DIFFRACTION OF WAVES BY A WEDGE

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[Received January 14th, 1915.—Read May 13th, 1915.]

THE following treatment of the wedge-problem starts from the idea of generalizing the familiar process of taking images, as used in electrostatics. It will be remembered that in the ordinary elementary problems, success depends on the fact that the angle outside the conductor is of the type π/n, where n is a positive integer: so that the angle of the wedge is then $2\pi-\pi/n$.

The method of generalization is to replace the sum of the effects of n images by a complex integral; and then to extend the integral so as to obtain a formula valid for any positive value of n (that is, for any angle of the wedge). The extension required is found to be a modification of the path of integration; the subject of integration remains unchanged. The discussion of the various points involved in this generalization occupies the greater part of the paper (§ 2).

Starting from § 2 it is possible to solve (§§ 3, 4) the diffraction-problem for impact on a wedge (i) of sound waves from a source, (ii) of electromagnetic waves from a Hertzian oscillator with its axis parallel to the edge of the wedge, (iii) of electromagnetic waves from an oscillating magnet, also parallel to the edge.

The problem (i) has been solved by Prof. Macdonald for a simple harmonic source; it is here extended to a source of any type. The problems (ii) and (iii) are new, so far as I know.

1. *A General Integral of the Wave-Equation.*

It is evident that the complex integral

$$U=\int\frac{f(R-ct)}{R}\,g(\xi)\,d\xi$$

is a solution of the fundamental wave-equation, where

$$R^2 = r^2+r_0^2+(z-z_0)^2-2rr_0\cos(\theta-\xi),$$

and (r, θ, z) are the cylindrical coordinates of a variable point, and (r_0, θ_0, z_0) are those of a fixed point.

In order to make R single-valued in the plane of the complex variable ξ, it will be convenient to make cuts as shown, parallel to the imaginary axis, from the branch-points of R; these branch-points are given by

$$\xi-\theta = 2k\pi \pm \iota a,$$

where $$2rr_0\cosh a = r^2+r_0^2+(z-z_0)^2,$$

and k is any integer (positive or negative) or zero.

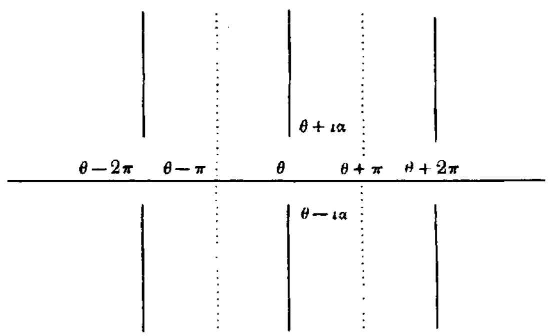

FIG. 1.

The value of R is fixed if we agree that R is to be positive, say, on the real axis; it is easy to see that R is then also positive along the lines (indicated by dots) which are midway between the cuts.

The phase of R will be $+\frac{1}{2}\pi$ on the right-hand edge of each of the upper cuts, and $-\frac{1}{2}\pi$ on the left-hand edge; for the lower cuts these signs will be reversed.

For the special cases of the potential equation and of simple harmonic waves,* this solution was first given by Sommerfeld.† Using the other methods developed by Sommerfeld in the paper quoted, we could proceed to the solution of § 2, for the special case $n = \frac{1}{2}$ (a half-plane or straight-edge) with very slight additional work: but the process fails for other values of n.

* Given by taking $f(R-ct)$ = const. and $f(R-ct) = e^{\iota\kappa(ct-R)}$, respectively.

† *Proc. London Math. Soc.*, Vol. XXVIII, 1897, pp. 405, 429.

2. *Solution of the Wave-Equation, corresponding to a Source outside a Wedge.*

Consider now the question of finding a solution V of the wave-equation corresponding to a source placed at the point (r_0, θ_0, z_0) between the two planes $\theta = 0$, $\theta = \pi/n$, subject to the condition that the solution is to vanish on both the boundary planes.

When n is an integer, the solution is given by the familiar image process of electrostatics, and can be written in the form

$$V = V_1 - V_2,$$

where
$$V_1 = \sum_{m=0}^{n-1} \frac{f(R_1 - ct)}{R_1}, \qquad V_2 = \sum_{m=1}^{n} \frac{f(R_2 - ct)}{R_2},$$

and
$$R_1^2 = r^2 + r_0^2 - 2rr_0 \cos\left(\theta - \theta_0 - \frac{2m\pi}{n}\right) + (z - z_0)^2,$$

while
$$R_2^2 = r^2 + r_0^2 - 2rr_0 \cos\left(\theta + \theta_0 - \frac{2m\pi}{n}\right) + (z - z_0)^2,$$

and the nature of the function f is given by the character of the source.

We shall now express the two functions V_1, V_2 by means of integrals of the type U defined above (§ 1). It is evident that

$$V_1 = \frac{1}{2\pi\iota} \int \frac{f(R - ct)}{R} g_1(\xi)\, d\xi,$$

provided that the path of integration is a closed loop surrounding the points
$$\xi = \theta_0 + 2m\pi/n \qquad (\text{where } m = 0, 1, 2, \ldots, n-1),$$

and that $g_1(\xi)$ is a function which has a pole of residue unity at each of these n points. It may be noticed that, since $0 < \theta_0 < \pi/n$, these n points all lie between $\xi = 0$ and $\xi = 2\pi$, on the real axis of our complex diagram; and so the path of integration may be *any* simple loop cutting the real axis at the points $\xi = 0$ and $\xi = 2\pi$.

In the subsequent argument we suppose that $g_1(\xi)$ has a period 2π in ξ, and further that $g_1(\xi)$ remains finite when ξ tends to infinity in either direction parallel to the imaginary axis (see p. 455 below). These two conditions, together with the specification of the poles, determine the function $g_1(\xi)$ completely, save for an arbitrary additive constant C: and, according to a known theorem on periodic functions, we have

$$g_1(\xi) = \tfrac{1}{2} \sum_{m=0}^{n-1} \cot \tfrac{1}{2}(\xi - \theta_0 - 2m\pi/n) + C = \tfrac{1}{2} n \cot \tfrac{1}{2} n (\theta - \xi_0) + C.$$

It is easy to see that the value of C does not affect the value of the integral for V_1: and so we may put $C = 0$.

Since the two functions R and $g_1(\xi)$ both have the period 2π in ξ, the path of integration may be any simple loop, provided that it cuts the real axis in two points whose distance apart is 2π.

We now change the variable of integration to $\zeta = \xi - \theta$, and then

$$V_1 = \frac{1}{2\pi\iota}\int \frac{f(R-ct)}{R}\, G_1(\zeta)\, d\zeta,$$

where $$R^2 = r^2 + r_0^2 + (z-z_0)^2 - 2rr_0 \cos\zeta,$$

and $$G_1(\zeta) = \tfrac{1}{2}n \cot \tfrac{1}{2}n\,(\zeta+\theta-\theta_0),$$

the path of integration still cutting the real axis in two points at a distance 2π apart. Of course, in the ζ-plane, the cuts extend from the branch-points $2k\pi \pm \iota\alpha$ parallel to the imaginary axis.

In order to have a standard diagram we suppose that the path of integration cuts the real axis at $\zeta = -\pi$, $\zeta = +\pi$ as indicated in Fig. 2.

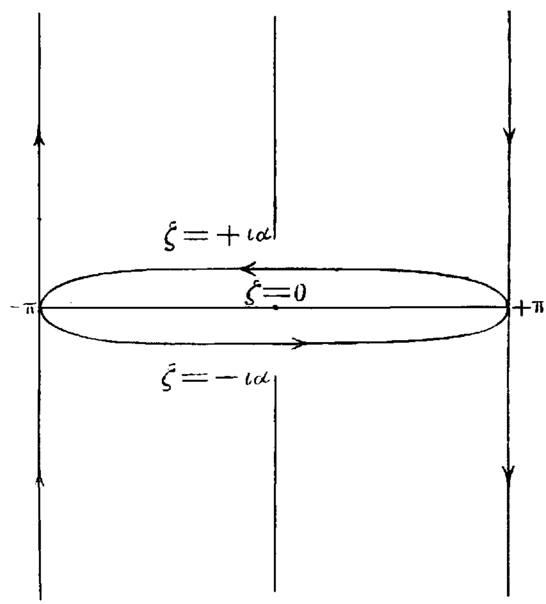

FIG. 2.

In like manner we see that

$$V_2 = \frac{1}{2\pi\iota}\int \frac{f(R-ct)}{R}\, G_2(\zeta)\, d\zeta,$$

where $$G_2(\zeta) = \tfrac{1}{2}n \cot \tfrac{1}{2}n(\zeta+\theta+\theta_0),$$

the path of integration being the same as for V_1.

We have now expressed V_1 and V_2 by means of integrals which contain n only through the two functions $G_1(\zeta)$, $G_2(\zeta)$: and it may therefore seem unnecessary to retain any longer the assumption that n is an integer. That is, we may be tempted to suppose that the corresponding solution for a wedge of *any* angle π/n, may be obtained by taking

$$V = V_1 - V_2,$$

as expressed above. It will be seen, in fact, that this function does satisfy the necessary boundary conditions, and that it has the correct infinity at the source. But, as a matter of fact, *the function V is no longer a continuous function of θ.* To explain the reason for this phenomenon, it is necessary to consider the poles of $G_1(\zeta)$, $G_2(\zeta)$, which are still given by the formulæ

$$\left.\begin{aligned} \zeta &= \theta_0 - \theta + 2k\pi/n, \\ \zeta &= -\theta_0 - \theta + 2k\pi/n, \end{aligned}\right\} \quad \begin{aligned} 0 &< \theta_0 < \pi/n, \\ 0 &< \theta < \pi/n, \end{aligned}$$

k being any integer.

Now, as θ varies, these poles travel along the real axis; but the number of them contained in the interval $(-\pi, +\pi)$ no longer remains *fixed.* Thus, when θ increases through such a value as brings one of the poles of $G_1(\zeta)$ to the point $\zeta = +\pi$, the effect on the integral V_1 is to introduce an abrupt change in its value: this is due to the presence of an additional pole in the interior of the path of integration. Similarly, when a pole comes to $\zeta = -\pi$, it will disappear from the interior of the loop, and again a discontinuity will occur in the value of V.

To illustrate the difficulty let us consider in detail the simple case $n = \frac{2}{3}$.
The poles of $G_1(\zeta)$ are then given by

$$\zeta = \theta_0 - \theta + 3k\pi,$$

and those of $G_2(\zeta)$ are given by $\qquad \zeta = -\theta_0 - \theta + 3k\pi,$

where k is any integer, and θ, θ_0 both lie between the limits 0 and $\frac{3}{2}\pi$.
It will be easily seen that

(i) If $0 < \theta_0 < \frac{1}{2}\pi$, $G_1(\zeta)$ has a pole ($k = 0$) at $\zeta = -\pi$, when $\theta = \theta_0 + \pi$; and $G_2(\zeta)$ has a pole ($k = 0$) at $\zeta = -\pi$, corresponding to $\theta = \pi - \theta_0$. Thus V_1 will have a discontinuity corresponding to $\theta = \theta_0 + \pi$, and V_2 will have one at $\theta = \pi - \theta_0$.

(ii) If $\frac{1}{2}\pi < \theta_0 < \pi$, $G_1(\zeta)$ has no pole at either limit; but $G_2(\zeta)$ has a pole ($k = 1$) at $\zeta = +\pi$, when $\theta = 2\pi - \theta_0$, and a second pole ($k = 0$) at $\zeta = -\pi$, when $\theta = \pi - \theta_0$. Thus here V_2 has two points of discontinuity.

(iii) If $\pi < \theta_0 < \frac{3}{2}\pi$, $G_1(\zeta)$ has a pole ($k = 0$) at $\zeta = +\pi$, when $\theta = \theta_0 - \pi$: and $G_2(\zeta)$ has a pole ($k = 1$) at $\zeta = +\pi$, when $\theta = 2\pi - \theta_0$. Thus again both functions V_1, V_2 are discontinuous.

In order then to solve our problem for any value of the angle π/n, it will be necessary to choose a path of integration which does not meet the real axis; and further the path chosen must be equivalent to the loop already used when n is an integer.

Such a path can be obtained by adding on to the loop the two vertical lines of the last figure (the directions of integration being indicated by the arrows); for, when n is an integer, the functions G_1, G_2, and R, have each a period 2π in ζ, and consequently the contributions to V_1 and V_2 from the two vertical lines will cancel each other.

Of course care must be taken to see that the integrals taken along the vertical lines are convergent. Now along these two lines, R is real and positive, and tends to infinity like $\sqrt{(2rr_0 \cosh \eta)}$, if η is the imaginary part of ζ. Thus $\int d\zeta/R$ is absolutely convergent: and $|G_1|$, $|G_2|$ tend to the common limit $\frac{1}{2}n$. Hence, provided that $f(R-ct)$ remains numerically less than some fixed value as R tends to $+\infty$ by real values, the integrals V_1 and V_2 will be absolutely convergent along the vertical lines.*

Now, keeping the ends of the path fixed at infinity, we can, by Cauchy's theorem, deform the path of integration into the shape given in Fig. 3. Care must be taken that neither portion of the path comes in contact with a cut: and both parts must be kept clear of the real axis, so as to avoid the poles which traverse the interval $(-\pi, +\pi)$ as θ varies.

We have still to prove that the solution $V = V_1 - V_2$ satisfies the prescribed conditions. It is evident that it satisfies the fundamental differential equation: and that it corresponds to the case of *divergent* waves, which have not been reflected back to the source. Also we have just seen that by using our new path of integration the function V will be in general continuous in the interior of the space bounded by $\theta = 0$ $\theta = \pi/n$.

We have now to see that the necessary boundary conditions are satisfied; at $\theta = 0$, we have

$$G_1(\zeta) = \tfrac{1}{2}n \cot \tfrac{1}{2}n(\zeta-\theta_0), \quad \text{and} \quad G_2(\zeta) = \tfrac{1}{2}n \cot \tfrac{1}{2}n(\zeta+\theta_0),$$

and so
$$G_1(\zeta) = -G_2(-\zeta).$$

But we can choose our two halves of the path so as to be symmetrical about the origin; and then, by changing the sign of ζ, we change one half into the other, *described in the proper sense.* Thus it is seen that $V_1 = +V_2$ at $\theta = 0$, because R does not alter when the sign of ζ is changed. Hence $V = 0$ at $\theta = 0$.

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* It is at this stage that we utilise the condition imposed on the functions $G_1(\zeta)$, $G_2(\zeta)$, that they must remain finite as η tends to either $+\infty$ or $-\infty$ (see p. 452).

Again, at $\theta = \pi/n$, we have

$$G_1(\zeta) = -\tfrac{1}{2}n \tan \tfrac{1}{2}n(\zeta-\theta_0), \qquad G_2(\zeta) = -\tfrac{1}{2}n \tan \tfrac{1}{2}n(\zeta+\theta_0),$$

so that
$$G_1(\zeta) = -G_2(-\zeta)\,;$$

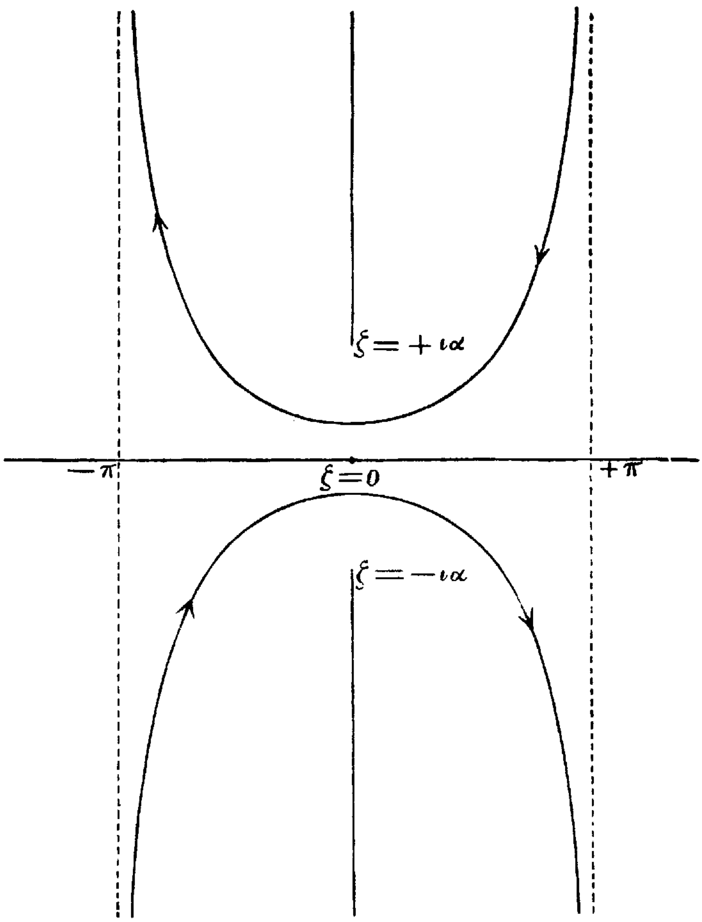

FIG. 3.—The curves are at present restricted to be asymptotic to the vertical lines; when more information as to the nature of the function f is available, we can often remove this restriction.

and thus $V_1 = V_2$ and $V = 0$ at both the planes $\theta = 0$ and $\theta = \pi/n$. That is, *the required boundary conditions are satisfied.*

The only point remaining is to prove that V has the right form near the source. When the point (r, θ, z) approaches (r_0, θ_0, z_0) it is evident that α will tend to zero because

$$2rr_0 \cosh \alpha = r^2 + r_0^2 + (z-z_0)^2.$$

Thus the two cuts of our diagram will tend to join; and since our path of integration must pass between the cuts (twice), we may anticipate that a special discussion will be needed. We can replace our path by the form given in Fig. 4. Then, as $\theta_0-\theta$ tends to zero, we can take the inner loop

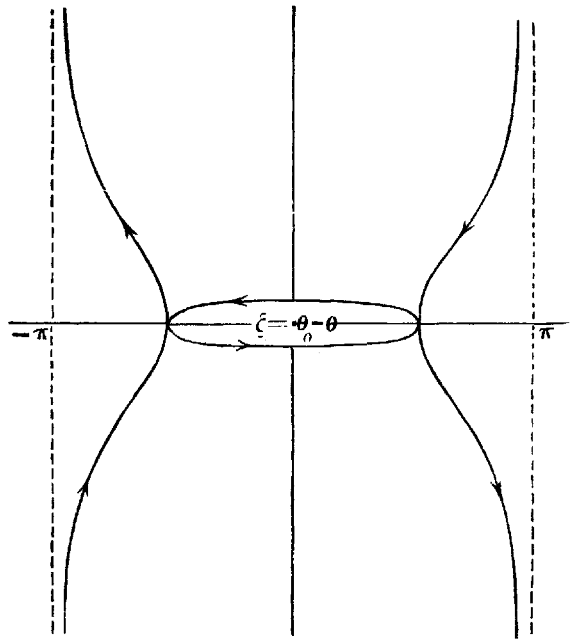

FIG. 4.

smaller and smaller. Now $G_1(\zeta)$ has the pole $\zeta=\theta_0-\theta$ inside the loop, and the residue there is unity; but $G_2(\zeta)$ has no pole there. Thus, by Cauchy's theorem, the contributions from the loop to V_1 and V_2 are respectively

$$\frac{1}{R_0}f(R_0-ct) \quad \text{and} \quad 0,$$

where
$$R_0^2 = r^2+r_0^2-2rr_0\cos(\theta-\theta_0)+(z-z_0)^2.$$

The remainder of the path indicated in Fig. 4 will contribute finite amounts to V_1 and V_2.

Thus in all we see that the character of V near the source is of the type $\frac{1}{R_0}f(R_0-ct)$: and by a proper choice of the function f, this will represent any assigned type of source (having an infinity of the *first* order only).

The proof that the function V satisfies all the prescribed conditions is now complete: and *our problem is accordingly solved, for all angles* π/n, *by the formula*

$$V = V_1 - V_2,$$

where $V_1 = \dfrac{1}{2\pi\iota}\displaystyle\int \frac{f(R-ct)}{R} G_1(\zeta)\,d\zeta, \qquad V_2 = \dfrac{1}{2\pi\iota}\displaystyle\int \frac{f(R-ct)}{R} G_2(\zeta)\,d\zeta,$

and
$$R^2 = r^2 + r_0^2 - 2rr_0\cos\zeta + (z-z_0)^2,$$

$$G_1(\zeta) = \tfrac{1}{2}n\cot\tfrac{1}{2}n(\zeta+\theta-\theta_0), \qquad G_2(\zeta) = \tfrac{1}{2}n\cot\tfrac{1}{2}n(\zeta+\theta+\theta_0),$$

the path of integration being that drawn in Fig. 3.

We may suppose that the two halves of the path are symmetrical about $\zeta = 0$; then the lower path changes into the upper (taken in the proper sense), by changing the sign of ζ; and so we may replace the integral along the two paths by the sum of two integrals along one path. Thus *we find*

$$V_1 = \frac{1}{2\pi\iota}\int \frac{f(R-ct)}{R} H_1(\zeta)\,d\zeta, \qquad V_2 = \frac{1}{2\pi\iota}\int \frac{f(R-ct)}{R} H_2(\zeta)\,d\zeta,$$

where
$$H_1(\zeta) = G_1(\zeta) - G_1(-\zeta) = -\frac{n\sin n\zeta}{\cos n\zeta - \cos n(\theta-\theta_0)},$$

and
$$H_2(\zeta) = G_2(\zeta) - G_2(-\zeta) = -\frac{n\sin n\zeta}{\cos n\zeta - \cos n(\theta+\theta_0)},$$

the path of integration being now reduced to the upper path only in Fig. 3.

The second formula has the advantage of making more obvious the fact that the boundary condition $V=0$ is satisfied at the two planes $\theta = 0$, $\theta = \pi/n$; for clearly at each of these planes $H_1(\zeta) = H_2(\zeta)$, and consequently $V_1 = V_2$ also.

3. *The Diffraction-Problem for Sound-Waves from a Source, impinging on a Wedge.*

We can now write down without difficulty the solution ϕ of the wave equation which satisfies the boundary condition

$$\frac{\partial\phi}{\partial\nu} = 0,$$

at the planes $\theta = 0$, $\theta = \pi/n$ (for any value of n), corresponding to a source between the planes.

As is well known, when n is an integer, the solution is given by

$$\phi = V_1 + V_2,$$

where V_1 and V_2 are the functions obtained above. Thus, by a corresponding generalization, we see that this formula will continue to represent the required solution for all values of n, provided that V_1, V_2 have the values given by the complex integrals of p. 458.

We note as a verification that in the second form of the solution

$$\frac{\partial H_1}{\partial \theta} + \frac{\partial H_2}{\partial \theta} = 0,$$

both for $\theta = 0$ and π/n.

For a simple harmonic source, we may take

$$f(R - ct) = e^{\iota\kappa(ct-R)},$$

the real or imaginary part of the results being finally selected, as may be convenient. In this case the solution reduces to that found by Prof. H. M. Macdonald.* We can then also, if we wish, suppose that the ends of the path are displaced so that the imaginary part of R is *negative*; that is the path may start from any point at infinity of the type $\iota\infty + \lambda$, where $\pi \leqslant \lambda < 2\pi$, and it may finish at any point of the type $\iota\infty - \lambda'$, where $0 < \lambda' \leqslant \pi$; instead of being asymptotic to the dotted lines $\pm\pi$, as indicated in Fig. 3.

Prof. Macdonald has indicated how the complex integrals can be expressed in other forms which lend themselves to numerical calculation when $n = \frac{1}{2}$ (the case of a *half-plane*); his formulæ may be written (in the notation used here),

$$V_1 = \frac{\iota\kappa}{\pi} \int_{-\infty}^{v_1} K_1(\iota\kappa R_0 \cosh v)\, dv,$$

where $$R_0^2 = r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0) + (z - z_0)^2,$$

and $$\sinh v_1 = \frac{2}{R_0} \sqrt{(rr_0)} \cos \tfrac{1}{2}(\theta - \theta_0).$$

An alternative form, suitable when v_1 is positive, is given by

$$V_1 = \frac{e^{-\iota\kappa R_0}}{R_0} - \frac{\iota\kappa}{\pi} \int_{v_1}^{\infty} K_1(\iota\kappa R_0 \cosh v)\, dv.$$

The value of V_2 is found by similar formulæ in which $\theta + \theta_0$ replaces $\theta - \theta_0$.

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 14, 1915, p. 418 above; the results quoted are given on p. 424, and the reduction of the integrals to Fresnel's form on p. 426.

4. *The Diffraction-Problem for Waves from a Hertzian Oscillator impinging on a Wedge.*

It is known that if V denotes a solution of the fundamental wave-equation, and if

$$W = l\frac{\partial V}{\partial x} + m\frac{\partial V}{\partial y} + n\frac{\partial V}{\partial z},$$

then the electromagnetic equations are satisfied by the solution

$$X = -\frac{\partial W}{\partial x} + \frac{l}{c^2}\frac{\partial^2 V}{\partial t^2}, \qquad \alpha = \frac{1}{c^2}\frac{\partial}{\partial t}\left(m\frac{\partial V}{\partial z} - n\frac{\partial V}{\partial y}\right),$$

$$Y = -\frac{\partial W}{\partial y} + \frac{m}{c^2}\frac{\partial^2 V}{\partial t^2}, \qquad \beta = \frac{1}{c^2}\frac{\partial}{\partial t}\left(n\frac{\partial V}{\partial x} - l\frac{\partial V}{\partial z}\right),$$

$$Z = -\frac{\partial W}{\partial z} + \frac{n}{c^2}\frac{\partial^2 V}{\partial t^2}, \qquad \gamma = \frac{1}{c^2}\frac{\partial}{\partial t}\left(l\frac{\partial V}{\partial y} - m\frac{\partial V}{\partial x}\right),$$

where (l, m, n) are any fixed direction-cosines, (X, Y, Z) is the electric force, and (α, β, γ) the magnetic force.

Assuming the wedge to be perfectly conducting, the condition to be satisfied at the boundaries is that the electric force shall be normal to the boundary. Now the boundaries are planes: say that one of them is the plane of yz. Then the conditions to be satisfied at the plane $x = 0$ are $Y = 0$, $Z = 0$; and both these conditions will clearly be satisfied provided that

$$V = 0, \qquad W = 0.$$

If we choose $V = 0$ at the plane $x = 0$, the value of W at that plane reduces to

$$l\frac{\partial V}{\partial x},$$

which can only be zero if $l = 0$: or if the axis of the oscillator is parallel to the plane boundary.

Similar reasoning applies at the second face of the wedge: and we conclude that our condition is satisfied by $V = 0$, provided that (l, m, n) is parallel to *both* planes; that is, provided that (l, m, n) is parallel to the edge of the wedge.

We can accordingly solve our problem for a Hertzian oscillator with its axis parallel to the edge of the wedge, by taking V to be the solution determined in § 2 (see p. 458), and $l = 0$, $m = 0$, $n = 1$; so that

$$X = -\frac{\partial^2 V}{\partial x\,\partial z}, \qquad \alpha = -\frac{1}{c^2}\frac{\partial^2 V}{\partial y\,\partial t},$$

$$Y = -\frac{\partial^2 V}{\partial y\,\partial z}, \qquad \beta = +\frac{1}{c^2}\frac{\partial^2 V}{\partial x\,\partial t},$$

$$Z = -\frac{\partial^2 V}{\partial z^2} + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}, \qquad \gamma = 0.$$

In cylindrical coordinates this solution can be written *

$$X' = -\frac{\partial^2 V}{\partial r\,\partial z}, \qquad \alpha' = -\frac{1}{c^2 r}\frac{\partial^2 V}{\partial \theta\,\partial t},$$

$$Y' = -\frac{1}{r}\frac{\partial^2 V}{\partial \theta\,\partial z}, \qquad \beta' = +\frac{1}{c^2}\frac{\partial^2 V}{\partial r\,\partial t},$$

Z and γ remaining unchanged.

If we assume that the source is an oscillating magnet, instead of a Hertzian oscillator, it is easy to see that the corresponding solution is

$$\alpha' = -\frac{\partial^2 V}{\partial r\,\partial z}, \qquad X' = +\frac{1}{r}\frac{\partial^2 V}{\partial \theta\,\partial t},$$

$$\beta' = -\frac{1}{r}\frac{\partial^2 V}{\partial \theta\,\partial z}, \qquad Y' = -\frac{\partial^2 V}{\partial r\,\partial t},$$

$$\gamma = -\frac{\partial^2 V}{\partial z^2} + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}, \qquad Z = 0,$$

where now the value of V is of the form $V_1 + V_2$ (as in § 3). For then it is evident that $X' = 0$ at the faces of the wedge; and consequently the electric force is normal to the boundary.

These two solutions can be adapted to numerical calculation by means of Prof. Macdonald's formulæ (quoted above, p. 459) for the case of a half-plane or straight-edge, the source being supposed simple harmonic.

It is not without interest to deduce from our solutions the known results for the case of simple harmonic *plane* waves. If we make r_0 tend to infinity, and take

$$\frac{f(ct-R)}{R} = \frac{A r_0}{R}\, e^{\iota\kappa(ct+r_0-R)},$$

it will be easily seen that $R - r_0$ tends to $-r\cos\zeta$ and that $R_0 - r_0$ tends

* Here X', Y' are the components of electric force along the directions of r, θ respectively; while α', β' are the corresponding components of the magnetic force.

to the form $-r\cos(\theta-\theta_0)$; consequently the limiting form corresponds to a value of

$$V = A\,e^{\iota\kappa\,[ct+r\cos(\theta-\theta_0)]}$$

in the incident wave. Thus the incident wave is given by

$$X' = 0, \qquad Y' = 0, \qquad Z = e^{\iota\kappa\,[ct+r\cos(\theta-\theta_0)]},$$

provided that

$$A = -1/\kappa^2.$$

The complete solution is then given by

$$Z = Z_1 - Z_2,$$

where
$$Z_1 = \frac{1}{2\pi\iota}\int e^{\iota\kappa r\cos\zeta}\,H_1(\zeta)\,d\zeta, \qquad Z_2 = \frac{1}{2\pi\iota}\int e^{\iota\kappa r\cos\zeta}\,H_2(\zeta)\,d\zeta,$$

$H_1(\zeta)$, $H_2(\zeta)$ being the functions defined above (p. 458), and the path of integration being the upper curve in Fig. 3, or its extension defined in § 3.

Similarly, by starting from the solution when the source is an oscillating magnet, we can deduce the solution corresponding to an incident wave

$$\alpha' = 0, \qquad \beta' = 0, \qquad \gamma = e^{\iota\kappa\,[ct+r\cos(\theta-\theta_0)]}.$$

The result is seen to be

$$\gamma = \gamma_1 + \gamma_2,$$

where γ_1, γ_2 are expressed by the same integrals as Z_1, Z_2 above.

These results are due to Macdonald,* and forms suitable for numerical calculation have been deduced from them by W. H. Jackson.†

5. *Electrostatic and Hydrodynamical Problems.*

The problem of electrostatics may be briefly mentioned here: it corresponds to the case in which $f(R-ct)$ is a constant, say unity. The solution then gives Green's function for the space defined by the limits

$$0 \leqslant \theta \leqslant \pi/n.$$

Under these conditions the ends of the path in Fig. 3 can be displaced to any extent (between adjacent cuts), without affecting the convergence of the integrals; and so the upper path may be reduced to two integrals along the edges of the upper cut. On the right-hand edge we then find

$$R = +\iota\sqrt{\{2rr_0(\cosh v - \cosh\alpha)\}},$$

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* *Electric Waves*, 1902, pp. 192, 195.

† *Proc. London Math. Soc.*, Ser. 2, Vol. 1, 1903, p. 393.

where $\zeta = \iota v$, and on the left-hand edge, the sign of R is changed; also we find on either edge

$$H_1(\zeta) = -\frac{n\iota \sinh nv}{\cosh nv - \cos n(\theta-\theta_0)}, \qquad H_2(\zeta) = -\frac{n\iota \sinh nv}{\cosh nv - \cos n(\theta+\theta_0)}.$$

Hence the formula reduces to $V = V_1 - V_2$, where

$$V_1 = \frac{n}{\pi}\int_a^\infty \frac{dv}{\sqrt{\{2rr_0(\cosh v - \cosh a)\}}}\,\frac{\sinh nv}{\cosh nv - \cos n(\theta-\theta_0)},$$

and V_2 is a corresponding integral with $\theta+\theta_0$ in place of $\theta-\theta_0$.

These formulæ were given by Prof. H. M. Macdonald,* and require no further development now.

It may be worth while to add the remark that the corresponding hydrodynamical problem (for a source in an incompressible fluid occupying the space between two planes) is given by the solution

$$\phi = V_1 + V_2,$$

the integrals for V_1 and V_2 being those given above. For it is then evident that $\partial\phi/\partial\theta = 0$ both at $\theta = 0$ and at $\theta = \pi/n$.

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* *Proc. London Math. Soc.*, Vol. XXVI, 1895, p. 160, and Ser. 2, Vol. 14, 1915, p. 412 above. See also Sommerfeld's paper, quoted above, for the special case $n = \frac{1}{2}$.