On the Musical Scale

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C. A. BARRY, ESQ.,

IN THE CHAIR.

ON THE MUSICAL SCALE.

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IN a footnote to Blaserna's " Theory of Sound in relation to
Music " (p. 94), it is stated that Euler was acquainted with
the importance of the numbers 2, 3, and 5, and that he
" established upon them a rule for the development of our
musical system."   To understand this statement it is
necessary to know that a musical note is an effect of regular
vibrations of a sounding body—that is to say, of vibrations
occupying equal intervals of time ; and that two notes cannot
be consonant unless the proportion subsisting between their
two rates of vibration is simple.   These things being under-
stood, it may be shown that the importance of the numbers
2, 3, and 5, employed for the purpose of expressing the ratios
of vibration rates, cannot be overestimated.   When the
intervals between musical notes are expressed by the ratios
of the vibration rates of the notes, these three numbers
alone, with their squares, cubes, and other powers, and
numbers arising from their intermultiplication, are capable of
expressing the ratios of consonant notes ; and not only so,
but they alone are capable of expressing the ratios of all
notes belonging to the same scale, and of all notes belonging
to all scales into which it is possible to pass from the original
scale by modulation.
    Any prime number higher than 5 occurring in the expression
for the ratios of two notes indicates that those two notes

* Mr Habens being resident in New Zealand, the paper was read by the
Assistant Secretary.

cannot both belong to the same musical system. For example, if for every 6 vibrations of one note another note has 7 vibrations, these notes are not musically related, they cannot even enter together into any discord recognised in music. Two notes having the ratio 14 : 7 are at the interval of an octave, but this ratio ought to be written 2 : 1, and 7. has nothing to do with the ratio.

The simplest ratio is 2 : 1, where the higher note is an octave above the lower. The ratio 4 : 1 expresses an interval of two octaves, and 8 : 1 three octaves. If we consider only the ratios of consecutive numbers, and take them in order, beginning with the simplest, the intervals are found to be as follows:—

Ratio 2 : 1   —   Octave.
  ,,   3 : 2   —   Perfect fifth.
  ,,   4 : 3   —   Perfect fourth $(2 \times 2 : 3)$.
  ,,   5 : 4   —   Major third $(5 : 2 \times 2)$.
 ,,   6 : 5   —   Minor third $(3 \times 2 : 5)$.
  ,,   7 : 6   —   No relation (7 is prime).
  ,,   8 : 7   —   ,,   (,,   ).
  ,,   9 : 8   —   Greater tone $(3 \times 3 : 2^3)$.
  ,,   10 : 9   —   Lesser tone $(5 \times 2 : 3^2)$.
  ,,   11 : 10   —   No relation (11 is prime).
  ,,   12 : 11   —   ,, $(11$,, $)$.
  ,,   13 : 12   —   ,, $(13$,, $)$.
  ,,   14 : 13   —   ,, $(7 \times 2$ and $13)$.
  ,,   15 : 14   —   ,, $(5 \times 3 : 7 \times 2)$.
  ,,   16 : 15   —   Semitone $(2^4 : 5 \times 3)$.

The next ratio in the series that is not to be rejected on account of primes higher than 5 is—

Ratio 25 : 24   —   Minor semitone $(5^2 : 3 \times 2^3)$.

And the next is—

Ratio 81 : 80   —   Comma $(3^4 : 5 \times 2^4)$.

This last interval is the difference between the intervals of the greater tone and lesser tone (the ratio of $\frac{9}{8}$ to $\frac{10}{9}$, that is, of $\frac{81}{72}$ to $\frac{80}{72}$). The ratio 25 : 24 belongs to the interval of the lesser semitone, which is the difference between the major third and the minor third (the ratio of $\frac{5}{4}$ to $\frac{6}{5}$, that is, of $\frac{25}{20}$ to $\frac{24}{20}$).

It will be seen that all the numbers that are not rejected are the numbers 2, 3, and 5, and numbers resulting from raising these to higher powers, or from their inter-multiplication. The intervals obtained from ratios of consecutive numbers alone are, as has been shown, the octave, perfect fifth, major third, minor third, greater tone, lesser tone, semitone, lesser semitone, and comma. Several other intervals are obtained from these by inversion. It is evident that such intervals must belong to the same system, and must rest on the ratios of 2, 3, and 5. For inversion means

substituting for one of the two notes whose pitch is under consideration the octave of that note, and the change thus effected is expressed by halving the higher number of the ratio, or doubling the lower. Thus, the ratio of the minor third is 6 : 5, and the ratio of the major sixth resulting from its inversion is 3 : 5. The major third expressed by the ratio 5 : 4 becomes by inversion the minor sixth, of which the ratio is 5 : 8. The inversion of the tone gives the minor seventh, of which there are two forms corresponding to the greater tone and lesser tone ; the ratio of one being $\frac{9}{16}$, and of the other $\frac{5}{9}$. The inversion of the octave gives an octave, and the inversions of the fifth and the fourth give respectively the fourth and the fifth.

There are other intervals in the scale, intervals of which the ratios are more complex, but all accordant with the law that admits no factors but 2, 3, and 5. Such intervals are—

Imperfect fifth   —   64 : 45, that is $2^6 : 5 \times 3^2$.
Tritone   —   45 : 32, „   „ $5 \times 3^2 : 2^5$.
A minor third $\left(\frac{F \text{ to } D}{\text{in key } C}\right)$ —   32 : 27, „   „ $2^5 : 3^3$.

From the consideration of intervals we may now advance to the construction of the scale. The ratios of 2, 3, and 5 indicate three closely related notes; for if in some definite short space of time, say the sixteenth part of a second, a sounding body executes one vibration and gives forth a low note, say C, then another sounding body executing two vibrations in the same time will give forth a note an octave above this C ; four vibrations in the same time are executed by a body giving forth a note an octave above the second C ; three vibrations by a body sounding G, five vibrations by one sounding E. The vibrations 1, 2, 3, 4, 5, occurring in the same time, belong to notes constituting a common major chord. This fact may be represented graphically by a line divided into equal parts, of which each represents one vibration in the given time (*see* Fig. 1, p. 4).

If now a common chord with G instead of C for a bass be found, that is to say, if two notes be ascertained having the same relation to G that E and G have to C, the two notes thus found are B and D. And again, if a common chord be found in which the three notes occur in the same order as C, E, and G, but C occupies in the new chord the position occupied by G in the original chord, the three notes of the new chord are F, A, C. Thus all the notes in the scale of C are found. In the diagram, as the octave between 3 and 6 has three equal divisions, let three equal divisions be taken in the octave from 2 to 4, and by this means F and A are found. As at 3 G falls half-way between C and C, insert a note D half-way between G and G, and half-way between G and this new note put in B, just as E falls half-way between C and G.

This statement may be graphically expressed as follows (*see* Fig. 2) :—

Fig. 1.

Fig. 2
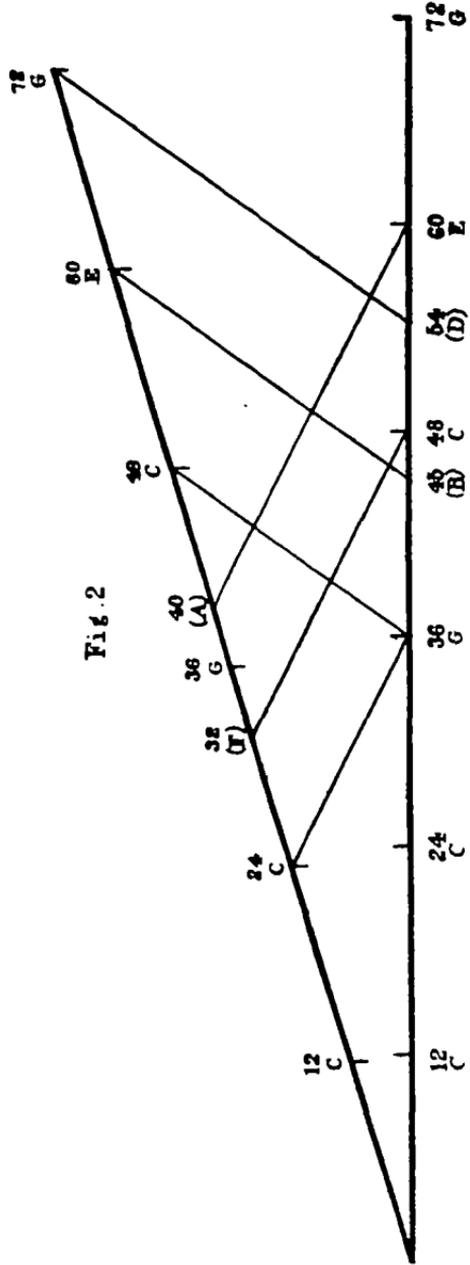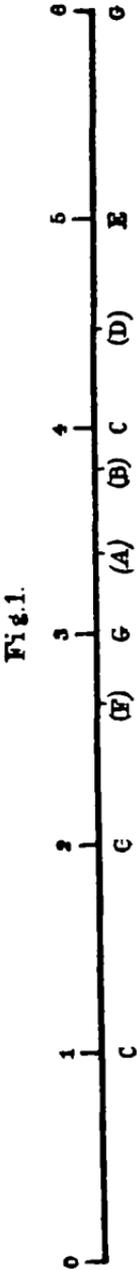
Let the horizontal line have the same meaning as before, but to avoid fractions suppose the time in which the vibrations occur to be twelve times as long as before, so that each of the six original divisions represents 12 vibrations. Draw a similarly divided line making any angle with the other, and complete a triangle by joining the points C 48 and G 36. Lines drawn parallel to this third line will cut the two other lines proportionally. Lines so drawn from E and G will therefore give the places of B and D (45 and 54).

Similarly if C 24 and G 36 are joined, two parallel lines from C and E will give the places of F and A (32 and 40).

In this way the seven notes of the scale in consecutive order from F to E are obtained, the ratios of their vibrations being as follows—

$$32, \ 36, \ 40, \ 45, \ 48, \ 54, \ 60.$$

These numbers are all compounded of numbers belonging to the group 2, 3, 5. Expressed in terms of this group they are—

$$2^5, \ 3^2 \times 2^2, \ 5 \times 2^3, \ 5 \times 3^2, \ 3 \times 2^4, \ 3^3 \times 2, \ 5 \times 3 \times 2^2.$$

To express the scale from C to C it is necessary to take the octaves of some of the notes. The octaves below the notes represented by the three highest numbers must be represented by the halves of these numbers, and the order will be—

$$24, \ 27, \ 30, \ 32, \ 36, \ 40, \ 45, \ 48.$$

Expressed in terms of the series 2, 3, 5, the vibrations of these notes are—

$$3 \times 2^3, \ 3^3, \ 5 \times 3 \times 2, \ 2^5, \ 3^2 \times 2^2, \ 5 \times 2^3, \ 5 \times 3^2, \ 3 \times 2^4$$

From this it is evident that if the vibrations of C and its octave are expressed by the ratio 1 : 2, the vibration numbers for the several notes of the scale from C to C are in the following proportion :—

$$1, \ \tfrac{3^3}{2^3}, \ \tfrac{5}{2^2}, \ \tfrac{2^2}{3}, \ \tfrac{3}{2}, \ \tfrac{5}{3}, \ \tfrac{5 \times 3}{2^3}, \ 2.$$

In this statement it appears very clearly that only the ratios of 2, 3, and 5 are concerned in the relations of the several notes of the scale to each other. But it is advisable to express these fractions in a more familiar way, as follows :

$$1, \ 1\tfrac{1}{8}, \ 1\tfrac{1}{4}, \ 1\tfrac{1}{3}, \ 1\tfrac{1}{2}, \ 1\tfrac{2}{3}, \ 1\tfrac{7}{8}, \ 2.$$

It must be borne in mind that the interval between two notes whose vibration numbers are to one another in the ratio of two fractions in this series cannot be adequately expressed by the difference of the two fractions. Thus the difference between 1 and $1\tfrac{1}{8}$ is $\tfrac{1}{8}$, but this does not express the relation between C and D. For in the next octave the vibration numbers of C and D are represented by 2 and $2\tfrac{1}{4}$, and the difference in this case is $\tfrac{1}{4}$ instead of $\tfrac{1}{8}$. Moreover, the difference which from one point of view is $\tfrac{1}{8}$, is from another point of view $\tfrac{1}{9}$, for the difference $\tfrac{1}{8}$ is one-eighth of the lesser quantity 1, but it is one-ninth of the greater quantity $1\tfrac{1}{8}$. The fundamental way of regarding the interval is to regard it as depending on a ratio of vibrations, the ratio of the

vibrations of D to those of C being 9 to 8. Looked at from this point of view the intervals in the scale of C from each note to the next, step by step, are indicated by ratios as follows:—

$$\text{C to D, } 24:27 = 8:9.$$
$$\text{D to E, } 27:30 = 9:10.$$
$$\text{E to F, } 30:32 = 15:16.$$
$$\text{F to G, } 32:36 = 8:9.$$
$$\text{G to A, } 36:40 = 9:10.$$
$$\text{A to B, } 40:45 = 8:9.$$
$$\text{B to C, } 45:48 = 15:16.$$

It thus appears that the intervals of the second are of three magnitudes. Between E and F the ratio is $15:16$, as also between B and C, and the interval indicated by the fraction $\frac{15}{16}$ or $\frac{16}{15}$ is the semitone. Between C and D, as also between F and G, and between A and B, the ratio is $8:9$, and the interval indicated by $\frac{8}{9}$ or $\frac{9}{8}$ is the greater (major) tone. The third magnitude is the ratio $9:10$ between D and E, and between G and A, and the interval indicated by $\frac{9}{10}$ or $\frac{10}{9}$ is the lesser (minor) tone. The order in which these intervals appear in ascending from C to C is—

Greater Tone, Lesser Tone, Semitone, Greater Tone, Lesser Tone, Greater Tone, Semitone.

In a rude way these intervals may be approximately compared by means of fractions, thus : if by $\frac{1}{8}$ we agree to understand $\frac{1}{8}$th of the vibration number of the lower of two notes we may say that the note which is a lesser tone above it is sharper by $\frac{1}{8}$th ; or, if by $\frac{1}{8}$ we agree to understand $\frac{1}{8}$th of the vibration number of the higher of two notes, we may say that the note which is a greater tone below it is flatter by $\frac{1}{8}$th. On the first understanding the tones and semitones in the ascending scale will appear as—

$$\tfrac{1}{8}, \ \tfrac{1}{9}, \ \tfrac{1}{16}, \ \tfrac{1}{8}, \ \tfrac{1}{9}, \ \tfrac{1}{8}, \ \tfrac{1}{16}.$$

On the other understanding they will appear as—

$$\tfrac{1}{9}, \ \tfrac{1}{10}, \ \tfrac{1}{16}, \ \tfrac{1}{9}, \ \tfrac{1}{10}, \ \tfrac{1}{9}, \ \tfrac{1}{16}.$$

These statements bring out in a rough way the inequality of the greater and lesser tones, and show that the semitone is greater than the half of either tone ; but the fact that the two inconsistent statements are equally tolerable is enough to show that they are both untrue.

In order to express intervals satisfactorily, we require a notation by which any number of intervals that together make up some greater interval may be expressed by a series of numbers, the addition of which will result in a total that will consistently express the greater interval. While our only consistent method of indicating intervals is the employment of ratios, we must multiply the ratios proper to the lesser intervals to obtain a consistent expression for the greater interval which together they make up.

For example, a greater tone, a lesser tone, and a semitone occupy the whole interval from C to F, which is the interval of a fourth. The ratios proper to the three constituent intervals are respectively $\frac{9}{8}$, $\frac{10}{9}$, and $\frac{16}{15}$, and the ratio proper to the whole interval (the fourth) is $\frac{4}{3}$. The product of the three smaller fractions is the greater fraction ($\frac{9}{8} \times \frac{10}{9} \times \frac{16}{15} = \frac{4}{3}$). Again, the ratio of the vibrations of the upper to the lower note at an interval of an octave is $2 : 1$, and this is the product of the ratios of all the separate intervals in the scale ($\frac{9}{8} \times \frac{10}{9} \times \frac{16}{15} \times \frac{9}{8} \times \frac{10}{9} \times \frac{9}{8} \times \frac{16}{15} = \frac{2}{1}$).

Every one that is familiar with the theory or with the use of logarithms will see at once that, in order to obtain expressions for these intervals of such a nature that addition may be substituted for multiplication in the compounding of intervals, we have only to substitute for the ratios the logarithms of those ratios and the problem is solved. For this purpose any system of logarithms will answer, the common system of logarithms to base 10 as well as any other.

In the common system of logarithms the logarithm of 2 is ·3010300. If this number is taken to represent the interval of an octave, then the logarithms of $\frac{9}{8}$, $\frac{10}{9}$, and $\frac{16}{15}$ will represent the intervals of the greater tone, lesser tone, and semitone respectively. These logarithms are—

$$\text{Log. } \tfrac{9}{8} = \text{·}0511525.$$
$$\text{Log. } \tfrac{10}{9} = \text{·}0457575.$$
$$\text{Log. } \tfrac{16}{15} = \text{·}0280287.$$

From this it appears that if the octave is represented by a line 301 inches in length, the greater tone is about 51 inches, the lesser tone about 46 inches, and the semitone about 28 inches. Three greater tones (153), two lesser tones (92), and two semitones (56) make up the octave (301). These numbers are only approximately proportional to the logarithms, and the logarithms themselves are only approximately correct, and would be only approximately correct if they were carried to any number of decimal places beyond the seventh, which is the last here given.

With the human voice or on any stringed instrument a skilful artist can render the intervals more correctly than any logarithmic expressions can indicate them. But on a keyed instrument tuned to suit fifteen major and twelve minor keys and having only twelve notes in the scale, it is impossible to perform with correct intonation. To say that an instrument is equally adapted to all these cases is in effect to say that it is equally unrelated to them all. If the octave, represented by the logarithm ·3010300, is divided into twelve equal intervals called semitones, each of these is represented by ·0250858, whereas the true semitone is represented by ·0280287. The tempered semitone therefore is about $\frac{24}{28}$ths of a true semitone, and the tempered tone (·0501717) is about $\frac{49}{51}$ of the greater

tone, and about $\frac{49}{44}$ths of the lesser tone. Of the intervals of
the tempered scale the fifth and the fourth (which is the
difference between the fifth and the octave) approximate most
nearly to the corresponding intervals of the true scale. The
true fifth is sharper than the tempered fifth (and the true
fourth flatter than the tempered fourth) to the extent of about
$\frac{1}{884}$th of an octave, the fraction being very nearly $\frac{1}{885\cdot 1100}$.
The difference is represented by the ratio of $\frac{886}{885}$; that is to
say, if a monochord is divided by a bridge in such a way as to
have 885 equal parts on one side of the bridge and 886 equal
parts on the other side, the difference of pitch between the two
lengths of the string is the correction a tuner has to make after he
has obtained G as a true fifth from C, this correction being
made for the sake of equal temperament. If the octave is
represented by the logarithm of 2, that is, by ·301029995664,
this difference is ·000490428252.

The errors of the notes of the tempered scale may be very
conveniently expressed in terms of this small quantity taken
as a unit. Taking the tonic of any key as the standard of
comparison for the other notes and using the sol-feggio
syllables we may state the errors of the notes of the tempered
scale as follows :—

Re	—	2 units flat
Mi	—	7 units sharp
Fa	—	1 unit sharp
Sol	—	1 unit flat
La	—	8 units sharp
Si	—	6 units sharp

Since the unit is about the 614th part of an octave, and the
tempered semitone is therefore equal to about 51 units, it
appears that the error of La is about the sixth or seventh part
of a semitone, an error of too serious a magnitude to be dis-
regarded, and the errors of Mi and Si are not much less
considerable.

The amount of error as stated above is absolutely correct
for Fa and Sol, and also for Re, but it is slightly inaccurate
for Mi, La, and Si, the extent of the inaccuracy being
exactly the same for each of these three notes. This error
is remarkably small, being only about $\frac{18}{49}$ths of the millionth
part of the octave. The comma, the difference between the
greater and lesser tones, and corresponding to the ratio $\frac{81}{80}$, is
greater than eleven units by ·000000321116 if the octave is
·301029995664, and this difference is the only correction that
is required to make the foregoing statement of the errors
of Mi, La, and Si perfectly accurate.

The question now arises—what number of notes within the
compass of an octave must a keyed instrument have in order
that within the limit of error last stated (rather more than
the millionth of an octave for each of three notes in every

scale) it may be possible to play fifteen true major scales and fifteen corresponding minor scales ? The number required is 52, more than four times the number now in use.

In passing from the key of C into the key of G it is not enough to substitute F♯ for F. It is necessary to alter A also; for in the scale of C the note A is a lesser tone above G, while in the scale of G the note A must be a greater tone above the tonic. Similarly, if the modulation is from the key of C into the key of the subdominant, it is not enough to substitute B♭ for B; the D also must be changed, so that between it and the unchanged C there may be a lesser tone instead of a greater tone. Thus (without taking minor keys into account for the present) to the seven notes of the scale of C must be added two new notes for every new key ascending by fifths, and two new notes for every new key descending by fifths, or twenty-eight new notes in all, bringing up the number to thirty-five. Seventeen other new notes are required as sharp fourths and fifths of major scales to be used in the relative minor scales. These notes must be regarded as intended to divide the interval between Mi and La in the same way as the sixth and seventh divide the interval between Sol and Do. The errors of these notes in the tempered scale are exceedingly large, amounting to 16 units and 14 units respectively, and it should be remembered that 16 units is nearly a third of a tempered semitone (51·15 units). The La is, as has been stated, 8 units sharp, and the semitone between it and the new leading note is in the tempered scale 6 units smaller than a true semitone, so that the whole error of this leading note is 14 units; and the tempered tone is too small by 2 units for the greater tone which is required between the two sharps, so that the error of the lower of the two sharps is 16 units. Of the seventeen new notes required for minor scales the three middle scales (F, C, and G major) require five—viz., F♯ and G♯ for C, C♯ for F and G, D♯ for G, and B♮ for F. Every other key requires two notes of this class, but only one of the two is original to each key, the other two being borrowed from a key nearer to C. These seventeen notes bring up the number to 52.

Of these 52 notes, three notes belonging to minor scales may be dispensed with—viz., C♯♯, G♯♯, and F made natural from F♭, seeing that the minor keys in which alone they occur are not in use. It is, however, convenient to show their places in a general scheme, and the scheme gains in consistency by taking them into account.

Moreover, 17 other notes out of the 52 are of very little value, because each of them lies very close to a companion note that must find a place in the scale. The difference between such a note and its companion is exactly one unit, the smallest error in the tempered scale, being the 51st part

of a tempered semitone.   Such a note is F flat, which differs
by a unit from its companion note, E.   If these notes are
disregarded there are (52 less 17) 35 notes required instead of
the 12 generally recognised, that is, three notes for every
existing note with the exception of one, and two notes for
that one.   The one exception is B♭ or A♯, for which two
forms instead of three will suffice if the small distinction
between a note and its companion note is disregarded.

The three forms of each note are—a middle form, a grave
form, and an acute form ; the interval between the middle
form and either of the others being a comma, the interval
corresponding to the ratio $\frac{81}{80}$, and containing about 11 units,
or about a fifth of a true semitone.   The interval therefore
between the gravest and the acutest form of the same note is
22 units, or nearly two-fifths of a true semitone (and more
than two-fifths of a tempered semitone), or three-fifths of the
difference between a major third and a minor third.   The
extreme forms of G♯, for example, are 22 units apart, and the
acutest form of G♯ is actually 23 units above the gravest
form of A♭.

It will be more satisfactory to provide for 52 notes
than for 35, and if every couple of companion notes
(such as G♯ and A♭) is to be split up into its two separate
notes, while we still regard these separate notes as forms of
one note of the series of twelve in a keyed instrument, we
shall require for some notes of the keyed instrument five
forms and for others four.

As the place of each note is to be indicated by its distance
in units from the note which represents it in a system of
equal temperament, it is necessary to assume one note as
being in unison with the corresponding note of the tempered
system.   The most convenient note for this purpose is E,
because the discrepancies between the true and the tempered
systems are equally balanced on both sides of that note, the
extreme case of flattening (to the extent of 16 units in D♯ as
a tonic) and the extreme case of sharpening (to the extent of
16 units in F♮ in a minor key) being cases of equal error.

The following table shows, opposite to the name of each
note of a keyed instrument, the several notes which that note
has to do duty for, and the number of units by which it
requires to be raised or depressed to represent each of those
several notes accurately, the signs + and — being used for
necessary sharpening and flattening respectively.   The
middle form of E being made to accord perfectly with that
note on a keyed instrument, the acute form of E is to be
obtained by means of a monochord divided by a bridge into
two parts in the proportion of 80 to 81, and tuned so that the
longer part shall give E in unison with the E of the keyed
instrument ; the shorter part will give the acute E, sharper

than the other by a comma. By a similar process the grave form of E, flat by a comma, is to be obtained. All the other notes are to be obtained from these by fifths ascending or descending. The grave E is E natural, made natural for the purposes of a minor key, and from it are obtained 11 notes by fifths ascending, the last of the 11 being G♯♯, and 5 notes by fifths descending, the last being F♮. All these notes are notes required only in minor keys, and they are indicated in the table by being enclosed in brackets. Not only does the number against each note indicate the number of units by which that note differs from the corresponding note of the keyed instrument, but also the difference between this number and the number proper to the E from which the note is derived shows at how many removes by fifths the note lies from E.

From the acute form of E are derived 5 notes by ascending fifths to D♯, and 12 notes by descending fifths to F♭. Every note used in a major scale as a tonic or as a dominant or subdominant or as the second note of the scale, and these same notes when they occur in the relative minor scale, must be chosen from this set of notes, and except in such relations the notes of this set are not to be used. The notes of this set are marked by square brackets.

The rest of the notes are derived from the middle form of E, by fifths ascending (eight times) to B♯, and by fifths descending (eight times) to A♭.

TABLE OF 52 NOTES TO THE OCTAVE.

|	GRAVE.		MIDDLE.		ACUTE.	
C	(C♮−15)		C−4	(B♯♯−3)	[C+7]	B♯♯+8
B		(B♮−10)	[C♭ ±0]	B+1		[B+12]
B♭	B♭−6	(A♯−5)	[B♭+5]	A♯+6		
A	(A♮−12)		A−1	(G♯♯±0)	[A+10]	
G♯	A♭−8	(G♯−7)	[A♭+3]	G♯+4		[G♯+15]
G	(G♮−14)		G−3	(F♯♯−2)	[G+8]	
F♯		(F♯−9)	[G♭+1]	F♯+2		[F♯+13]
F	(F♮−16)		F−5	(E♯−4)	[F+6]	E♯+7
E		(E♮−11)	[F♭−1]	E ±0		[E+11]
D♯	E♭−7	(D♯−6)	[E♭+4]	D♯+5		[D♯+16]
D	(D♮−13)		D−2	(C♯♯−1)	[D+9]	
C♯		(C♯−8)	[D♭+2]	C♯+3		[C♯+14]

The accuracy of this table may be tested by selecting from it the notes belonging to any one scale, and seeing whether the numbers expressing the distances of the several notes from the corresponding notes of the equal temperament scale are consistent with the statement that has been made of the amounts of error of the several notes of the equal temperament scale. According to that statement the errors of the equal temperament require the following corrections, if the error of Do is assumed to be nil:—Re+2, Mi−7, Fa−1, Sol+1, La−8, Si−6. Therefore if in the scale of C the keynote has for its number of units of error, +7, the number shown in the table, then the numbers for the other notes of the scale in order ought to be the numbers arising from the addition of 7 to the numbers just assigned to Re, Mi, Fa, &c. The numbers so arising are as follows: +9, +0, +6, +8, −1, +1. It will be found on inspection that these are the numbers given in the table for [D], E, [F], [G], A, and B, respectively. Similar results will attend a similar test applied to the other scales, if it is borne in mind that in every scale Do, Re, Fa, and Sol must be selected from the notes marked with square brackets.

In the following diagram (Fig. 3, p. 13) the inadequacy of the system of equal temperament and the character and extent of its errors are illustrated in a visible form. The perpendicular heavy line represents a semitone of the equal temperament scale, from C to C♯. The faint lines in continuation of it show the extent to which the portion of the octave included between this C and this C♯ must be expanded, as it were, to make it include the places of all the notes for which these two notes have to do duty. The horizontal lines show (by their intersections with the perpendicular) the places of all the forms which C and C♯ justly assume in the course of modulation with true intonation. As the octave is equal to about 613·8 units, the heavy line for the equal temperament semitone represents 51·15 units.

The line drawn from [D♭+2] to C−4 represents a true major semitone. It is so placed as to be readily compared with the heavy line which represents an equal temperament semitone. The dotted line from (C♯−8) to [C+7] represents a minor semitone. The line drawn from C−4 to (C♮−15) represents a comma. The interval between C♯+3 and [D♭+2] is the unit of which so much use has been made in this paper, as also is the interval between B♯+8 and [C+7].

The advocates of equal temperament are accustomed to regard as insignificant the errors of which the nature and magnitude have been thus exhibited. It may be doubted whether they have ever adequately understood the extent to which the errors reach, and the degree in which voices and

Fig . 8.
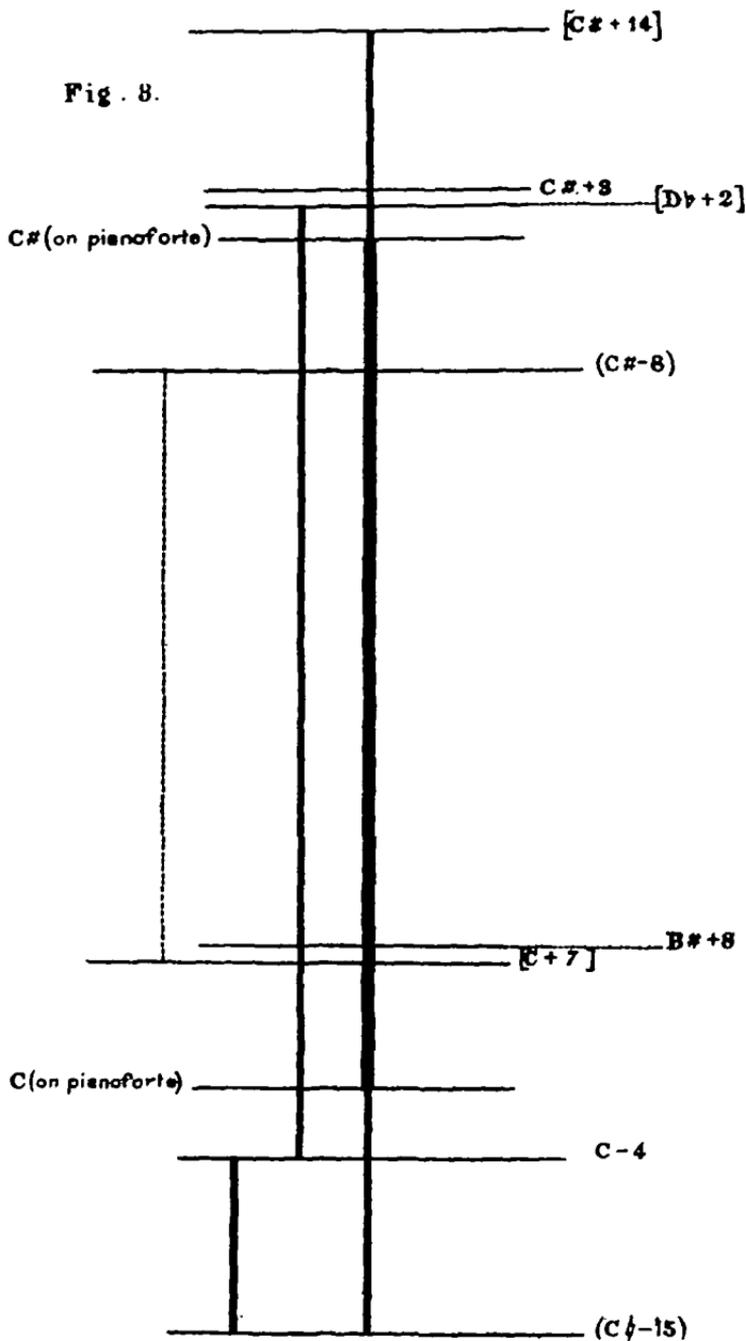
stringed instruments are compelled to depart from true intonation under the tyranny of the pianoforte and the modern organ.

No doubt the complexity of a keyed instrument made to satisfy the demands of just intonation must be very great. Even if the difficulties of manipulation were reduced to a minimum by means of a moveable keyboard, which would lend itself to the application of the Tonic Sol-fa notation to music for the pianoforte and organ, the great number of strings or pipes would constitute a great difficulty with respect to construction and to expense. But unless difficulties of this kind can be surmounted it is to be hoped that a study of the true relations of the notes of the scale in melody and in harmony will lead to the cultivation of vocal and orchestral music, and of music of every kind for stringed instruments, in ways that will render them independent of the pianoforte and organ.

NOTE.—In this paper the fullest deference has been paid to the consensus of authorities by which the place of the supertonic is fixed at a greater tone above the tonic. According to this rule the interval between the tonic and the mediant is divided into a greater and lesser tone, the greater tone coming first in the ascending order. It is a question worth considering, however, whether this rule is without exception. It has the merit of exhibiting a certain symmetry in the order of the vibration numbers, inasmuch as if the numbers for the tonic and mediant are 24 and 30, the number for the supertonic is by this rule 27 and falls exactly midway between 24 and 30. But the relations of the supertonic with some other notes of the scale are not of the best if this rule is absolute. It is impossible to have a D which is at the same time a fifth above G and a fifth below A. Are there not occasions when the supertonic and the submediant ought to be separated by an interval of a true fifth, that is to say, are there not occasions when the graver form should be substituted for the acuter form of the supertonic? Is there not sometimes a slide made from one form to the other in artistic vocalisation? And in the key of A minor does it not seem desirable to have a note in the scale a true fourth above and a true fifth below A? It is remarkable that the graver form of the supertonic is given by a process very similar to that by which the acuter form is found, the difference of process being that lengths of string are considered instead of vibrations.

It will be remembered that the relations of C, E, G, having been determined by the ratios of their vibration numbers, which are as 4, 5, and 6, the other notes of the scale were found by constructing two chords similar to the chords C, E, G, one chord beginning with G as its lowest note and the other

ending with C as its highest note. Now, if lengths of string are considered, lengths 6, 5, and 4 are the lengths of three notes forming a minor chord, A, C, E. If the lowest of these notes be made the highest of a similar chord, and the highest of them be made the lowest of another similar chord, the new chords are—first, D, F, A ; and second, E, G, B. The D found by this process is the graver D, flatter by a comma than the recognised supertonic in the major key of C. With respect to length of string, this note exhibits the same kind of symmetry as the other D exhibits with respect to vibrations, that is, it lies just half-way between C and E. If the lengths for C and E are 30 and 24 (as 5 to 4), the length for grave D is 27, the length for acute D is 26$\frac{2}{3}$, and the ratio of the two forms is 81 : 80, the ratio of the comma. There is a strong temptation to argue that as the string is in the order of nature and time before the vibrations which are only affections of the string, a scale derived from a consideration of the symmetry of the proportions of lengths of string has an *a priori* claim to regard. The answer, of course, is that the phenomena of sound do not emerge until vibrations are set up, and that the vibrations therefore deserve the chief consideration.

It is, however, interesting to observe that from the simplest relations of string lengths the chord first derived is the minor chord, and that, as if with an appreciation of that fact, the author of our musical notation gave a singular prominence to the minor relations of the scale by using the first letter of the alphabet, not for the tonic of the open major scale, but for the keynote of the open minor scale ; and it is also interesting to observe that, if the grave supertonic is adopted, the intervals of the major scale expressed in lengths of string afford a singular apparent justification for the use of the terms tone and semitone. Thus, if 60 parts of a string give C, the parts cut off successively in a diatonic ascent to the octave are—

$$\underline{6, \quad 6, \quad 3,} \quad 5, \quad \underline{4, \quad 4, \quad 2.}$$

In this succession of intervals, such a group as $\underline{6, 6, 3,}$ or $\underline{4, 4, 2,}$ appears to warrant the recognition of the first two intervals in the tetrachord as equal intervals, and the next interval as exactly the half of either of them. It is to be observed also that with this construction a scale has two exactly similar tetrachords, 6, 6, 3, and 4, 4, 2 ; and that these are separated by a tone exactly intermediate in value between 6 and 4. Are such considerations as these to be altogether ignored ? Does it not seem that there ought to be some scientific recognition of a kind of elasticity about the supertonic, or of a necessity for two forms of it ? From this point of view it may be necessary to modify the rule of

selecting supertonics from the set of notes from which tonics, dominants, and subdominants must be selected ; in that case a new note must be added to the table—viz., D♭ —9 for occasional use in the key of C♭.

NOTE 2.—The ordinary logarithms to seven places are not quite full enough for the investigation of this subject. The logarithms of 2, 3, and 5 to 12 places are therefore given here :—

Log. 2 = ·301029995664.
Log. 3 = ·477121254720.
Log. 5 = ·698970004336.

With ·301029995664 for the interval of an octave the most necessary logarithms deduced from the above are—

12th of an octave	—	·025085832972.
True Semitone	—	·028028723600.
Minor Tone	—	·045757490560.
Major Tone	—	·051152522448.
Comma	—	·005395031888.
Unit	—	·000490428252.

DISCUSSION.

THE CHAIRMAN.—Ladies and gentlemen, I am sure you will all be agreed that we have listened to a paper which represents an immense amount of study, thought, and calculation ; but I think we shall all be agreed it is one of more interest to scientists than to practical musicians. Before discussing this point, however, I think we should record a vote of thanks to the writer of the paper. [The vote was passed unanimously].

Mr. HERBERT.—I came here this evening imagining that an attack would be made on equal temperament, and, having had a great deal to do with it for the last thirty odd years, I wished for the opportunity of giving my experience on the subject. I will not enter into the circumstances which led me to take charge of the Roman Catholic choir in Farm Street, I only tell you I did so in September, 1852, and that my first step was to have the organ re-tuned equally by Messrs. Hill. I was totally alone, everyone was against me, and I was about the best abused man in London for some time. All the builders opposed me, and the organists mostly did not know· much about it. My own master, George Cooper, one of the best players and best masters that ever lived, came to try the organ, and was so pleased that he had his own at Christ's Hospital tuned equally also. Shortly after that I presume Sir John Goss went to hear the Christ's Hospital organ, and the consequence

of that was that St. Paul's was tuned equally. I met old Mr. Bishop in the street very shortly after St. Paul's had been tuned, and I said : "Well, Mr. Bishop, I hear you have tuned St. Paul's equally." "Yes," he said, "we have, and I am bound to say the more I hear it the more I like it ; and at the same time," added he, "I think they sing better in tune." Now, here was a man over seventy years of age, who had done nothing all his life but tune perfect thirds and flat fifths, and heard nothing else, and yet that was his opinion to me. Another striking anecdote is this from Dr. Hopkins, who told it me himself. When he and Mr. Rogers, of Doncaster, went abroad to gather materials for their great work on the organ they took letters of introduction everywhere, and went to compare the organs and stops with those at home. Hopkins said : "Rogers, that is our diapason, that is our principal, that is this and that, but somehow it sounds different ; I like it, Jerry" (Rogers' name was Jeremiah), "don't you ?" and Jerry said he did. After some time Hopkins said, "What if it should be the temperament after all ?" Now I will tell you on what grounds I defended the equal temperament. The old mean temperament had what was called a *wolf;* it had many wolves, but the special wolf was that sharp 5th in four flats—*i.e.*, A♭ and E♭, which rendered the key horrible; the third, A♭ and C, was also insufferable; and so the key of four flats was almost tabooed on the organ, though Bach wrote one Fugue in F minor for it. Accordingly what I had to combat was a temperament where, in the good keys, C, G, D, and so on (I ignored the wolves), the thirds were perfect, or nearly so, the fifths very flat (¼ of a comma), and the fourths equally sharp. Now, I said, I take you on your good keys, and I maintain that your diatonic scale, the Doh, Rey, Mi, Fa of the Tonic Sol-fa people, is something so vile, so bad, that no singer and no violin or cello player would tolerate it for a moment. My own master, the late Mr. Hancock, always used to abuse the organ, and declare it was never in tune. He did not mean the wolf, he meant the organ generally. Of course the thing cannot now be heard, but in those days it could, and if you compared the diatonic scale of the good keys in an organ tuned on the mean temperament with an organ tuned on the equal temperament, there could be but one opinion as to which was the best. On one occasion, when I had an organ tuned equally, I left the choir unequal, then called the clergy up into the organ loft, and without telling them which was which I said : "Fathers, I will play the same passages in harmony and melody on the two rows of keys, and you shall tell me which you like best." They said, "We like *that* the best"—they could not tell why, but they said it was more musical. Another curious experiment was tried that morning. I took

a mixture of 3, 5, 8 (of course they were tuned perfectly), I
then drew a principal in the choir and played the same notes
in a succession of chords, and let them hear the difference
between those with mathematically perfect intervals and
tempered ones, and the difference was very peculiar.   In the
perfect intervals we took scarcely any cognisance of the notes
themselves, they sounded like one note; whereas in the
tempered intervals you heard the parts distinctly, and I
suggested at the time that it might be on that account that
the equal temperament in part-music on the organ was
more agreeable.   In those days of the mean temperament,
organists were taught to, and as a matter of fact did, play as
full-handed as possible.   With the introduction of the German
organ with the 16-ft. stop on the manuals and German music
generally, organists had to learn to play in parts, not less than
four generally, if they could in five, and so on, so that it was
part playing and no longer playing handfuls of notes.   If you
want to see what organ arrangements were in those days look
at Dr. Crotch's arrangement of the Amen Chorus, a piece he
was very fond of giving to those who came to him for a
testimonial.   That is one reason why I think the equal
temperament now seems far more acceptable than it would
possibly do in those days.   Then came another question about
the mixtures.   Hill's partner said, Mr. Herbert had spoilt the
Farm Street organ : " What do you do with the mixtures?"   I
said " I do not know ; I suppose you tune them perfect."   Will
it be believed, Helmholtz in his magnificent work on Musical
Acoustics talks of the infernal noise (Höllenlärm) made by the
mixtures in equal temperament ?   Almost all the mathematical
musicians, or musical mathematicians, I do not care which,
seem to have gone crazy on this point.   I have brought with
me the most learned of all the books I know on the subject,
Colonel Perronet Thompson's " Treatise on Just Intona-
tion," and you can see for yourselves what he proposed to do
to play perfectly in twenty keys.   There are three manuals,
not as in an organ, but simply to play in different keys.
There is one point in the paper which I should like to allude
to, where the writer speaks of the second of the scale.
Colonel Thompson entitled that book in the first edition,
" The Duplicity of the Dissonances."   One dissonance was
the second of the scale, and he showed that that note cannot
be in tune with the dominant and subdominant both.   All
subdominant harmonies require the flat form of the super-
tonic or the minor tone, and all dominant harmonies require
the major tone.   Well, as I said at the time, what are you to
do in a very common harmony indeed to the supertonic—
namely, a $\frac{6}{4}$ upon F say, followed by the common chord or
the seventh on G ?   In that case four perfect voices would
raise the supertonic one comma, you would get the two

forms of the supertonic perfectly distinct—first of all the flat form, and then the other. As to the possibility of introducing additional notes or keys to keyed instruments, Colonel Thompson's is one form. Mr. Bosanquet, whom I believe you have heard here, has invented and had built another, which is, I believe, at Oxford. A clergyman, the Rev. Henry Liston, constructed another, which was thought a good deal of; he, I think, made the changes with a pedal. Helmholtz himself has invented one; but I must ask you, gentlemen, whether, in the present state of pianoforte and organ playing, it is practicable to add notes to those already existing? To return to equal temperament, I have been at it half my life. Here is a table of intervals which I copied out fifty years ago. There are the compositions of 2, 3, and 5, really copied out of Rees's Cyclopædia, and this has been my gospel about temperament ever since. In 1852, when I had that fight with the old temperament, I drew up a table showing the errors of both systems. In the equal temperament, the third is certainly very sharp, the fifths and fourths are nearly perfect, the sixth is sharp, the minor third is very flat. Granted; but now we come to a very curious thing. There is a tendency in violin players to sharpen certain notes and to flatten others. Almost all violin players make the third and seventh sharper than the mathematical note, and in the case of flats, for example, in the key of E♭ they will make E♭ lower than D♯. It ought not to be so, but they generally do it. Dr. Pole alludes to it in that very charming book, " The Philosophy of Music," and in this equal temperament it is a singular thing that all the notes that first-rate players are inclined to make a little sharp are already a little sharp, and those notes which players, and singers too, are inclined rather to humour and make a little flat, are already a little flat. I remember the Temple organ in the old days when it had the so-called quarter-tones. Of course they were not quarter-tones, or anything of the sort; what was there was this: the key which usually represents D♯ and E♭ was divided into two, and the key which represents G♯ and A♭ was also divided into two, and one was a little higher than the other—I forget which was which. As the tuner had not got the fear of the wolf before his eyes, because on the Temple organ you could play in four flats perfectly well, he tuned the major thirds absolutely perfect, and the consequence was the diatonic scale in the good keys on the Temple organ was worse than ever, and about the worst that ever was heard. As soon as Dr. Hopkins came there he did all he could to get equal temperament introduced, and to get those quarter-tones abolished; but the old Benchers were very proud of their organ, and thought them very fine. After a time Hopkins had the organ tuned equally, but let the

quarter-tones remain, so that the good Benchers imagined
they still had their old friends; but at last they were
abolished, and now it is all equal together.

MR. BLAIKLEY.—I have had an opportunity of reading this
paper of Mr. Habens', otherwise I should certainly not
attempt to take up your time, for I think it is not one that
can be considered just on the first hearing. What I should wish
to say will not be in the way of criticism of this particular
paper, setting forth Mr. Habens' view, for I do not think
that can be profitably entered upon at such short notice; but
I differ a little from Mr. Herbert, who has put the present
position of equal temperament and the real idea of it before
us.   I think the subject brought forward should be regarded
in the first place from the scientific and artistic point of view.
I couple the two together, taking them as standing in
opposition to the merely utilitarian point of view of the
possibility of mechanical construction, or the cost of
instruments, and so forth.   It appears to me that when we
are dealing with any matter of artistic or scientific theory we
may eliminate from our consideration altogether, if we hold
up the principles of art or science, practical questions of cost
or mechanism; they can come afterwards and qualify the
results otherwise obtained.   It appears to me that to force
the theory of musical intervals and harmony into an
agreement with the equal tempered scale of twelve semitones
is contrary to the scientific point of view, and it seems to me
it is as contrary and as wrong to attempt such a forcing as it
would be to force, let us say, the theory of the prismatic spectrum,
or the art of painting in water or oil colours, to fit in with the
number of pigments which the chromo-lithographer can
practically use, and which, for the sake of analogy, we may
assume to be twelve.   Therefore, I think it is worth while
for an Association like this to have such questions brought
before it from time to time, to be viewed simply from the
artistic and scientific point of view.   I was a little struck
(but possibly Mr. Habens' desire to keep the paper within
reasonable limits may be the reason for it) with the omission of
any reference to the work already done in this matter. Some
instances were quoted by Mr. Herbert, and particularly Mr.
Bosanquet's instrument.   There is also an instrument, not so
ambitious in its character, by Mr. Colin Brown, Ewing
Professor at the Andersonian University, Glasgow.   The
distinction between the two chief systems proposed for prac-
tical work with respect to an approximation to the theory of
the intervals of the scale in just intonation is, as you know,
that one is cyclic, of which the division of the octave into
twelve equally tempered semitones is an example; and the
other system makes no attempt at dealing with keys in a
cycle; of this Mr. Brown's instrument is an example.   It is

now some years since he showed and explained to me his instrument, but my remembrance of it is that the volume, power, and purity of tone of the chords, due to the tuning in just intonation, was something marvellous when compared with the ordinary results obtained from a single set of reeds on the common harmonium. Of course, in cyclic systems we have to consider some practical limit of accuracy. In the system proposed by Mr. Habens, the difference known as the schisma, which is that between the equally tempered fifth and the true fifth, in the ratio of 3 to 2, is used as a unit for comparing errors. This very slight difference is not very material, as we have it on the ordinary tempered instrument. It is not material on the fifths and fourths, but the thirds and sixths are very far from being harmonious when sounded in sustained chords. I did not quite gather from Mr. Habens' paper whether the system suggested by him is a cyclic system or not; I think it is not, but simply a modification of Mr. Colin Brown's system, as it were. I would say a word with respect to what is commonly understood as the natural scale, or the scale of nature. We are commonly informed that the basis for that term is to be found in the fact that a cylindrical tube will give forth tones in harmonic succession, and a stretched string also; but we must remember that neither cylindrical tubes nor stretched strings are at all natural objects. Therefore we should have some other basis for the term, the natural scale. I think the true basis is to be found in the nature of any powerful tone. When it is resolved into its component parts they are found to be in the harmonic scale, as you all know. Now with respect to the intervals in the diatonic scale we are usually told that the semitone in the ratio of $\frac{16}{15}$ is the natural semitone. I cannot find any sufficient reason for calling that interval a natural semitone; it is no more natural than the ratio $\frac{17}{16}$ or $\frac{18}{17}$. Indeed, these two as dividing the major tone from 8 to 9 have a better title to the name of semitones than the ratio of $\frac{16}{15}$, which does not divide a tone of any kind, major or minor. Then with respect to the scale as a whole, F, the subdominant in the key of C, cannot be derived from the natural harmonic series. If we take the natural harmonic scale with vibrations in arithmetical progression from 1 upwards, it is evident that no addition or multiplication or subtraction of whole numbers can give us a number comprising a fraction such as the subdominant of the scale must include if we take the tonic as being 1, 2, 4, and so on. Therefore, the subdominant is really foreign to the natural harmonic scale. Should we not therefore rather view the diatonic scale as a combination or an interweaving for artistic purposes of elements drawn from two harmonic scales? It appears to me that that is the simplest way to build up theories concerning

the diatonic scale, and from this point of view it follows that the germ, as it were, of all transitions and modulations, either through the dominant or subdominant, is inherent in the scale as thus formed from elements chosen from two harmonic scales. Also we get that variation in the pitch of the super-tonic of the D, which has been alluded to in the paper. If we have it as part of the scale of the subdominant F it is certainly a comma flatter than it is if viewed as part of the scale of C, and if we look upon the scale of C as an inter-weaving of the harmonic scale in C with the harmonic scale in F, by choosing elements out of each of those two scales we shall have the D in its two forms.

Mr. WESCHÉ.—It may be interesting to know that the late Dr. Sebastian Wesley always insisted on his organ being tuned in the old mean temperament. This was told me by the organ builder, Mr. Bryceson. He said that he had considerable trouble at the Victoria Rooms, Bristol, and with the Gloucester organ on that account. I have asked practical musicians that have tried these organs of Perronet Thompson, and other people, and from what I could make out the result was confusion.

Mr. HERBERT.—Did Dr. Wesley never come round to equal temperament ?

Mr. WESCHÉ.—No.

Mr. HERBERT.—I thought as much. I may perhaps say one word more that occurred to me. Does anyone believe that in an orchestra where there are ten or twelve first fiddles, and all the rest in proportion, all these mathematical intervals are used—or in a chorus ? The thing is simply impossible. It is all very interesting, but, as applied to keyed instruments, of no practical use. Four voices, if they are very well trained, may, and possibly do, sing mathematically correct. It is quite possible to write a melody or a harmony of such a nature that if the intervals are sung perfectly correctly the singers are bound to rise or to fall. If the melody or harmony has a preponderance of ascending fifths or descending fourths they will rise, and if it have a preponderance of descending fifths and ascending fourths they will fall. But they do not fall if they sing well. The Leeds singers did not fall, and the Bach Choir at the last Concert sang an unaccompanied Motet of Bach's and they did not fall. Then comes the question—how do good singers and good players temper themselves; how do they prevent this falling ? At Farm Street every year during Lent we had to sing what are called Lamentations, which is done entirely without accompaniment by single voices, and the Cantor—a most excellent singer—and I used to take them alternately. He could not make out how it was that he used to sink and I did not. I told him he always took the second of the scale too flat. If you take the

second of the scale always in the sharp form you can keep up.

Mr. BLAIKLEY.—I should like to add with reference to the remark about just temperament in an orchestra, that I believe, as a matter of fact, we do get much nearer just intonation than we do to equal temperament in the orchestra. I cannot speak with respect to stringed instruments from any special knowledge, but with respect to wind instruments they are so much under the power of the lip that the slight difference of a comma can easily be made, and I should imagine that if a tonic and dominant were played, and the flute or clarinet player had to put in the third to complete the chord, he would naturally put it in just intonation; in fact, he could not hold it in tempered intonation.

Mr. WESCHÉ.—There are constantly intervals written quite incorrectly for instruments in the orchestra on the old mean temperament.

Mr. HERBERT.—Do you know, by the way, that Wheatstone's concertina for twelve or fifteen years has been tuned equally? When Wheatstone brought it out he told me it was perfect in 6 keys. Some years ago I had to choose a concertina for a friend by Wheatstone, and to my astonishment I found it was tuned equally. They have the two keys for D♯ and E♭, and G♯ and A♭, but they give the same note. I suppose they found they could not play with the pianoforte. Giulio Regondi, who was the finest concertina player, had two always, one for the orchestra and one for the piano.

Mr. WEBB.—I made some experiments once with my piano with regard to tuning, and can bear out Mr. Blaikley's assertions. I tuned two octaves in the key of C perfect, and certainly the perfect fifths and thirds had a delightfully satisfying effect; but, of course, the other keys went to smash. I think one thing we have lost sight of is, that although we have more or less imperfect instruments, and out of tune, on the other side the ear accommodates itself and compensates very much for it. If we have it not, the ear in many cases practically approaches perfect intonation.

Miss PRESCOTT.—The whole theory of modulation and enharmonic change is founded on that principle, that the ear accepts what it wants and not what it gets.