ON LINEAR DIFFERENTIAL EQUATIONS OF RANK UNITY

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THE present paper is concerned with linear ordinary differential equations with rational coefficients. If such an equation be

$$y^{(n)}+p_1 y^{(n-1)}+\ldots+p_n y = 0,$$

and p_r has a pole of order ϖ_r at infinity (ϖ_r being negative if p_r is zero at infinity), and if p is the least integer greater than the greatest of the quantities ϖ_r/r ($r = 1, \ldots, n$), the equation may be written

$$y^{(n)}+P_1 x^p y^{n-1}+\ldots+P_n x^{np} y = 0$$

where P_r is finite and developable in powers of $1/x$ near $x = \infty$.

If $p = -1$, the integrals are regular in the neighbourhood of $x = \infty$, and the nature of their singularity is determined.

If $p \geqslant 0$, expansions of the integrals in the neighbourhood of $x = \infty$ are possible in the form of normal and subnormal series, but these expansions are, in general, divergent, and give little information about their nature.

For $p = 0$ an explicit solution is known in the form of Laplace's definite integral; but for $p > 0$ this, too, fails. An extension of this definite integral solution is here obtained in the case of $p = 1$, which appears to be capable of extension to greater values of p, though the analysis would be cumbrous.

The form of solution suggested by the known normal series is the following :—

$$\iint e^{tx+\frac{1}{2}ux^2}\, U\, du\, dt$$

where u is a function of t and u, and an appropriate contour is to be determined for each integration.

If this expression be substituted for y in the given equation, and the double integral obtained as the left-hand member be transformed by integration by parts, a partial differential equation is obtained which takes the place of the ordinary differential equation known as the Laplace transformation for $p = 0$.

The complete integration of this equation is not required, but particular integrals are obtained which lead to the divergent normal series already known to exist.

The equation considered will be

$$P(y) \equiv \phi_\lambda(x)\,y^{(n)} + \phi_{\lambda+1}(x)\,y^{(n-1)} + \ldots + \phi_{\lambda+n}(x)\,y = 0$$

where the affix of each coefficient denotes its degree in x, each being supposed a polynomial.

Let λ be an even integer—(if not, the equation may be multiplied throughout by x)—and let $\lambda + 2n = 2m$. Then, if

$$y = \iint e^{tx+\frac{1}{2}ux^2}\, U\, du\, dt,$$

the contours being independent of x,

$$y' = \iint e^{tx+\frac{1}{2}ux^2}(t+ux)\, U\, du\, dt,$$

$$y'' = \iint e^{tx+\frac{1}{2}ux^2}\{(t+ux)^2+u\}\, U\, du\, dt,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

In general, $$y^{(r)} = \iint e^{tx+\frac{1}{2}ux^2}\,\omega_r\, U\, du\, dt$$

where the quantities ω_r are given by

$$\omega_r = \frac{\partial \omega_{r-1}}{\partial x} + \omega_{r-1}\,\omega_1, \qquad \omega_1 = t+ux.$$

It is important in the sequel to notice that

(i.) ω_r is a polynomial of degree r in u and t combined, and of degree r in x alone;

(ii.) that the *two* highest powers of x in ω_r arise only from the term $(t+ux)^r$ in ω_r.

Multiply the expression $P(y)$ by x^n, and then substitute the values of $y^{(r)}$ $(r = 1, \ldots, n)$.

$$x^n P(y) = \iint e^{tx+\frac{1}{2}ux^2}(\phi_\lambda x^n \omega_n + \phi_{\lambda+1} x^n \omega_{n-1} + \ldots + x^n \phi_{\lambda+n})\, U\, du\, dt$$

$$= \iint e^{tx+\frac{1}{2}ux^2}\,\Pi(t, u, x)\, U\, du\, dt.$$

Π is now a polynomial of degree $\lambda+2n$ in x and of degree n in t and u combined.

If ϕ_{sr} denotes the coefficient of x^s in ϕ_r, and π_h the coefficient of x^h in Π,

$$\pi_{\lambda+2n} = u^n \phi_{\lambda\lambda} + u^{n-1}\phi_{\lambda+1,\lambda+1} + \ldots + \phi_{\lambda+n,\lambda+n};$$

$$\pi_{\lambda+2n-1} = \{nu^{n-1}\phi_{\lambda\lambda} + (n-1)u^{n-2}\phi_{\lambda+1,\lambda+1} + \ldots\} + \{\phi_{\lambda-1,\lambda}u^n + \ldots + \phi_{\lambda+n-1,\lambda+n}\}$$

$$= t\frac{\partial \pi_{\lambda+n}}{\partial u} + \psi_{\lambda+n-1}.$$

Similarly, $$\pi^{\lambda+2n-2} = t^2\frac{\partial^2 \pi_{\lambda+n}}{\partial u^2} + t\psi_{\lambda+n-2} + \chi_{\lambda+n-2},$$

and so on.

Now

$$\iint e^{tx+\frac{1}{2}ux^2} U x^r \, du\, dt = \int e^{tx} dt\, [2e^{\frac{1}{2}ux^2} x^{r-2} U]_u - 2\iint e^{tx+\frac{1}{2}ux^2} x^{r-2} \frac{\partial U}{\partial u}\, du\, dt$$

where the square brackets in the first integral on the right-hand side denote that the difference of the values of the contained function at the extremities of the contour is to be the subject of integration with respect to t.

In the same way a single integration with respect to t gives

$$\iint e^{tx+\frac{1}{2}ux^2} U x^r \, du\, dt = \int e^{\frac{1}{2}ux^2} du\, [e^{tx} U x^{r-1}]_t - \iint e^{tx+\frac{1}{2}ux^2} x^{r-1} \frac{\partial U}{\partial t}\, du\, dt.$$

By a repeated application of these two equations the expression $x^n P(y)$ is reducible to the form

$$\iint e^{tx+\frac{1}{2}ux^2} f(U, u, t)\, du\, dt + \bar{P}$$

where $\bar{P}$ represents an aggregate of semi-integrated terms, and $f(U, u, t)$ contains derivates of U with respect to u and t. If, then, U be chosen to be a solution of the partial differential equation $f(U, u, t) = 0$, and the contours of the integrations can be assigned so that $\bar{P}$ vanishes identically, the integral, if existent, will be a true solution of the equation $P(y) = 0$.

The highest power of x in $\Pi(t, u, x)$ being $x^{\lambda+2n}$, by integrating m times with respect to u, the corresponding term in $f(U, u, t)$ is

$$(-2)^m \frac{\partial^m}{\partial u^m}\{U \pi_{\lambda+2n}\}.$$

Similarly, the term arising from the term involving $x^{\lambda+2n-1}$ in Π is

$$(-1)^m 2^{m-1} \frac{\partial^m}{\partial u^{m-1}\partial t}\{U \pi_{\lambda+2n-1}\}$$

All the remaining terms may be integrated after the same manner, no derivatives of more than the $(m-1)$-th order in respect to u occurring; but there is a certain amount of arbitrariness in dealing with the later terms which requires some explanation. The equation $f(U, u, t) = 0$ is, however, in any case of the form

$$\pi_{\lambda+2n}\frac{\partial^m U}{\partial u^m} = \Sigma a_{rs}\frac{\partial^{r+s} U}{\partial u^r\partial t^s} \qquad (r = 0, 1, \ldots, m-1),$$

and the coefficients a_{rs} are polynomials in u and t. It appears, therefore, that in the neighbourhood of any pair of values $u = \alpha$, $t = \beta$ a function can be found satisfying this equation, and developable in a double power series in u and t, but that the region of convergence is always limited by the points at which $\pi_{\lambda+2n}$ vanishes.

As in Laplace's solution, however, it is the special integrals related to these singular points which lead to the required result.

Let the roots of the equation $\pi_{\lambda+2n} = 0$ be assumed all different. Let a be a root of this equation, and let $v = u-a$. Further, if

$$\left(\frac{\partial\pi_{\lambda+2n}}{\partial u}\right)_{v=0} = \beta \qquad \text{and} \qquad (\psi_{\lambda+2n-1})_{v=0} = \gamma,$$

let $s = t+\gamma/\beta$. Then the term independent of v in $\pi_{\lambda+2n-1}$ is simply βs. We have then

$$P(y) = e^{\frac{1}{2}ax^2-(\gamma x/\beta)}\iint e^{\frac{1}{2}vx^2+sx}\,U\Pi'(s, v, x)\,dv\,ds,$$

Π' being the result of changing the variables in Π.

A part of the last integral arising from a term $x^k v^r s^{r'}$ in Π' will be reduced to a double integral independent of x by means of r' or $r'+1$ integrations in respect to s, together with $\frac{1}{2}(k-r')$ or $\frac{1}{2}(k-r'-1)$ in respect to v, according as $k-r'$ is an even or odd integer. This is always possible, since, the equation having been multiplied by n, the least value of k is n, and the greatest value of r' is m.

The result of this is that the function within the double integral becomes of the form

$$\left[(-2)^m\frac{\partial^m}{\partial v^m}\{\pi_{\lambda+2n}(v)\,U\} - (-2)^{m-1}\frac{\partial^m}{\partial v^{m-1}\partial s}\{U\beta s+v\,(s, v)_{1,n-1}\}\right.$$
$$+(-2)^{m-2}\frac{\partial^m}{\partial v^{m-2}\partial s^2}\{U(s, v)_{2,n}\}$$
$$-(-2)^{m-3}\frac{\partial^m}{\partial v^{m-3}\partial s^3}\{U(s, v)_{3,n}\}$$
$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

where $(s, v)_{r,n}$ denotes a polynomial of degree r in s and n in v.

On expansion this equation is of the form

$$2^m \pi_{\lambda+2n} \frac{\partial^m U}{\partial v^m} + 2^{m-1}\{\beta s + v(s, v)_{1, n-1}\} \frac{\partial^m U}{\partial v^{m-1}\partial s}$$
$$+ \sum_{r=0}^{n+1} \sum_{k=2}^{m} \{s^r(1, v)_n + s^{r-1}(1, v)_n\} \frac{\partial^{m-k+r} U}{\partial v^{m-k}\partial s^r} = 0.$$

In connection with the singular point $v = 0$, a particular solution of this equation is sought in the form

$$v^\rho \{f_0(t) + v f_1(t) + v^2 f_2(t) + \ldots\}.$$

Substituting in the equation, and equating to zero the coefficients of the successive powers of v, beginning from the lowest, the following equations are obtained :—

(1) $$\rho(\rho-1)\ldots(\rho-m+1)\, 2^m \beta f_0 + \rho \ldots (\rho-m+2)\, 2^{m-1}\beta s \frac{\partial f_0}{\partial s}$$
$$+ \rho(\rho-1)\ldots(\rho-m+2)\, a_0 f_0 = 0,$$

(2) $$(\rho+1)\rho\ldots(\rho-m+2)\, 2^m\beta f_1 + (\rho+1)\ldots(\rho-m+3)\, 2^{m-1}\beta . s \frac{\partial f_1}{\partial s}$$
$$+ (\rho+1)\ldots(\rho-2m+3)\, a^0 f_1$$
$$= (a_1 s^2 + b_1 s)\frac{\partial^2 f_0}{\partial s^2} + (c_0 s + d_0)\frac{\partial f_0}{\partial s} + e_0 f_0,$$

... ...

The first equation gives

$$s\frac{\partial f_0}{\partial s} + \left\{2(\rho-m+1) + \frac{a_0}{\beta}\right\} f_0 = 0$$

or
$$f_0 = A s^{-\sigma}$$

where
$$\sigma = 2(\rho-m+1) + \frac{a_0}{\beta}.$$

Substituting this in the right-hand side of equation (2), this equation becomes

$$(\rho+1)\ldots(\rho-m+3)\, 2^{m-1}\beta\left\{s\frac{\partial f_1}{\partial s} + (\sigma+2) f_1\right\} = Bs^{-\sigma} + Cs^{-\sigma-1},$$

giving
$$s^{\sigma+2} f_1 = \frac{Bs^2}{2} + Cs$$

or
$$f_1 = s^{-\sigma}\left(\frac{C}{s} + \frac{B}{2}\right).$$

The next equation gives f_2 of the form

$$s^{-\sigma}\left(1, \frac{1}{s}\right)_2,$$

and, in general, $$f_r = s^{-\sigma}\left(1, \frac{1}{s}\right)_r.$$

Thus a formal solution of the equation is obtained in the form

$$v^\rho s^{-\sigma}\left\{1+v\left(1, \frac{1}{s}\right)_1+v^2\left(1, \frac{1}{s}\right)_2+\ldots+v^r\left(1, \frac{1}{s}\right)_r+\ldots\right\}$$

where $$\sigma = 2(\rho-m+1)+\frac{a_0}{\beta}.$$

Of this solution it will now be proved that it is absolutely convergent within a finite circle about $v=0$ and for all values of s greater in modulus than any given finite quantity s_0.

For the proof of this we may, without loss of generality, assume that $k=0$ and $\rho=0$. If otherwise, let $U = u^\rho s^{-\sigma} U'$; then U' will satisfy a precisely similar equation.

If the variable $t = \frac{1}{s}$ be used in place of s, the equation becomes

$$2^m \pi^{\lambda+2n}\frac{\partial^m U}{\partial v^m} - 2^{m-1}\{\beta t+vt(t, v)_{1, n-1}\}\frac{\partial^m U}{\partial v^{m-1}\partial t}$$
$$+\Sigma\Sigma\{t^r(1, v)_n+t^{r+1}(1, v)_n\}\frac{\partial^{m-k+r} U}{\partial v^{m+r}\partial t^r} = 0,$$

and the solution becomes

$$1+v(1, t)_1+v^2(1, t)_2+\ldots.$$

Consider now the equations giving the successive polynomials $(1, t)_r$. They are of the form

(A) $$-t\frac{\partial f_r}{\partial t}\, 2rf_r = \sum_h \sum_{k=0}^{r-1} \frac{\partial^h f_k}{\partial t^h}(a_{hk}t^h+b_{hk}t^{h+1})$$

and give f_r as a polynomial of degree r in t.

Compare with these the equations obtained by making the coefficients all positive.

(B) $$-t\frac{\partial \phi_r}{\partial t}+2r\phi_r = \Sigma\sum_{k=0}^{r-1}\frac{\partial^h \phi^k}{\partial t^h}(|a_{hk}|\, t^h+|b_{hk}|\, t^{h+1}).$$

Then ϕ_r will be a polynomial in t whose coefficients are the moduli of those of f_r, and therefore $$|f_r^{(h)}(t)| \leqslant \phi_r^{(h)}|t|$$

for all values of t. Again, if $t>1$, $t^h < t^{h+1}$.

Now compare with equations (B) the equations

(C) $$-t\frac{\partial \psi_r}{\partial t}+2r\psi_r = \Sigma\Sigma\frac{\partial^h \psi_k}{\partial t^h}(|a_{hk}|+|b_{hk}|)\, t^{h+1}.$$

Then, if t is a positive quantity greater than 1, and if

$$\left|\frac{\partial^h \psi_k}{\partial t^h}\right| \geqslant \left|\frac{\partial^h \phi_k}{\partial t^h}\right| \quad (k = 0, \ldots, r-1),$$

the equations shew that $\quad |t^{-2r}\psi_r| \geqslant |t^{-2r}\phi_r|,$

and therefore that $\quad |\psi_r| \geqslant |\phi_r|.$

But also the solution of equations (C), taking

$$\psi_0 = \phi_0 = f_0 = 1,$$

gives $\psi_r = c_r t^r$, c_r being positive, while $\phi_r = (1, t)_r$, the coefficients being positive. Hence, if $\psi_k \geqslant \phi_k$ for all values of $t > 1$,

$$\frac{\partial^h \psi_k}{\partial t^h} \geqslant \frac{\partial^h \phi^k}{\partial t^k};$$

so that we deduce successively that

$$\frac{\partial^h \psi_1}{\partial t^h} \geqslant \frac{\partial^h \phi_1}{\partial t^h}, \quad \frac{\partial^h \psi_2}{\partial t^h} \geqslant \frac{\partial^h \phi_2}{\partial t^h}, \quad \ldots,$$

and therefore that $\quad \dfrac{\partial^h \psi_r}{\partial t^h} \geqslant \left|\dfrac{\partial^h f_r}{\partial t^h}\right|.$

Now the expression $\quad V = 1 + v\psi_1 + v^2\psi_2 + \ldots$

becomes $\quad 1 + c_1 vt + c_2 v^2 t^2 + \ldots,$

and satisfies an equation of the form

$$\frac{\partial^m V}{\partial v^m} - 2t\frac{\partial^m V}{\partial v^{m-1}\partial t} = \sum_h \sum_k t^{h+1} P_{hk}(v) \frac{\partial^{m-k+h}}{\partial v^{m-k}\partial t^h} V$$

where P_{hk} is a power series converging up to the root of $\pi_{\lambda+2n}$ nearest to the origin, say for $|v| < a$.

If the variable v be replaced by $w = vt$, a partial differential equation is obtained which is satisfied by a function of w alone and whose coefficients are developable in power series converging for $|w| < at$. Thus V satisfies an ordinary differential equation in w whose coefficients converge if $|w| < at$ and are finite for $w = \infty$, and hence the series V converges provided $|w| < at$, *i.e.*, provided $|v| < a$; and this is proved only under the assumption that $t > 1$. Hence the series

$$U = 1 + vf_1(t) + v_2 f_2(t) + \ldots$$

converges absolutely if $|v| < a$ and $|t|$ is finite and greater than unity.

Clearly, moreover, since $f_r(t)$ is a polynomial in t, the restriction that $|t| > 1$ may be removed. Hence U converges for all finite values of t, that is, for all non-zero values of s, including $s = \infty$, and for $|v| < a$. Also, if the real part of sz is negative, $\operatorname*{Lt}_{s=\infty} |e^{sz} U| = 0$ if $|v| < a$.

We have now to consider the continuation of this function to values of v outside the circle $|v| = a$, in order that we may know its behaviour for large values of v.

We may, without loss of generality, suppose that no root of the equation $\pi_{\lambda+n}(v) = 0$ is a real positive quantity. Let c be a real positive quantity less than a, and let $v = v_1 + c$. Then the expansion of U in powers of v_1 is

$$U_c + v_1\left(\frac{\partial U}{\partial v}\right)_c + \tfrac{1}{2}v_1^2\left(\frac{\partial^2 U}{\partial v^2}\right)_c + \ldots,$$

while that of V is
$$V_c + v_1\left(\frac{\partial V}{\partial v}\right)_c + \tfrac{1}{2}v_1^2\left(\frac{\partial^2 V}{\partial v^2}\right)_c + \ldots,$$

and it has been shown that, for any value of $t > 1$, the sum of the absolute values of the terms of the power series in t which constitutes the expression $\dfrac{\partial^{h+k} U}{\partial v^h \partial t^k}$ is less than the sum of the absolute values of the terms of $\dfrac{\partial^{h+k} V}{\partial v^h \partial t^k}$ which are powers of t with positive coefficients.

Now, since V satisfies a certain ordinary differential equation, its development in powers of v_1 will converge within a finite circle extending to the nearest root of $\pi_{\lambda+n} = 0$, and so, too, will its differential coefficients with respect to v and t.

It follows, therefore, that the expansion

$$U_c + v_1\left(\frac{\partial U}{\partial v}\right)_c + \ldots$$

will converge within the same circle, and likewise its differential coefficients.

Again, taking a point $v_1 = d$ within this circle, V and its differential coefficients are at this point power series in t with positive coefficients, and the above reasoning applies again; so that, in general, we find that U is developable within a finite region containing the real axis, and that it and its differential coefficients are at all points on the real axis less in modulus than V or its corresponding differential coefficient.

Again, since, as remarked above, the differential equation in $w\ (= vt)$ satisfied by V has its coefficients finite for $w = \infty$, a finite quantity λ exists such that

$$\operatorname*{Lt}_{w=\infty} \{e^{-\lambda w} V\} = 0;$$

or, if t is other than zero, $\operatorname*{Lt}_{v=\infty} \{e^{-\lambda v t} V\} = 0.$

Under the same conditions, also,

$$\underset{v=\infty}{\mathrm{Lt}}\,\{e^{-\lambda vt}|U|\} = 0,$$

and therefore, if t be restricted to be less than a finite quantity τ, a finite quantity z can always be found so that

$$\underset{v=\infty}{\mathrm{Lt}}\,\{e^{-\frac{1}{2}vx^2}|U|\} = 0$$

for all values of t greater than unity.

But, since U is absolutely convergent and contains only positive powers of t, the restriction that $t < 1$ may be removed, and the same is true for all values of $|t| < \tau$, including zero.

Similarly, also, it is found that

$$\underset{v=\infty}{\mathrm{Lt}}\left\{e^{-\frac{1}{2}vx^2}\left|\frac{\partial^{h+k}U}{\partial v^h\,\partial t^k}\right|\right\} = 0.$$

If z^2 is not real and positive, the same result will be obtained if we proceed to infinity in a direction such that the real part of vz^2 is negative, provided that direction does not pass through a root of $\pi_{\lambda+2n} = 0$.

Reverting to the original variable s, therefore, we see that, provided $|s|$ exceeds any definite finite quantity, s_0 (that is, τ^{-1}),

$$\underset{v=\infty}{\mathrm{Lt}}\left[e^{-\frac{1}{2}vx^2}\frac{\partial^{h+k}U}{\partial v^h\,\partial s^k}\right] = 0,$$

and also that

$$\underset{s=\infty}{\mathrm{Lt}}\left[e^{-sx}\frac{\partial^{h+k}U}{\partial v^h\,\partial s^k}\right] = 0,$$

it being assumed in both cases that $|x|$ is sufficiently large and that v and s are made infinite with such arguments that the real parts of vx^2 and sx are both positive. This is always possible. Hence it is always possible to assign contours of integration, for v and s respectively, consisting of loops encircling the points $v = 0$ and $s = 0$ and extending to infinity in appropriate directions, so that $\iint e^{sx+\frac{1}{2}vx^2}\,U\,dv\,ds$ exists, and so that when this is substituted in the equation, and the integration by parts indicated on p. 376 is carried out, the integrated part vanishes at the infinite limit. The double integral is therefore an integral of the given equation.

Recalling the form of U, the integral is

$$e^{\frac{1}{2}\alpha x^2-\beta x/\gamma}\iint e^{sx+\frac{1}{2}vx^2}\,v^\rho\,s^{-k}\left\{1+v\left(1,\ \frac{1}{s}\right)_1+v^2\left(1,\ \frac{1}{s}\right)_2+\dots\right\}dv\,ds.$$

Consider in particular

$$\iint e^{sx+\frac{1}{2}vx^2} v^{\rho} s^{-k} \left\{\frac{v^h}{s^l}\right\} dv\, ds.$$

With the contours found this is equal to

$$x^{(l+k-1)-2(\rho+h+1)} \int_{-\infty,\,0} \int_{\infty,\,0} e^{\xi+\eta} \xi^{\rho+h} \eta^{-(k+l)} d\xi\, d\eta$$

$$= \pm x^{l-2h+k-2\rho-3} \Gamma(\rho+h+1)\, \Gamma(1-k-l)$$

$$= \pm x^{2m-a_0/\beta-1-(2h-l)} \Gamma(\rho+h+1)\, \Gamma(1-k-l).$$

Now the coefficient of $v^{\rho+h}$ in U is $s^{-k}\left(1, \frac{1}{s}\right)_h$; so that l varies from 0 to h.

Hence the expansion of the integral is of the form

$$e^{\frac{1}{2}\alpha x^2 - \beta x/\gamma} x^{2m-a_0/\beta-1} P\left(\frac{1}{x}\right),$$

$P\left(\frac{1}{x}\right)$ being a series of ascending positive integral powers of $\frac{1}{x}$, which will in general be divergent, but which, exactly as in Poincaré's development of equations of rank 1, can be shewn to represent the value of the integral asymptotically.

If the expansion is to be a valid representation of the function, it must either converge or terminate. The latter will be the case if, and only if, the function u terminates.

The necessary modification of the foregoing in the case of equal roots of the equation $\pi_{\lambda+2n} = 0$ will not be carried out here. There appears to be no essential difficulty, but the analysis is cumbrous and does not illustrate any new fact of importance.

The extension to equations of rank p greater than 1 consists in the adoption of a trial solution in the form

$$\iiint \dots e^{tx+\frac{1}{2}ux^2+\dots+vx^{p+1}/(p+1)}\, U\, dv \dots du\, dt,$$

U being a function of $t, u, \dots, v$; and, again, the difficulty is in the writing rather than the reasoning.

Particular integrals of linear partial differential equations with rational coefficients can be investigated by means of a similar analysis in the form

$$\iint \dots e^{tx+uy+\dots+vz}\, U\, dv \dots du\, dt,$$

there being many points of similarity with the foregoing analysis (*v.* Picard, *Rend. del Circ. Mat. di Palermo*, v.).