It may suffice, therefore, that attention has been called to a theory alternative to that of v. Helmholtz, and which seems, to me at least, much simpler and more probably true.

An undoubted knowledge of even the sign of the potential of mercury in contact with an electrolyte would go a great way towards settling the question at issue. According to my view, it is probably negative. V. Helmholtz's theory is founded on the assumption that it is positive; but, notwithstanding the rather decided way in which this assumption is at first stated *, no real proof is given; and the subsequent remark ("wäre z. B. das Queksilber positiv" †) indicates the recognized provisional character of the assumption. It lies with those who would give to the provisional theory of v. Helmholtz the character of an ascertained law of Nature to provide a knowledge of the true value of the hypotheses on which it is based before it can carry the weight they propose to attach to it, or serve as a reliable support for the further researches already dependent on it.

XX. *Problems in Probabilities: No. 2, Competitive Examinations.* *By* Professor F. Y. Edgeworth, *D.C.L.* ‡

THE following study is related to the first number of the series § as being another instance of the Probability-calculus applied to a practical interest. In a paper on "The Statistics of Examinations," which was published in the September number of the Journal of the Royal Statistical Society for 1888, and in a sequel to that paper which will shortly be published in the same Journal, I have made an estimate of the extent to which the results of competitive examinations depend upon the accident of one examiner rather than another equally competent being appointed to mark the work of the candidates. Referring to those papers for a fuller exposition of the statistical data and the practical conclusions, I propose here to abstract the mathematical reasoning.

The fundamental axiom is the proposition, evidenced by analogy and specific experience, that the marks given by different examiners to the same piece of work are apt to be arranged according to some law of error or facility-curve which is constant for the same class of examiners and work examined. The analogy between errors of observation and discrepancies in marking is evident. But, as the transition is

* *Wissenschaftliche Abhandlungen*, i. p. 934. † *Ibid.* p. 936.

‡ Communicated by the Author.

§ See Philosophical Magazine, October 1886.

rather abrupt from objective quantities observed by the senses to degrees of intellectual quality estimated by the judgment, I have confirmed the analogy by trying an experiment in an intermediate case, where the quantity to be determined is objective, but the operation by which it is determined is an estimate rather than a simple observation. Of this sort are the answers which I have obtained by asking a number of persons separately: "What is my weight?" Having collected 96 estimates made in reply to this question, I find that they are constant to a definite law of frequency. That law is exhibited in the annexed table or scheme; in which the ordinary numerals denote pounds above nine stone (thus 25 means 10 st. 11 lb.), and the Roman numerals *above* denote the number of estimates which are entered in each of the spaces bounded by the ordinary numerals (thus there were four estimates below 9 st. 10 lb., and nineteen between that figure and 10 st. 6 lb.) *. The verification of our axiom is

IV.		XIX.		XXIV.		XXVI.		XIX.		IV.
	10.		**20.**		**25.**		**30.**		**40.**	
II.		XIV.		X.		XV.		VI.		I.
† I.		VII.		XIII.		XII.		XII.		III.

shown by the *lower* rows of Roman numerals which respectively designate the distribution of the first 48 answers which I received, and of the second batch. I add the following verifications. Regarding as an error the deviation of any estimate from the average of all ‡, namely 25, I find that the average of the errors in defect is 7·2; the average of the errors in excess is 6·8. Now split up the forty-seven errors in defect into two batches as nearly as may be; the average of the first twenty-four—first in the order of arrival—is 7·5; of the remaining twenty-three, is 6·8. So the average of the first twenty-five errors in excess is 5·6, of the remaining twenty-four 8.

Even the batches of sixteen show considerable steadiness. The following table exhibits this constancy. The first column designates the position of each batch of sixteen in the accidental order in which it was received and entered. Thus

* Where a number of estimates coincided at one of the boundaries, *e. g.* 20, I gave half to one compartment, *e. g.* IV.-XIX., half to the other, XIX.-XXIV. Where the number was odd I gave the benefit to the compartment nearer to the centre.

† The apparent anomaly that the whole of certain compartments contains more or less than the sum of the parts is explained by the preceding note.

‡ *Average* in this paper stands for Arithmetic Mean.

I. designates the first sixteen estimates which I received. The second column contains the average of each batch, expressed in pounds above nine stone. The third column gives the average of the sixteen errors measured without regard to *sign* from the average of each batch, and multiplied by the constant $1{\cdot}25\left(=\sqrt{\frac{\pi}{2}}\text{ nearly}\right)$ for the sake of comparison with the entries in the fourth column, each of which is the *Mean* of similarly reckoned errors, in the Gaussian sense of *Mean Error*: that is, the square root of the sum of squares of errors divided by the number thereof less by one, that is here 15.

Order of entry.	Average.	Average error × 1·25.	Mean error.
I.	24·5	8·2	8·1
II.	23	7·6	7·8
III.	22·5	8	8·6
IV.	22	10	9·8
V.	29	6·6	6
VI.	29	9	9·6
Means ...	25	8·2	8·3

It will be observed that these estimates obey not only *a* law, but *the* law of error; according to which the Mean Error ought to be equal to the Average Error $\times\sqrt{\frac{\pi}{2}}$. For further verification of this incident I may refer to the scheme on page 172, where it appears that the probable error, as deduced from the distribution of the observations, is 5. Now, according to the Tables compiled for the Error-function, the number of observations outside a distance on either side from the centre of three times the probable error ought to be 4·3 per cent. That is exactly what occurs. It is true that the average and mean error do not perfectly fit the probable error. But the imperfection is hardly greater than might be expected in dealing with a number of observations so limited as 96. Nor would I contend for a perfect fulfilment of the law of error—more perfect than in the case of human statures and other natural groups*. Indeed I am not concerned to show that *the* law of error is fulfilled; except so far as this incident may afford some guarantee of greater stability. It is on this account only, if at all, that I am concerned with another striking incident, namely that the Mean of all the estimates†, the apparent weight, 9 st. + 25 lb., coincides with the real weight, which is exactly, or oscillates about, 10 st. 11 lb.

A less perfect, but still I think sufficient, verification of our axiom is afforded by another series of 96 estimates which I obtained by asking the additional question: "What is my

XIX.		XXXII.		XXVI.		XIX.
	7·5		9		10	
XIII.		XVI.		XI.		VIII.
VI.		XV.		XVI.		XI.

height"? The grouping of the series is as before exhibited by the first two rows of figures in the annexed scheme. The correspondence of the parts with the whole and with each other is shown by the third and fourth rows, referring respectively to the first and second batches of forty-eight estimates. The average of the thirty-five errors in defect—measured from the average of the ninety-six observations, namely 8·6—is 1·73. The average of the first eighteen errors in defect is 1·55. The average of the remaining seventeen errors in defect is 1·9. Again, the average of the sixty-one errors in excess is ·98. The average of the first thirty-one of those errors is 1; the average of the remaining thirty is ·96.

As in the case of the weights, the apparent ‡ and true measures coincide. But there does not exist that guarantee of stability which may be afforded by conformity to a Probability-curve. That hypothesis is negatived by the protuberance of the lower limb which has just now been evidenced. It may be added that, whereas the lower quartile is distant from the Median by less than 1·5, there occur (in so small a set) three observations distant respectively from the Median 5, 7, 8. This occasional darting out of the lower limb is unfavourable to that steadiness in the average of small batches which we noticed in the case of the weights. The Medians of component batches are sufficiently steady §.

* The aggregation of observations at round numbers is one of the vitiating causes in both cases.

† The Arithmetic Mean and Median coincide.

‡ Taking as the apparent weight $8\frac{3}{4}$, intermediate between the *average* which is $8\frac{1}{2}$ nearly and the *median* which is 9 nearly.

§ For further details see the companion paper.

A still less perfect verification is presented by a series of

X. X. XX. XX. X. X
8·5 14 50 100 225

V. V. X. X. V. V.
7 17·5 55 122·5 225

V. V. X. X. · V.
10 12·5 30 87·5 300

eighty answers which I have obtained to the question : "How many five-pound notes are equal in weight to a sovereign?" The grouping of these estimates is represented in the first of the annexed schemes *; the grouping of two component batches by the second and third scheme. It will be seen that there is a general resemblance between the two parts and the whole. The displacement of the Median seems not inconsistent with the hypothesis of a constant facility-curve. I thus conclude, partly from a rough application of the formula which I have cited from Laplace in the predecessor to this paper for the error of the Median of any facility-curve †, partly by a still rougher reasoning as to the divergency that might be expected, if we were dealing with a genuine Probability-curve.

But it must be admitted that the upper extremity of the curve defies law. The maximum of the first batch of 40 is 1500; the maximum of the second batch of 40 is 20,000 ‡. These fluctuations are so violent that we could not expect to determine their law of frequency without statistics more copious than I have attained for examination marks. It should be considered, however, that at examinations the maximum and minimum are usually fixed, so that enormous vibrations of the extremities are impossible. In so far as the abnormal or incalculable element in the fluctuation of the maximum or minimum may make itself felt, it should be held that my estimates of the extent to which chance affects examinations are underrated.

These experiments in an intermediate case seem to warrant our applying with caution the Theory of Errors to the more specific experience which I shall now adduce. It consists of

* *Cf.* p. 172.
† Phil. Mag. October 1886, p. 375.
‡ The true figure is 6!

two kinds: marks given by two examiners to several papers, and marks given to the same paper by several examiners. The former kind of statistics are more copious; and in some respects more valuable. For they admit of being freed from a constant difference, of the nature of a "personal equation," between the marking of two examiners; which equally affecting all the competitors does not disturb their *order;* and therefore perhaps ought not to be reckoned *.

My method of dealing with these data may be described by taking as an example the most perfect specimen which I obtained, namely the marks given by two examiners to 400 pieces of English composition. First I took the difference between the two marks given to each paper; then squared those differences, found the Arithmetical Mean of those squares, and took the square root of that Mean Square as the *Mean Deviation* in a sense analogous to Gauss's use of *Mean Error*. The peculiar propriety of this coefficient as a measure of discrepancy is that it not only represents, as well as other sorts of average error, the deviation between marks in any particular subject; but also leads to the coefficient of the Probability-curve which measures the deviation between *sums* of marks. Thus the Mean Deviation for the 400 pairs of marks in English composition proved to be 67, which coefficient not only gives a general idea of the discrepancy to be expected between any two marks, but also yields a precise system of measures for the discrepancy between the sum of several marks assigned by two examiners to the same pieces of work *in pari materiâ*. That discrepancy would fluctuate according to a Probability-curve whose *modulus* is $\sqrt{2 \times 10} \times 67$, or whose *probable error* is $\cdot 674 \ldots \times \sqrt{10} \times 67$.

For our purpose it is generally convenient to express the Mean Deviation as a percentage of the mean mark for a whole set of papers. Thus in the case before us the average of the 800 marks was 227; and accordingly the Mean Deviation per cent. in round numbers 30.

This result requires to be corrected for a certain "personal equation." The constant difference between the two sets of marks is about 20, nearly ten per cent. of the average mark. The square of this constant difference is to be subtracted from the uncorrected Mean Square; of which 67 was the square root. The corrected Mean Deviation is 64; expressed as a percentage of the average mark, 28 nearly.

The worth of this result may be appreciated from the state-

* See on this point the companion paper in the 'Journal of the Statistical Society' for 1890.

ment that, whereas the (uncorrected) Mean Deviation for the whole set of 400 pairs is 67, the corresponding determinations for the first, second, and third batches of 133 papers were respectively 63, 67, 72. I have similarly verified other results obtained by the same or a similar method. Thus the Mean Deviation for marks given by two examiners to fifteen papers in Greek Prose is 25 per cent.; for marks given by the same pair of examiners to thirty pieces of Latin Prose, 26 per cent. For sixty pieces of Greek and Latin Prose and Verse composition (including some of the Latin Prose but not any of the Greek Verse before mentioned) I obtained,—by a more summary, but in the particular instance at least sufficiently safe, method *—again 25 per cent.

When the statistics are in the form of marks given by several examiners to the same piece of work, I have extracted the Mean *Error* according to the usual rules; and then multiplied the coefficient by $\sqrt{2}$ in order to obtain the Mean *Deviation* as above defined. In the only case in which I have been able to compare the two methods of determination the results yielded are fairly consilient. I refer to Latin Prose Composition, for which, according to the first method, I extracted from thirty pairs of marks given to as many pieces of prose the Mean Deviation 26 per cent. By the second method I obtained from twenty-eight marks given to the same piece of prose by as many highly competent examiners the Mean Deviation 20 per cent.—of the average of the twenty-eight marks; which, being four fifths of the maximum, is not exactly comparable with the *general* average referred to in the first method.

I annex a summary statement of the results obtained by one or other of those methods †:—

* Using the formula: Mean Deviation (in the sense above defined) $= \sqrt{\frac{\pi}{2}}$ *Average* Deviation; which relation had held good for a great number of marks given by the same examiners in a variety of Classical subjects including Composition.

† These computations derive some confirmation from an experiment which Mrs. Bryant, D.Sc., of the North London Collegiate School, has communicated to me. Having examined forty Geometry papers, she re-examined them after some weeks. The discrepancy between the two sets of marks (corrected for a certain difference of *scale*) proves to be only 12 per cent. For further remarks on the Table see the companion paper.

Designation.	Number of Marks on which the Computation is based.	Mean Discrepancy per cent. of mean mark.
High School, Geometry and History ...	160	15
Cambridge Honours in Classics, Translation, History, Composition (mixed)	480	18
India Civil Service, Latin Prose Composition..................................	28	20
Oxford Honours in Literæ Humaniores, Philosophy (alone)...........	480	21
Cambridge Honours in Classics, Composition (alone)	150	25
English Composition	800	28

These data are adapted to certain problems which are of practical interest.

I. The first problem is, What is the probability that any particular candidate who has come out successful at an examination would have been successful (or *vice versâ*) if the candidate's work and that of his competitors had been appraised by a different though equally competent examiner (or examiners)? This general problem may be variously subdivided. First (A) success may be defined by the attainment of a predetermined number of marks, a fixed Honour Line. Or (B) the number of prizes, say n, may be predetermined; and the first n candidates, without respect to the absolute number of their marks, may obtain prizes. Other distinctions turn on the presence or absence of an attribute which is particularly favourable to the calculation of probability: namely, a certain plurality which renders applicable the laws of large numbers; the attribute in virtue of which the movement of multitudinous atoms is more tractable than the problem of three bodies. We may inquire whether a candidate would be displaced, if (x) the mark assigned to each paper in each subject had been what may be called the *true mark*—namely the mean of the marks given by an indefinite number of equally competent examiners; or ($\bar{x}$), if the marks in each subject had been given by a single examiner (or a few) different from the one (or two) who acted on the given occasion. Again (y) the number of competing candidates may be large, or ($\bar{y}$) not so. Lastly (z) there may be several subjects, the candidate's place being determined by the sum of his marks in each; or ($\bar{z}$) there may be only one or two subjects.

Of the immense number of cases formed by the combinations of these attributes I shall discuss only the most interesting. First in the order of simplicity is Axz. In this case there is a fixed Honour-line, say H; the comparison is between the place actually obtained by the candidate and the place which he would have obtained if in each subject the marks had been determined by a numerous jury of competent examiners; the number of competitors may be either many or few; the number of subjects is large, say S. Since the number of subjects is large and the marking in each subject fluctuates with the change of examiners according to *a definite* law of frequency, it follows that sums of S marks fluctuate according to *the* law of error, the Probability Curve. Let C be the Mean Deviation for the marking in each subject; in the sense above defined, that is $\sqrt{2}\times$ the Gaussian Mean Error. Then the Probability-curve, according to which the compound mark determined by any set of S examiners will fluctuate, has for its Modulus the coefficient C; which is ascertainable by observation. Now suppose the candidate has obtained the mark $H+l$. The problem may be likened to the familiar one: If the average of S observations of given precision be $H+l$, what is the probability that the true value is less than H. By received reasoning the probability in question is

$$\frac{1}{\sqrt{\pi}C}\int_{l}^{\infty} e^{-\frac{x^2}{C^2}}\,dx\,;$$

which may be calculated from the usual tables.

$A\bar{x}z$. This case differs from the preceding in that the comparison of the actual compound mark is not with the true mark, but with the mark which any other set of S competent examiners might have assigned. It is as if, in the parallel problem, we sought the probability, not that the true value is less than H, but that any second measurement made under similar conditions should fall below H. According to a well-known theory the solution is obtained by substituting $\sqrt{2}\times C$ for C in the solution of the preceding problem.

$Bxyz$. This is the case in which a fixed large number, say n, prizes are assigned to the n candidates who come out first (irrespectively of the absolute number of their marks); and the inquiry is whether any particular candidate would have his status changed from successful to unsuccessful, or *vice* *versâ*, if each of the marks in each subject had been determined by a jury; the number of candidates, say m, and the number of subjects, S, being large. The number of candidates and prizes being large, we may assume that the mark of the nth candidate in a descending order of merit, that is the Honour line, will be constant for any particular set of examiners; for much the same reason that, if two large batches of similar objects, *e. g.* statures of the same nation, be taken at random, the quartiles, octiles, deciles, &c. remain constant. Thus the problem is reduced to $Axyz$ which—as one case of Axz—has been solved.

$B\bar{x}yz$. This problem is related to the preceding $Bxyz$, as $A\bar{x}z$ to Axz.

$Bx\bar{y}z$. This case differs from $Bxyz$ in that the number of the competitors (and prizes) is small. First, let there be only two competitors, and one prize. The problem is: What is the probability that the verdict of any particular set of S examiners would be reversed, if the two papers in each of the S subjects had been marked by a jury. Let the (compound) marks of the two candidates differ by l. The probability of reversal is identical with the probability that the difference between the candidates in a particular direction, or with its *sign*, according to the actual set of examiners, should differ by as much as l from the difference in the same direction under the jury-system. If, as before, C is the Mean Deviation in each subject, then, upon principles to which allusion has been made, the required probability is

$$\int_l^\infty \frac{1}{\sqrt{\pi 2 \mathrm{S}}\,\mathrm{C}} e^{-\frac{x^2}{2\mathrm{S}\mathrm{C}^2}} dx.$$

Next, let there be three candidates; and, to fix the ideas, let there be two prizes, and let the question be, What is the probability that the second prize-man would fail to obtain a prize, if the work were marked by other examiners? In order that the original second should become third, it is evident that he must come out below the original third. Thus, l being the distance between the original second and third, the solution would be the same as for the simple case, if the original first were not liable to move relative to the original second. But, in virtue of this liability, a certain proportion of cases in which the original second comes out below the original third are not failures for the original second. Thus l, the distance from the first of the unsuccessful, or Honour-line as it may be called, being the same, the probability of failure decreases with the increased number of candidates. The limiting case where, the candidates and prizes being indefinitely numerous, the Honour-line may be regarded as fixed, our $Bxyz$. It will be observed that the measure of discrepancy for $Bxyz$ is less than the corresponding measure for the extreme case of $Bx\bar{y}z$ in the ratio $1 : \sqrt{2}$. We have thus a rough measure of the inaccuracy which we commit in treating intermediate cases according to the rule proper to either extreme.

$B\bar{x}\bar{y}z$. This relation of this case to the preceding is like the relation of $Bxyz$ to $B\bar{x}yz$; which has already been considered. The coefficient of discrepancy which is proper to $Bx\bar{y}z$ should be multiplied by $\sqrt{2}$ for $B\bar{x}\bar{y}z$.

$Bx\bar{y}\bar{z}$. This case differs from the preceding in that the number of the papers is small. At this stage therefore the Probability-curve which has hitherto illumined our course disappears. A certain twilight may, however, be derived from that source of illumination. Take the extreme case of two candidates examined in one subject for a prize. What is the probability that the award of one examiner would be reversed by an equally competent examiner? As we saw under a preceding head, the answer turns upon the variation with the change of examiners in the difference between the marks of the two candidates. Such a difference between differences of marks will in general fluctuate according to the same law as a sum of *four* marks taken at random from under the facility-curve, according to which by our postulate the marks of different examiners fluctuate. But a sum of four observations taken from under any facility-curve will in general fluctuate according to a law which is getting on for a Probability-curve, unless indeed the given facility-curve be exceedingly abnormal. But so far is the facility-curve with which we have to deal from being exceedingly abnormal, that it is presumably getting on for a Probability-curve. Accordingly the rule for $Bx\bar{y}z$ may pretty safely be extended to $Bx\bar{y}\bar{z}$; especially where we have specific experience that the facility-curve in question does not violently rebel against the normal law of error—experience which I have obtained with respect to several subjects. For example, let a prize be given to the one of two candidates who obtains the higher mark for a piece of Latin Prose, of about the same calibre as the Composition at the India Civil Service or Cambridge Classical Tripos Examinations. Even if the successful candidate exceeds his rival by *fifty* per cent. (of the mean between the two marks), there is some probability, say one in a hundred, that the verdict would be reversed by another equally competent examiner.

II. The answers which have been given to the first problem are required for the solution of the second problem: At any examination of which the circumstances are given, how many of the candidates are *uncertain* in this sense, that there is an appreciable chance of any assigned one of them who is now successful coming out unsuccessful, and *vice versâ*; if the work were marked by different but equally competent examiners? If we confine ourselves to the general case of several candidates and several subjects (yz), we have only to measure from the Honour-line in both directions a distance such that the probability of any candidate at this distance being displaced is very small, say less than one in a hundred. This *improbable* error, or discrepancy as it may be called, is found by multiplying the *probable* error, or discrepancy, proper to the case by 3·5. The candidates above that limit may be described as "safe."

The reader who applies this formula to statistics of examinations, such as those which are given in the Reports of the Civil Service Commissioners, may be surprised to find that the number of the uncertain unsuccessful is greater than that of the uncertain successful; although, in the case of a determinate number of prizes (B), every instance of a successful candidate being in the wrong box involves an instance of an unsuccessful candidate being misplaced. The explanation of this anomaly is that, in applying the received formula, we have made the common assumption that the *à priori probability* of the candidate's real mark, so to speak, being one figure rather than another is constant. The nature of that assumption and the caution with which it must be made* are well illustrated by these problems. In the present case the *à priori* probabilities are not constant. In general the marks of candidates at an examination are not distributed equably between the positions of the senior and the man at the bottom; but are heaped up in the form of a Probability-curve†. Now the scene of our operations is the upper extremity of this Probability-curve; whence it follows that the *à priori* probability (for each point or degree) diminishes as we ascend

* See my paper "On *A priori* Probabilities" in the Philosophical Magazine for September 1884; also 'Metretike' (London, Temple Co., 1887).

† With respect to this statement and others which may seem to require proof the reader is again referred to companion papers in the Journal of the Statistical Society, Sept. 1888 and Sept. 1890. I have sometimes in those papers used the term "true mark" for the mean of the marks given by an indefinite number of examiners—a conception which is not absolutely essential to the $\bar{x}$ variety of our problems.

from below towards the Honour-line and above it. The effect of this consideration is that we have somewhat underrated the probability of displacement for positions above the Honour-line, and somewhat overrated it for positions below. It may be assumed, I think, that these errors will compensate each other when we determine the total number of the uncertain in the manner which I have indicated.

III. I have made a similar assumption in solving the following *third* problem. At any examination of which the circumstances are given, what number of candidates is *most probably* displaced? I proceed as follows. Having ascertained the coefficient of the probability-curve which governs the case, I determine numbers corresponding to equal increments of that coefficient above and below the Honour-point. In each of the degrees so constituted I find how many candidates are comprehended; and I assign to the candidates in each degree the probability of displacement which is found by Problem I. (with the aid of the proper tables) to appertain to the centre of the degree. The product of the number of candidates and the probability of their displacement gives for each degree the number most probably displaced; which numbers being added give the total number most probably displaced. It is assumed that the underrating of the probability above the line will fairly well be compensated by the overrating below.

An example will make my meaning clear. With reference to an examination for 50 clerkships of the second class, of which the statistics are given in the Twentieth Report of the Civil Service Commissioners*, how many would most probably be displaced if the work has been marked by another set of equally competent examiners. The problem is of the species $B\bar{x}yz$, the candidates being numerous and the papers about ten in number. The Honour-line is at 1720, and the *probable error* for the regulating Probability-curve (what I have elsewhere called the probable discrepancy) is taken as 50; upon the assumption that the *Mean Error* for each of the ten papers is 15 per cent., that is the lowest coefficient which I have actually observed. Accordingly the intervals 1720–1730 &c. correspond each to a fifth of the Probable error †. The computation is shown in the annexed Table.

* Parliamentary Papers, 1876, xxii. p. 180.

† Here called probable *error* with reference to the tables in the books, elsewhere in connexion with the subject-matter probable *discrepancy*; being $\sqrt{2}\times$ probable divergence of a mark from the "true mark."

1.	2.	3.	4.	5.	6.	7.	8.	9.
	Successful.		Unsuccessful.					
Degrees of probable error.	Marks.	Number of candidates.	Marks.	Number of candidates.	Total number of candidates.	Half-integral of Error-function.	Probability of displacement.	Number of candidates × Probability of displacement.
·0-·2	1720-30	1	1720-10	4	5	·027	·473	2·3
·2-·4	1730-40	2	1710-1700	2	4	·080	·42	1·7
·4-·6	1740-50	3	1700-1690	2	5	·132	·37	1·8
·6-·8	1750-60	1	1690-80	5	6	·181	·32	1·9
·8-1·0	1760-70	2	1680-70	4	6	·228	·27	1·6
1·0-1·2	1770-80	3	1670-60	3	6	·270	·23	1·4
1·2-1·4	1780-90	2	1660-50	1	3	·309	·2	·6
1·4-1·6	1790-1800	2	1650-40	4	6	·344	·16	·9
1·6-2·0	1800-20	3	1640-20	4	7	·387	·11	·8
2·0-2·4	1820-40	6	1620-1600	7	13	·431	·07	·9
2·4-2·8	1840-60	6	1600-1580	10	16	·46	·04	·6
2·8-3·2	1860-80	2	1580-1560	2	4	·47	·03	·1
								14·6

In this Table the first column denotes degrees of probable error corresponding to intervals of marks designated in columns 2 and 4. Columns 3 and 5 give the number of candidates successful and unsuccessful whose marks fall in those intervals; column 6 the total of those numbers. Column 7 contains the values of half of the integral $\frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx$ for x = Probable Error* $\times$ ·1, ·2, &c.—1·8, 2·2 respectively (values obtained from the fourth table appended to DeMorgan's 'Calculus of Probabilities'). Column 8 gives the difference between ·5 and each of these values, the corresponding integral between limits ∞ and x; which represents the probability of displacement for candidates in the corresponding compartment. Column 9, the product of column 6 and column 8, gives the most probable of number of those who would be displaced, for each degree or interval. The sum of these numbers is the most probable number displaced, out of all the candidates. I take the half of this number as the most probable number of successful candidates who would be displaced on re-examination.

In conclusion I submit a Table containing answers to Problems 2 and 3 for certain cases which seem to me fairly typical of the various statistics which I have inspected.

In this Table the *first* column designates a service to which appointment is made by competitive examination. The *second* column contains references to the Reports of the Civil Service Commissioners, in which are published the marks given at public examinations. The Reports referred to are in the 22nd volume of the Parliamentary papers for 1875 and for 1876. The *third* column gives the number of candidates at each of the examinations referred to in this table. The *fourth* column gives the corresponding numbers of successful candidates. The *fifth* column contains the mark of the lowest successful candidate at each of the examinations, or of the highest unsuccessful, or some intermediate number (figures differing from each other by quantities which may be neglected). The *sixth* column contains the aggregate marks which occupy the halfway position in the order of merit at each examination. Thus at the second examination (referred to in the second row of the table) there being 171 candidates, the aggregate mark which is 86th in the order of merit is 1601; in round numbers 1600. At the first examination, the number of candidates being even, viz. 150, the Median is intermediate between the 75th mark, which is (in the descending order of merit) 1601, and the

* See footnote, p. 183.

1.	2.	3.	4.	5.	6.	7.	8.	9.		10.		11.	12.	13.
Designation.	Reference to Report of Civil Service Commission.	Total number of candidates.	Number of successful.	Honour line.	Median.	Probable discrepancy.	Improbable discrepancy.	Limits of uncertainty.		Number of the uncertain.		Proportion of the successful quite safe.	Most probable number of displacements.	Proportion of the successful most probably displaced.
								Upper.	Lower.	Successful.	Unsuccessful.			
Clerks No. 2.	XX. p. 168.	150	24	1850	1600	48	168	2018	1682	19	25	·2	5	·2
Clerks No. 2.	XX. p. 180.	171	50	1720	1600	48	168	1888	1552	33	49	·34	7	1·4
India Civil Service.	XIX. p. 522.	206	38	1055	674	27	95	1150	960	15	14	·60	3	·08
India Civil Service.	XX. p. 431.	198	37	1076	740	30	105	1181	971	13	16	·65	3	·08
Cavalry and Infantry.	XIX. p. 328.	329	152	2872	2737	82	287	3159	2585	32	25	·8	4·5	·03
Cavalry and Infantry.	XX. p. 245.	351	110	3301	2560	77	269	3570	3032	29	23	·74	4	·04

76th which is 1597; in round numbers 1600. In the *seventh* column, each entry is the discrepancy which is as likely as not to occur between the sum of ten marks given to any candidate's papers at the examination referred to and the sum of the marks which might have been given to the same work by any other set of equally competent examiners. This figure is calculated from the formula: Mean Discrepancy = Mean Error $\times \sqrt{2} \times \cdot 477$ (or Mean Error $\times \cdot 67 \ldots$) $\div \sqrt{10}$. Here 10 is the number (or greater than the number) of the papers answered by a candidate. The other figures are explained in the books on Probabilities. The Mean Error is a coefficient determined by observation in the manner described in this paper (above 176). For most of the examinations the lowest figure actually observed, viz. 15, has been taken for the Mean Error. For the India Civil Service the higher coefficient 18 has been used; partly because that figure actually has been observed for that examination; and partly because the examination includes more advanced and speculative subjects than the Examinations for which the coefficient 15 was observed. The *eighth* column gives the discrepancy which in each case is very unlikely to occur, against the occurrence of which the odds are about 100 : 1. The improbable discrepancy is by the Theory of Errors equal to the probable discrepancy multiplied by 3·5 nearly. In the *ninth* column each upper limit is formed by adding the improbable discrepancy to the Honour-Line, the lower limit by subtracting the same figure from the same. In the *tenth* column the number of the successful who are uncertain is ascertained by counting the number of candidates whose marks are between the honour-line and the upper limit of uncertainty; the number of the unsuccessful who are uncertain is found by counting the number of candidates between the honour-line and the lower limit of uncertainty. To form the *eleventh* column subtract the number of the uncertain successful from the total successful; the remainder is the number of those who are "safe" in this sense that for any assigned one of them the odds against his being displaced upon a reexamination of his work are about 100 to 1. The number of the safe divided by the number of the successful at each examination is entered in the eleventh column. The laborious formation of the *twelfth* column is described above at page 185. To form the *thirteenth* column divide each entry in the twelfth column by the corresponding entry in the fourth. The average of the figures in the thirteenth column relating to the same class of examination gives the proportion of the successful candidates which would most probably be displaced upon a reexamination of their work—most probably in the same sense as we may say that the average death-rate represents the proportion of the population who will most probably die in any proximate year. Thus in the case of the India Civil Service we may say—or rather might have said at the period to which the statistics relate, twelve years ago—that the most probable proportion of displacement—the degree of failure of justice which may be expected—amounts to 8 per cent. of the successful; or rather $8 \div \sqrt{2}$, say 6, per cent., if we define the just verdict as that which would be found by taking the average of the results obtained by a variety of competent examiners.

XXI. *Some Electrical Properties of Flames.* *By* MAGNUS MACLEAN, *M.A.*, *F.R.S.E.*, *and* MAKITA GOTO (*Japan*) *.

[Plate V.]

IN connexion with our experiments on the "Electrification of Air by Combustion," we were led to make some experiments on the electricity of different parts of the flame itself. A Bunsen burner was used, and the potentials at different points, both inside and outside of the flame, were examined. In fig. 1, Plate V., is shown the arrangement for examining the inside of the flame. AA and BB are platinum wires insulated from the burner and projecting into the flame 5 millim. above the upper end of the burner. These wires can be adjusted by bending so as to lie in various positions from the middle line of the flame to its boundary. In our experiments one of them was left insulated (with its end free in air), while the other was connected to a terminal of a Thomson Quadrant Electrometer. Or, again, one wire was connected to one terminal of the electrometer, and the other to the other terminal. The Bunsen burner itself was always connected to earth. The sensitiveness of the electrometer was generally such that a difference of potential of $\frac{1}{200}$ volt between the terminals could be observed.

In fig. 2 is shown the arrangement for examining the outside of the flame. C is a platinum wire fused into a glass tube which covers the wire except a very small portion of its end. D is the scale for measuring the distance of the point of the platinum wire from the boundary of the flame. E is the index.

By these arrangements it was found that the flame is negatively electrified, while the film of air surrounding the flame is positively electrified. These results were already obtained by Elster and Geitel. Our results agree with what they found, though our method of examining the different parts of the flame is different from their method. (See an abstract of their experiments by S. P. Thompson in 'Nature,'

* Communicated by the Authors.