular on T, cutting it in A'. In the plane BAA' draw the parallel $A'B'$ to AB through A', which, being perpendicular to AA', must belong to T. But AB and $A'B'$ have in common their point of intersection with I. AB being perfectly arbitrary, it is evident that the S_{n-2} in which S intersects I must also belong to T.

S, T, I are therefore connected by a linear equation

$$aS + bI = T.$$

To find the significance of a and b we compose this equation with P, where P is any point of S.

Thus we obtain $\quad b = [TP],$

showing that b is the perpendicular distance of S from T. Composing with any point D of I,

$$a[SD] = [TD].$$

It will afterwards be shown that $[SD]$ is the sine of the angle that S forms with the lines that pass through D. So then

$$a = 1,$$

since parallel spaces form the same angle with any line, and

$$bI = T - S$$

(where T and S are in their normal form).

Let $A, B, \dots L$ be any pyramid fixing a S_n. Since $A-B, A-C, \dots$ $\dots K-L$, all belong to I, it is evident that I contains all the points

$$xA + yB + \dots + zL,$$

for which $$x + y + \dots + z = 0.$$

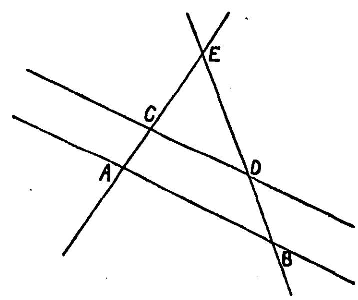
The latter is therefore the equation of I when coordinates are used.

Let A, B, C D be four points in a plane connected by the relation

$$A - B = \lambda\,(C - D)\,;$$

AB must be parallel to CD. From the above we obtain

$$A - \lambda C = B - \lambda D,$$



showing that the point E of intersection of AC and BD divides the segments AC and BD in the same ratio $\lambda : 1$, which is also $= \dfrac{[AB]}{[CD]}$. This makes it evident that the symbol $A-B$ is expressive of a certain direction as well as a certain magnitude; the magnitude being $[AB]$ and the direction being marked by the point at infinity of the line AB.

$A-B$ denotes therefore a certain "sect," parallel to AB and equal to it in length; $\lambda\,(A-B)$ denotes, similarly, a sect parallel to AB and equal to $\lambda\,[AB]$ in length (although, to be quite strict, such symbols should be regarded as denoting the point at infinity $A-B$ common to all lines parallel to AB, multiplied into a certain magnitude).

If P is a point, D any sect $= \lambda\,(A-B)$, then $P+D$ is a point, easily constructed by drawing through P in the plane PAB the line PQ parallel to AB and equal in length to $\lambda\,[AB]$. The magnitude of a sect may be positive or negative, according to the law of signs introduced in § 2. It may be denoted by the symbol $[\dots]$. Thus $[D]$ denotes the magnitude of the sect D. In applications where only its absolute value is considered, regardless of the sign, that value may be written, in conformity with a notation already in vogue, thus, $|\,D\,|$.

Let $D_1, D_2, \dots D_n$, be any n sects; $Q = P + D_1 + D_2 + \dots + D_n$ is found, as is easily seen, by describing a polygon whose one corner is P

and whose sides are successively parallel and equal in length to $D_1, D_2, \ldots D_n$. The ultimate point of this polygon is Q. It closes when

$$D_1+D_2+\ldots+D_n = 0.$$

For some applications of the preceding theory (although not affecting our present purpose) it may be well to remark that

$$| D_1+D_2+\ldots+D_n |$$

is, at most, equal to

$$| D_1 | + | D_2 | + \ldots + | D_n | ;$$

and that $$D_1+D_2+\ldots+D_n+\ldots, \lim n = \infty,$$

converges towards a definite sect of finite length whenever the series $| D_1 | + | D_2 | + \ldots + | D_n | + \ldots, \lim n = \infty$, is convergent; and *vice versa*. This is immediately seen by considering the geometrical significance of $D_1+D_2+\ldots+D_n$.

Points at I obey the same laws, so far as their composition is concerned, as points in the finite portion of the space S_n. So, then, D_1D_2 denotes the line at infinity joining the points at infinity D_1, D_2. We may represent D_1D_2 by means of triangles of a certain magnitude whose plane is parallel to a certain plane. The magnitude in question is $[P(P+D_1)(P+D_2)]$; the plane is PD_1D_2, P denoting any point whatever.

Generally the geometrical substrate of $D_1D_2 \ldots D_n$, *i.e.*, the space S_{n-1} of these points at infinity, is a pyramid of n manifoldness, whose space is parallel and which is equal in magnitude to the pyramid

$$P(P+D_1)\ldots(P+D_n),$$

P being perfectly arbitrary. The magnitude of the pyramid

$$P(P+D_1)\ldots(P+D_n)$$

may, for shortness, be denoted by

$$[D_1D_2\ldots D_n],$$

which is allowable, this value being quite independent of how P is chosen (as may, for instance, be shown by the theorem of composition).

If I' is any space, D any point, both at I, then $[I'D]$ or the magnitude formed with space I' by point D is the sine of the angle which the direction of D forms with that of the space I'; the angle DI'

being defined as the angle formed by any line PD with its perpendicular projection on space PI'. For this is exactly the magnitude of a pyramid whose base is a pyramid of magnitude 1 in PI' and whose vertex is $P+D$ (where D is of length 1). $[D_1 D_2 \dots D_n]$ is therefore calculated by multiplying the $[D_1][D_2] \dots [D_n]$ by the sine of the angle $D_1 D_2$, this again by the sine of the angle that $D_1 D_2$ forms with D_3 &c., according to the theorem of composition.

The conclusions of § 3 applied to sects show that the finite points of the space $PD_1 \dots D_n$ are expressible in the form

$$P + x_1 D_1 + \dots + x_n D_n,$$

where the x_i may assume any values whatever.

If S is any S_{n-1} in a S_n, then $[SP]$ denotes the length of the perpendicular from P on S. Interpreted in this manner, $[IP]$ would not $= 1$, but be infinite. The explanation is that I belongs to a class of spaces (of which it is the only real representative) to which the conception of normal form as originally given does not apply. The reason for this will very soon appear.

If P is any finite point, D a variable point at I in its normal form, then $P+D$ will cover one half of the surface of a spherical manifoldness whose centre is P, the other half being represented by $P-D$. The geometry on the surface of a spherical manifoldness is therefore identical, in metrical as well as projective relations, with the geometry of points at I.

7. The calculus whose outlines have been laid down in the preceding paragraphs may be divested of its geometrical meaning; and it will then become a calculus of linear forms and of determinants.

Indeed, let A, B, ... L be n linear forms in n homogeneous variables, and let $AB \dots L$ denote the corresponding determinant; AB a matrix of two rows, the other rows C', D', ... L' left indeterminate; ABC similarly a matrix of three rows, &c. And let any equation such as, for instance,

$$\mathrm{a}.AB + \mathrm{b}.CD + \mathrm{c}.EF = 0,$$

if valid at all, be understood as an abbreviation of an identity between matrices (AB, CD, EF) where the rows left indeterminate in these matrices are supposed to be identical. Then it is, indeed, easily enough seen that the laws of the geometrical calculus are expressions of elementary properties of determinants. For instance,

$$AA = 0$$

would signify that the equality of two rows in a determinant causes it to vanish,

$$AB = -BA$$

that the transposition of two rows makes it change its sign.

$$(A+B)\,C = AC+BC$$

is easily seen to follow from the elementary fact that a determinant is a linear function of the terms of each row or column. And, finally, the significance of

$$ABC \ldots L = 0$$

is obviously that $A, B, C, \ldots L$ are connected by some linear equation; that they are not "linearly independent."

8. The calculus is applicable to the geometry whose elements are:

(1) Plane spaces S_k through a fixed S_{k-1}.

(2) Plane spaces S_{n-1} in a fixed space of n manifoldness.

Indeed, any two S_k having S_{k-1} in common may be linearly connected so as to form another S_k through that S_{k-1}. Any element of this geometry may be generated by composing the fixed S_{k-1} with points outside of it; any linear manifoldness of elements of this geometry by composing the S_{k-1} with plane spaces outside the S_{k-1}. A space S_n will obviously be of manifoldness $n-k$ in regard to the elements of this geometry, the fixed S_{k-1} being contained by the S_n. If

$$E_1 = S_{k-1}A_1,$$
$$E_2 = S_{k-1}A_2,$$
$$\ldots \quad \ldots \quad \ldots$$
$$E_h = S_{k-1}A_h$$

are elements of this geometry, we need only identify

$$[E_1 \ldots E_h] \text{ with } [S_{k-1}A_1 \ldots A_h],$$
$$E_1 \ldots E_h \text{ with } S_{k-1}A_1 \ldots A_h,$$

and the whole theory of point geometry is at once transferred to this geometry.

(2) is proved by reference to §5, whence it appears that the S_{n-1} of a S_n form a linear manifoldness of degree n; that they are linearly expressible by the $n+1$ border S_{n-1} of any non-vanishing pyramid in the S_n; and that any linear form of these border S_{n-1} represents again, unconditionally, a S_{n-1}.

As regards geometry (1) $[E_1 \ldots E_h]$ (in the new sense) was found to be $= [S_{k-1}A_1 \ldots A_h]$ (in the old sense). If S is any one of its spaces,

$$S = S_{k-1}\xi$$

in its normal form (according to the original definition), and E any element

$$E = S_{k-1}A,$$

also in its normal form, then

$$[SE] = [S_{k-1}\xi A]$$

is, according to the theorem of composition, the perpendicular distance of A from S. If then the angle formed by two lines emanating from a point in the S_{k-1} into S and E respectively, and perpendicular to the S_{k-1}, is called the angle $\angle SE$, then it is easily seen that

$$[ES] = \sin \angle SE.$$

If the S_{k-1} lies entirely at I, then $[SE]$ is similarly seen to be the perpendicular distance of the two parallel spaces S and E. It is a remarkable fact that the same is true in geometry (2), as will be shown by the following line of reasoning.

The geometry (2) may be called the "reciprocal" geometry, and its composition be denoted by a vertical line (so that A/B would designate the space composed in this geometry by A and B).

Let then $ABCDE \ldots L$ be any pyramid in the fixed S_n; and take for definiteness

$$X = ABCDE,$$

$$Y = CDEFG \ldots L.$$

Then X and Y will have the plane CDE in common, and no point besides; since, if

$$P = aA + bB + cC + dD + eE$$

were a point of X also contained by Y,

$$YP = 0$$

would necessarily imply

$$[AB \ldots L] = 0,$$

or else $$a = 0, \quad b = 0.$$

Let, further, $A_1 \ldots A_{n+1}$ be another pyramid, and

$$A = a_{1,1} A_1 + \ldots + a_{1,n+1} A_{n+1},$$
$$\vdots$$
$$L = a_{n+1,1} A_1 + \ldots + a_{n+1,n+1} A_{n+1}.$$

Then the coordinates of CDE expressed by the border planes of the A_i are determinants of the third order in the $a_{i,j}$. The coordinates of X and Y are determinants of order 5 and $n-1$ in the $a_{i,j}$. Now X/Y is formed in accordance with the rules of the calculus. If

$$X = c_1 P_1 + c_2 P_2 + \ldots,$$
$$Y = c_1' P_1' + c_2' P_2' + \ldots,$$

the c being constants and the P border-spaces of the A_i, then

$$X/Y = c_1 c_1' P_1/P_1' + c_1 c_2' P_1/P_2' + \ldots.$$

The result must be CDE, as we found before, multiplied by some constant. This constant must be in the $a_{i,j}$ of order $n+1$. And it will never vanish so long as the assumption made is complied with, *i.e.*, so long as $[A \ldots L]$ is different from zero. Therefore it cannot be different from $[A \ldots L]$ itself.

If, then, BX is a S_{n-1}, A a point outside of it,

$$AX/BX = [ABX].X,$$

as is seen by considerations similar to the above. From this the proposition to be demonstrated (which might be called the sine theorem) follows exactly as in case (1). The factor of composition (in the reciprocal geometry) of ξ/η, where η is a S_{n-1}, and ξ any space, is the sine of the angle formed by ξ and η. If ξ is parallel to η, it is their distance; and the same is true when ξ is a point.

The S_{n-1} I forms with any finite S_{n-1} in its normal form the magnitude 1. Indeed, let A, B be two parallel S_{n-1} in their normal form, such that their distance is $= 1$.

Then $$I = A - B;$$

therefore $[IB]$ in the reciprocal sense

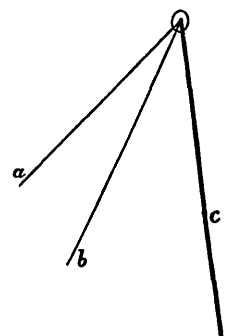
$$= [AB] \text{ in the reciprocal sense} = 1,$$

and, similarly, $$[IA] = -[BA] = 1.$$

With this the formal laws of the calculus are complete, since, by means of the theorem of composition in the original and reciprocal form, the coefficients occurring in any piece of work can always be determined.

9. Infinity is represented in the two geometries introduced in the last section in a manner very different from that in which it was expressed in point geometry.

Let O be a fixed point through which pass three rays a, b, c, situated in the same plane.



Let $\angle ab$ be denoted by γ,

$\angle bc$ „ „ α,

$\angle ca$ „ „ β,

a, b, c may be in their normal form. They are then connected by the relation

$$[bc]\,a+[ca]\,b+[ab]\,c=0,$$

$$\sin\alpha\,.\,a+\sin\beta\,.\,b+\sin\gamma\,.\,c=0.$$

If a, b are fixed, and c varies, then the angles α, β will vary. As long as α and β remain real, $\sin\alpha$ and $\sin\beta$ will also remain real and determinate quantities. There is no reason why we should restrict ourselves to real values only, the right of existence of imaginary quantities and geometrical entities in geometry having been long affirmed. If α and β assume, then, complex values, $\sin\alpha$ and $\sin\beta$ will still remain definite. This ceases only when α and β become infinite.

Let us now investigate the meaning of $\sin\alpha$ and $\sin\beta$ becoming infinite. α, β, γ being connected by the relation

$$\alpha+\beta+\gamma=0,$$

where γ is constant, $\sin\alpha$ and $\sin\beta$ will become infinite simultaneously. From

$$\sin^2\alpha+\cos^2\alpha=1,$$

$$\lim\frac{\cos\alpha}{\sin\alpha}=i \qquad (i^2=-1),$$

when

$$\lim\sin\alpha=\infty.$$

From $\sin\alpha\cos\gamma+\cos\alpha\sin\gamma+\sin\beta=0$, dividing by $\sin\alpha$,

$$\lim\frac{\sin\beta}{\sin\alpha}=-(\cos\gamma\pm i\sin\gamma),$$

$$\lim\sin\alpha=\infty.$$

In words: Let a, b be any two lines in a plane, so that $\lambda a+\mu b$, where λ and μ are any two constants, is $\equiv$ a line in their plane, and through their point of intersection. If γ is the angle formed by a and b, then $\lambda : \mu = -(\cos\gamma \pm i \sin\gamma)$ determines two lines of this pencil, which form with a and b, and therefore with any other line of the pencil, an infinite magnitude. These two lines are always distinct from each other, since $\sin\gamma$ must be different from zero. The two values $\lambda : \mu$ to which they belong are given by the equation

$$\lambda^2+2\lambda\mu\cos\gamma+\mu^2 = 0.$$

If $$\gamma = R,$$

then $$\cos\gamma = 0,$$

and the equation becomes $$\lambda^2+\mu^2 = 0,$$

showing that the two exceptional lines—isotropic lines as they are called—divide any pair of lines a, b perpendicular to each other in a harmonic ratio. These two lines are therefore the double lines of an involution, determined by pairs of lines through O at right angles to each other. The involution of these lines is projected on to line I of plane ab into an involution of points independent of O; and there are therefore two points (the circular points) on I determined as the double points of the involution at I of points at right angles with each other. Through one of these two points all isotropic lines must pass.

According to one of our elementary propositions the totality of lines situated in space S_n which are perpendicular to a line l and pass through a certain point A on l form a plane space of manifoldness $n-1$. This may be put differently by considering only the I of the space thus: to any point D of I corresponds a certain Σ_{n-2} situated at I called perpendicular to D; and *vice versa*, Σ being given, D is uniquely determined. A correspondence of such a nature may be conceived, as is well known, as resulting from polarization upon some quadric surface. The quadric surface thus determined at I will be denoted by J. It has received different names, one of which is "the imaginary spherical manifoldness at infinity." But we shall avoid giving this quadric a special name, only reserving the letter J for it.

We are not dependent for its definition on projective geometry. The following is an independent investigation, to prove the (abstract) existence, and to show the significance of that formation.

Let a, b, c be three rays emanating from a point O in a space S_3. Then any other ray d in the S_3 through O is a linear form in the a, b, c,

$$d = \lambda a + \mu b + \nu c.$$

Each plane through O will contain two isotropic lines; the totality of the isotropic lines through O in the S_3 form therefore a cone, which is cut by any plane through its vertex in two lines, and is therefore of the second order. The angle a, b being γ, that of b, c being α, and that of c, a being β, the isotropic lines situated in the planes ab, bc, ca, respectively, are the three pairs defined by

$$\nu = 0, \quad \lambda^2 + \mu^2 + 2\lambda\mu \cos\gamma = 0,$$

$$\lambda = 0, \quad \mu^2 + \nu^2 + 2\mu\nu \cos\alpha = 0,$$

$$\mu = 0, \quad \nu^2 + \lambda^2 + 2\nu\lambda \cos\beta = 0,$$

respectively. It follows that the equation of the isotropic cone must be

$$\lambda^2 + \mu^2 + \nu^2 + 2\lambda\mu \cos\gamma + 2\mu\nu \cos\alpha + 2\nu\lambda \cos\beta = 0.$$

This cone will cut I in a conic, determined by the three point-pairs in which it is cut by the lines infinity of ab, bc, ca, respectively; and which is therefore quite independent from O. It is this conic which we designate by J. If D_1, D_2, D_3 are any three points at I, forming with each other angles γ, α, β, respectively, then the conic J will contain all points $\lambda D_1 + \mu D_2 + \nu D_3$, for which above equation is satisfied. More especially, if γ, α, β are all equal to R, then the equation of J will be

$$\lambda^2 + \mu^2 + \nu^2 = 0.$$

If D is any point at I in its normal form, then the "cond" that D should belong to J (by which we denote that function of the coordinates of the formations considered, or that magnitude, which must vanish whenever the condition in question is satisfied), if D does not belong to J, is $= 1$. Indeed, let D_1, D_2, D_3 be three points at right angles at I. Let

$$D = a_1 D_1 + a_2 D_2 + a_3 D_3.$$

Then the cond $\qquad JD = a_1^2 + a_2^2 + a_3^2.$

Now, if P be any finite point, $P+D = Q$ another, then

$$JD = J(Q-P)$$

gives, according to the theorem of Pythagoras in space, the square of the distance from Q to P, which is $= 1$, since D is in its normal form.

In exactly the same manner it may be shown in regard to any space S_n.

(1) That the totality of the isotropic lines through any finite point P in a S_n cuts I in a certain surface J, of the second order, independent of P.

(2) That this surface contains all points

$$\lambda_1 D_1 + \lambda_2 D_2 + \ldots + \lambda_n D_n,$$

for which
$$\Sigma \lambda_i^2 = 0,$$

where the D_i are any sects in their normal form at right angles to each other (a configuration whose existence is easily shown by induction).

(3) That for any point D at I in its normal form we have

$$JD = 1,$$

with the exception, of course, of the points belonging to J (for which a normal form does not exist).

Let now D_1 and D_2 be any two points at I. If λ, μ be any two values, $J(\lambda D_1 + \mu D_2)$ is a homogeneous form of the second order in λ, μ; and, since

$$J(D_1) = J(D_2) = 1,$$

it must be
$$J(\lambda D_1 + \mu D_2) = \lambda^2 + \mu^2 + 2\lambda\mu \,.\, K,$$

where K is some constant depending solely on D_1 and D_2. It is this constant K that H. Grassmann calls "the inner product" of the two sects D_1 and D_2. To find its significance consider the values of $\lambda : \mu$ for which the quantic of the second order vanishes. Its two roots obviously indicate the position of the two points $\lambda D_1 + \mu D_2$ in which the line $D_1 D_2$ cuts J. But their equation is, as we know,

$$\lambda^2 + \mu^2 + 2\lambda\mu \cos(D_1, D_2) = 0.$$

So, then,
$$K = \cos(\angle D_1 D_2).$$

If, generally, D_1, D_2 are any two sects, not necessarily in their normal form, then their "inner product" is the factor of $2\alpha\beta$ in the development of $J(\alpha D_1 + \beta D_2)$, according to powers of α, β. Geometrically, it is the length of the one multiplied by the perpendicular projection of the other upon it; or $[D_1][D_2]\cos\angle D_1 D_2$. Originally H. Grassmann introduced the sign $\times$ to denote the inner product. Later on he abandoned this way of writing. In mechanics, where the surface J and the inner product are probably destined to be of much use, such a short sign would have its advantages.

Let $D\times E$ denote the inner product of two sects D, E. Let F be any other sect. Then

$$(D+F)\times E = D\times E + F\times E.$$

This follows readily, for instance, from the algebraical definition, since obviously the factor of $2\alpha\beta$ in the development of

$$J\{\alpha(D+F)+\beta E\}$$

is the sum of the factors of $2\alpha\beta$ in the corresponding development of $J(\alpha D+\beta E)$ and $J(\alpha F+\beta E)$.

The equation of J may be written in a very simple form. Let $D_1, \dots D_n$ be any n linearly independent points at I. Then a point $\lambda_1 D_1 + \dots + \lambda_n D_n$ will belong to J if

$$(\lambda_1 D_1 + \dots + \lambda_n D_n)\times(\lambda_1 D_1 + \dots + \lambda_n D_n) = 0.$$

If the D_i are in their normal form, this is equivalent to

$$\Sigma\lambda_i^2 + \Sigma 2\lambda_i\lambda_j \cos\angle D_i, D_j = 0,$$

a form which might have been found by our original process.

To bring any sect $\quad D = \lambda_1 D_1 + \dots + \lambda_n D_n$

to its normal form, it is necessary to divide it by the square root of JD, *i.e.*, by

$$\sqrt{\lambda_1^2 + \dots + \lambda_n^2 + 2\lambda_1\lambda_2\cos\angle D_1 D_2 + \dots}.$$

The investigation carried on so far might be pursued on the same lines for the geometry of spaces S_k through a fixed S_{k-1}, or of S_{n-1} in a fixed S_n. We shall designate by the name of isotropic spaces the two spaces of any pencil that form an infinite magnitude with any other space of that pencil. If A, B are any two spaces of k manifoldness having a S_{k-1} in common, and φ the angle they form, then, just as before, $\lambda A + \mu B$ will be an isotropic space when

$$\lambda^2 + \mu^2 + 2\lambda\mu\cos\phi = 0.$$

It is then shown, in exactly the same manner as before, that

$$\lambda_1 s_1 + \ldots + \lambda_{n+1} s_{n+1}$$

(the $s_1 \ldots s_{n+1}$ denoting a system of S_{n-1} in a fixed S_n) is an isotropic space, when

$$\lambda_1^2 + \ldots + \lambda_{n+1}^2 + \Sigma\, 2\lambda_i \lambda_j \cos \angle S_i S_j = 0,$$

$\angle S_i S_j$ denoting the angle formed by S_i and S_j. An isotropic space, it will be noticed, is one which cuts J in a quadric surface having a double point. The above is therefore, if projected into I, the reciprocal equation of J.

The "cond" that a space S_{n-1} in its normal form should touch J is 1. It will be sufficient to consider the state of things at I. Assume n S_{n-2} at I at right angles to each other $a, b, \ldots l$. Put

$$S_{n-2} = \lambda a + \mu b + \ldots + \nu l.$$

The "cond" in question is

$$= \lambda^2 + \mu^2 + \ldots + \nu^2.$$

The S_{n-2} and the $a \ldots l$ being supposed to be in their normal form

$$[ab \ldots l] \text{ is } = 1,$$

$$\lambda = [b \ldots l S_{n-2}], \quad \mu = -[ac \ldots l\, S_{n-2}], \text{ \&c.}$$

$[b \ldots l\, S_{n-2}]$ is simply the sine of the angle which the point $b \mid c \mid \ldots \mid l$ forms with S_{n-2}, &c. According to the "Pythagoras" for a space S_n, the value of $\lambda^2 + \mu^2 + \ldots$ is seen to be 1.

If we form the "cond" in question for $\alpha s + \beta t$, where α, β are any two constants, s and t any two S_{n-2} at I, the result will be a quadratic function of α, β of the form $\alpha^2 + \beta^2 + 2\alpha\beta K$. Similarly, as before, K may be denoted as the inner product of s and t, written $s \times t$. We have then, if s and t are in their normal forms,

$$s \times t = \cos \angle s, t,$$

and $$(as + b.s') \times t = as \times t + bs' \times t,$$

the a, b being constants. If the $a_1, a_2, \ldots a_n$, are the border-spaces of any pyramid at I, the reciprocal equation of J is simply

$$(\lambda_1 a_1 + \ldots + \lambda_n a_n) \times (\lambda_1 a_1 + \ldots + \lambda_n a_n),$$

which is $$= \Sigma \lambda_i^2 + \Sigma \lambda_i \lambda_j \cos \angle a_i a_j,$$

and it also follows that, to bring

$$\lambda_1 a_1 + \ldots + \lambda_n a_n$$

to its normal form, it is necessary to divide it by the square root of that expression.

Among the many properties of J none seem so interesting as the one which brings it into intimate connexion with the theory of the potential function W in any space S_n. It is that J is "apolar" to W.

10. To find the trigonometrical formulæ of plane and spherical manifoldnesses, and related problems, can now be easily solved. The following is a brief account of what might be said under this head.

Let $A, B, \ldots L$ be a pyramid in space of n manifoldness; $\bar{A}, \bar{B}, \ldots \bar{L}$ the border-spaces opposite to the corner-points $A, B, \ldots L$,

$$\bar{A} = \quad BCD \ldots L,$$

$$\bar{B} = -ABD \ldots L,$$

$$\bar{C} = \quad ABD \ldots L,$$

&c.,

so that
$$A\bar{A} = B\bar{B} = C\bar{C} = \ldots = \Delta = 1,$$

where the magnitude Δ of the pyramid is assumed for convenience equal to unity.

For any point P we shall then have

$$\pm P = [\bar{A}P]\, A + [\bar{B}P]\, B + \ldots + [\bar{L}P]\, L,$$

for
$$\bar{A}A = \bar{B}B = \ldots = \pm 1;$$

therefore this will be true when $P \equiv A, B, \ldots L$; hence also when P is a linear form of multiples of $A, B, \ldots L$. Similarly, if s be any S_{n-1} in the S_n,

$$s = [As]\, \bar{A} + [Bs]\, \bar{B} + \ldots + [Ls]\, \bar{L}.$$

As a special case,
$$IA = IB = \ldots = 1,$$

$$I = \pm (\bar{A} + \bar{B} + \ldots + \bar{L}).$$

If, then, we cut this by I,

$$\text{(E)} \quad \bar{A}/I + \bar{B}/I + \bar{C}/I + \ldots + \bar{L}/I = 0,$$

$\bar{A}, \bar{B}, \ldots \bar{L}$ are not in their normal form, but appear multiplied by the magnitude of the border-pyramid in their space. This is not altered by the intersection with I. Bringing $\bar{A}/I$ to the right-hand side and forming the inner product of each side with itself, we obtain

$$[\bar{A}]^2 = [\bar{B}]^2 + \ldots + [\bar{L}]^2 + 2\,[\bar{B}]\,[\bar{C}] \cos \angle \bar{B}\bar{C} + \ldots,$$

the generalized cosine theorem for point geometry. By treating (E) differently, it can of course be given different forms.

It may be shown without any difficulty that all the magnitudes connected with a pyramid are known when the $\frac{n \cdot n+1}{2}$ quantities, representing the distances of any corner-point from any other, are known; and that these quantities are perfectly independent among themselves.

The cosine of the angles formed by the directions of any two edges which do not intersect can easily be discovered.

From $$A-B = A-D+B-C+D-C,$$

forming the "inner square" of each side,

$$\begin{aligned}[A-B]^2 = [A-D]^2 + [B-C]^2 + [D-C]^2, \\ +2\,[A-D]\,[B-C] \cos \angle AI, \\ +2\,[A-D]\,[D-C] \cos \angle ADC \\ +2\,[B-D]\,[D-C] \cos \angle BCD,\end{aligned}$$

giving $\cos \angle AD, BC$ in terms of known quantities.

The distance of any point

$$P = aA + \ldots + \lambda L$$

from any other $$Q = a'A + \ldots + \lambda L$$

is the square root of the inner product of $P-Q$; and therefore expressible by the $\frac{n+1 \cdot n}{2}$ quantities.

The cosine of the angle formed by any two spaces

$$\lambda_1 a_1 + \ldots + \lambda_{n+1} a_{n+1},$$

$$\lambda_1' a_1 + \ldots + \lambda_{n+1}' a_{n+1},$$

the $a_1, \ldots a_{n+1}$ being supposed to be in their normal form, is

$$\cos\phi = \frac{\lambda_1\lambda_1' + \ldots + \lambda_i\lambda_j' \cos \angle (a_i, a_j) + \ldots}{\sqrt{\lambda_1^2 + \ldots + 2\lambda_i\lambda_j \cos \angle (a_i, a_j)}\,\sqrt{\lambda_1'^2 + \ldots}},$$

where (a_i, a_j) denotes the angle formed by a_i, a_j. For this is the inner product of the two spaces after they have been brought to their normal form.

Connected with n S_{n-1}, having a point P in common, is a certain magnitude, which is the sine of the angle formed by any two of them, multiplied into the sine of the angle formed by any third one with the S_{n-2} common to the first two, &c. It is the factor of composition in the reciprocal geometry, the composed space being the point P. If the n S_{n-1} are

$$\lambda_{i,1}a_1+\lambda_{i,2}a_2+\ldots \equiv S_i$$

(the a being supposed to be in their normal form), then P is found

$$P \equiv L_1A_1+L_2A_2+\ldots,$$

where the L are expressed according to the rules given by the $n+1$ determinants of a matrix of $n+1$ rows and n columns, whose coefficients are the $\lambda_{i,j}$. The factor of composition of P is therefore

$$x = \frac{L_1+L_2+\ldots+L_{n+1}}{\sqrt{\lambda_{1,1}^2+\ldots+2\lambda_{1,i}\lambda_{1,j}\cos(a_i, a_j)}\sqrt{\lambda_{2,1}^2+\ldots}\cdots},$$

the n square roots being necessary to bring the n S_{n-1} to their normal form. If, for instance, $n=2$, x represents the sine of the angle formed by two lines.

The discriminant of

$$(x_1D_1+\ldots+x_nD_n)\times(x_1D_1+\ldots+x_nD_n)$$

considered as a quantic in the x_i, the D being sects of any kind, is the square of $[D_1D_2\ldots D_n]$. Indeed, if $E_2\ldots E_n$ are sects in the space $D_1\ldots D_n$ at right angles to each other, and of length 1, then

$$(x_1E_1+\ldots+x_nE_n)\times(x_1E_1+\ldots+x_nE_n) = x_1^2+x_2^2+\ldots+x_n^2,$$

a form whose discriminant may have the value 1. If we replace

D_1 by its linear equivalent in the E_i,
D_2 ,, ,, ,, E_i,
&c.,

then $$(x_1D_1+\ldots+x_nD_n)\times(x_1D_1+\ldots+x_nD_n)$$

will be expressed by the sum of n squares of forms y_i linear and homogeneous in the x_i, whose determinant is not different from the determinant of the linear forms by which the D_i were expressed by the E_i; the discriminant in question is therefore the square of this determinant. But $[E_1 \dots E_n] = 1$; therefore the discriminant is the square of $[D_1 \dots D_n]$.

Let it be required to find the distance of a point P from a line l in space S_3 by means of the coordinates of P and l.

If A, B, C, D are any pyramid in S_3,

$$P = Ap_1 + p_2 B + p_3 C + p_4 D;$$

$$l = a_1 AB + a_2 AC + \dots + a_6 CD,$$

then
$$lP = (a_1 p_3 - a_2 p_2 + a_4 p_1)\, ABC + \dots .$$

If l is in its normal form, then the weight of lP will be $[lP]$. Therefore
$$[lP]^2 = c\left\{(a_1 p_3 - a_2 p_2 + a_4 p_1)^2 + \dots\right\} = lP \times lP,$$
where c is a constant whose value is 1, when l is in its normal form, and solely dependent on the a_i.

The right-hand side will vanish only when this algebraical expression for the distance of P from l vanishes, that is, when lP is one of the isotropic planes of the pencil in the S_3 through l; then it will vanish always. The value of the perpendicular from P on l may therefore be found by constructing these two isotropic planes, and forming the "cond" that P may be contained by any one of them.

If P, Q, R are any three collinear points, l any line not intersecting PQ in the space S_3; if, further, lP forms with lQ the angle ϕ, and

$$[lP] = p, \quad [lQ] = q,$$

$l(aP + \beta Q)$ will be an isotropic plane, when

$$a^2 p^2 + \beta^2 q^2 + 2a\beta pq \cos\phi = 0.$$

Should therefore $R = \lambda P + \mu Q$, then the square (r^2) of the distance of R from l is
$$r^2 = \lambda^2 p^2 + \mu^2 q^2 + 2\lambda\mu pq \cos\phi.$$

From
$$\lambda P + \mu Q = R$$
it follows that
$$\lambda lP + \mu lQ = lR.$$

If, therefore, the angles which lP and lQ form with lR are denoted by χ and ψ, then
$$\lambda p : \mu q = \sin\psi : \sin\chi.$$

In the same way, it is hardly necessary to mention, many other apparently more complicated metrical problems may easily be solved. In the remaining portion of this section we shall investigate the perpendicular distance of any two plane spaces that have no point in common.

If A and B are any two spaces of manifoldness h and k respectively, having no point in common with each other, then one point of A and one point of B are in such relation to each other that their distance is shorter in absolute length than that of any other two points belonging to A and B respectively. The line PQ joining them is perpendicular to A as well as B, and there is no other line of this nature. Indeed, the I of the space AB is of manifoldness $h+k$, I/A is of manifoldness $h-1$, and I/B is of manifoldness $k-1$. To I/A corresponds a space Σ_1 at I, of manifoldness k, whose every point is perpendicular to I/A; and similarly to I/B a Σ_2, of manifoldness h. Σ_2 and Σ_1 have a point D in common, perpendicular to I/A as well as I/B.

Through any point of the space AB it is possible to draw one line to cut A as well as B. This was shown in § 3. And only one such line can be constructed if $[AB]$ is different from zero, an assumption with which we started. The line l thus belonging to D is therefore the only line cutting A as well as B at right angles.

This line l will intersect A and B in points P, Q. The distance of P to Q is measured by the perpendicular distance of P from the space of manifoldness $h+k$ composed by B and I/A, parallel to A. This distance is the same from every point on A. It is, to write it symmetrically, the magnitude formed by $B\,I/A$ with $A\,I/B$, and thus, as a rule, easy to calculate when the coordinates of A and B are given.

11. To make this essay somewhat complete it will be necessary to discuss, in a few words, the theory of projection, or of linear transformation, as it presents itself in plane spaces of any manifoldness.

Many words are unnecessary on this subject, since it has long been exhaustively treated. Let any two spaces of the same manifoldness, for definiteness point-spaces, be put into a projective correspondence with each other. Then to any point A in the one space S corresponds one point A' in the other space S'. If any two pyramids are fixed in S, S', the coordinates of the points of S and S' are mutually expressible as linear functions of each other. It follows, that to any point A in S will generally correspond the multiple of some point A' of S'. And, if

$$A \text{ corresponds to } aA',$$

$$B \quad \text{,,} \quad \text{,,} \quad bB',$$

then $$\alpha A+\beta B \quad \text{,,} \quad \text{,,} \quad \alpha aA'+\beta bB'.$$

Let now

$$\begin{array}{llll} A & \text{correspond} & \text{to} & aA', \\ B & ,, & ,, & bB', \\ \dots & ,, & ,, & \dots, \\ L & ,, & ,, & lL', \end{array}$$

where $A \dots L$ is any pyramid in S, $A' \dots L'$ the corresponding one in S'.

Any point $M = \alpha A + \dots + \lambda L$

will correspond to $M' \equiv \alpha a A' + \dots + \lambda l L'$.

It is therefore seen that $n+2$ points determine the correspondence. For let

$$\begin{array}{llllll} A & \text{be projectively related to} & & & & A', \\ B & ,, & ,, & ,, & ,, & B', \\ \dots & ,, & ,, & ,, & ,, & \dots, \\ L & ,, & ,, & ,, & ,, & L', \\ M & ,, & ,, & ,, & ,, & M', \end{array}$$

$$M = \alpha A + \dots + \lambda L,$$

$$M' = \alpha' A' + \dots + \lambda' L';$$

then $a = \alpha' : \alpha \dots l = \lambda' : \lambda$,

and everything is known.

If S and S' are brought to coincidence with each other, then one pyramid $PQR \dots S$ will exist, whose corner points correspond to themselves.

Indeed, assume $P = \alpha A + \dots + \lambda L$,

$$P' = \alpha a A' + \dots + \lambda l L'.$$

If, then, P' and P coincide, for some value of ρ we must have

$$\alpha (\rho A - aA') + \beta (\rho B - bB') + \dots + \lambda (\rho L - lL') = 0.$$

If such equation connects

$$\rho A - aA', \quad \rho B - bB', \quad \dots,$$

then that pyramid must vanish,

$$\left[(\rho A - aA')(\rho B - bB') \dots (\rho L - lL') \right] = 0,$$

an equation in ρ of order $n+1$, which has therefore $n+1$ roots.

To each value of ρ will belong a system of values $\alpha, \beta, \dots \lambda$, connecting

$$\rho A - aA' \dots \rho L - lL',$$

so that the corner-points P of the pyramid whose existence was asserted can be found. In general, the ρ will be distinct from each other. If they are not, limiting processes will explain what then takes place. But this is not of much importance for the immediate objects of this essay.

Thursday, December 10th, 1896.

Prof. ELLIOTT, F.R.S., President, in the Chair.

Present—twenty-two members and four visitors.

The following gentlemen were elected members:—John Borthwick Dale, B.A., late Scholar of St. John's College, Cambridge; Charles Samuel Jackson, M.A., Instructor in Mathematics, Royal Military Academy, Woolwich; Arthur William Ward, M.A. St. John's College, Cambridge, Professor of Mathematics and Physics, Canning College, Lucknow, India. Mr. S. S. Hough was admitted into the Society.

The Auditor (Mr. Terry), having read his report, complimented the Treasurer on the way in which he had performed his duties. Mr. Kempe moved, and Mr. Bickmore seconded, a vote of thanks to the Auditor for his services. The vote was carried unanimously. A motion was then made by the President, and seconded by Lt.-Col. Cunningham, and carried unanimously, for the acceptance of the Treasurer's report. Dr. Larmor suitably acknowledged the compliment.

Major MacMahon gave a sketch of the result arrived at in Prof. Sylvester's "Note on a Discovery in the Theory of Denumeration." In connexion with this paper the President announced that Prof. Sylvester had given permission to the Society to publish the "Outline of Lectures on the Partitions of Numbers," which he read at King's College, London, in 1859, and which had never been published; and that the Council had arranged to print the "Outlines" as a companion to the late President's Valedictory Address.

Mr. Burbury communicated a paper "On the Stationary Motion of a System of Equal Elastic Spheres of Finite Diameter."

Mr. Hough read a paper "On the Influence of Viscosity on Waves and Currents."

Mr. Macfarlane Gray gave a description of his Multiplying apparatus. Messrs. C. V. Boys, Dewar, and Greenhill joined in a discussion of points connected with the subject of Mr. Gray's communication, and a cordial vote of thanks was passed to these gentlemen.

The multiplying apparatus consists of two principal parts, a sole frame and a grid. In the sole frame the product cards for the multiplicand are set up in order. These are the same as what are called "Napier's rods," being each the products in one column of the multiplication table, up to 9 times 9, with a card for the 0 column. The grid is a frame fitted with a number of sliders, each of the same breadth as the product cards. Each slider has at mid length a pane of glass. The edges of the sliders coincide with the vertical centres of the cards, when superposed, so that each pane lies over the unit place of one card, and the place of tens in the adjacent card. The sliders are set to bring the panes each over the product lines for one figure in the multiplier, taking the figures in the order the reverse of that in which the multiplicand has been set up. The grid frame is fitted to slide over the card frame upon stepped guides, the steps insuring the proper relative positions when reading the products. The sliders may be of leather with the glass panes cemented on. There is a figured plate for setting the sliders by.

To obtain the product of two multidigital numbers, the cards for the figures in one of them are set up on the sole frame, and the sliders in the grid are set for the figures in the other. The grid is then moved linearly over the sole frame, moving one figure at a time, and at each step the components of the products in one of the vertical columns of the ordinary multiplication are exhibited at the panes of the multiplier sliders. These are added together, giving one figure of the required product; the grid is then slid on to the next figure, and the next vertical column is then shown. In this way the final product is obtained without transcribing the intermediate products.

							0
	6		3		7		1
1	2		6	1	4		2
1	8		9	2	1		3
2	**4**	**1**	2	2	8		4
3		1	5	3	5		5
3	6	1	**8**	**4**	2		6
4	2	2	1	4	9		7
4	8	2	4	5	**6**		8
5	4	2	7	6	3		9

shows three cards of the multiplicand 637 being multiplied by 864. The multiplier is set to the required figures by adjusting the sliders till the figures in the column to the right (under 0) are seen through the windows.

Each figure in the product is the sum of the black figures seen through the windows. Through each window two figures are seen, but they are not on the same card. The 3 in the product is given in this position.

By the Machine.

```
   637
   864
  ----
   428
  212
  ----
   682
  314
  ----
   846
  425
------
550368
```

By Ordinary Multiplication.

```
   637
   864
  ----
  2548
 3822
5096
------
550368
```

Lt.-Col. Cunningham stated some results arrived at in his paper "On the Connexion of Quadratic Forms." Upon a portion of these results Mr. Bickmore made some supplementary remarks.

The following papers were taken as read :—

Concerning the Abstract Groups of Order $k!$ and $\frac{1}{2}k!$ Holoedrically Isomorphic with the Symmetric and the Alternating Substitution Groups on k Letters: Prof. E. H. Moore.

On a Series of Co-Trinodal Quartics: Messrs. H. M. Taylor and W. H. Blythe.

On Finite Variations: Mr. E. P. Culverwell.

The following presents were received for the Library:—

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xx., St. 10; Leipzig, 1896.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome xxx., Liv. 3; Harlem, 1896.

"Wiskundige Opgaven met de Oplossingen door de Leden van het Wiskundig Genootschap," Deel VII., St. 2; Amsterdam, 1896.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. III., No. 2; New York, 1896.

"Festschrift der Naturforschenden Gesellschaft in Zürich, 1746–1896," Teile 1, 2; Zürich, 1896.

"Bulletin des Sciences Mathématiques," Tome xx., Oct., 1896; Paris.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. II., Fasc. 8–10; Napoli, 1896.

"Transactions of the Canadian Institute," Vol. v., Pt. 1, No. 9; Toronto, October, 1896.

"Rendiconti del Circolo Matematico di Palermo," Tomo x., Fasc. 5; 1896.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 2, Vol. v., Fasc. 9, 10; Roma, 1896.

"Journal of the College of Science, Tokyo," Vol. x., Pt. I.; 1896.

"Journal für die reine und angewandte Mathematik," Bd. CXVII., Heft 2; Berlin, 1896.

"Annales de la Faculté des Sciences de Toulouse," Tome x., Fasc. 3, 4; Paris, 1896.

"Educational Times," December, 1896.

"Indian Engineering," Vol. xx., Nos. 17–20, Oct. 24–Nov. 14, 1896.

Presented by Mr. J. Hammond:—

"Commercium Epistolicum D. Johannis Collins et aliorum de Analysi promota"; Londini, 1722.

"The Method of Increments, wherein the Principles are Demonstrated, and the Practice thereof shown in the Solution of Problems" (by W. Emerson); London, 1763.