Equivalent Singular Points of Ordinary Linear Differential Equations.

By

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In a paper published in 1909*) I introduced the notion of equivalent singular points of ordinary linear differential equations. But the theorem of classification to which this notion gave rise was incomplete because certain exceptional cases were laid to one side.**) In the present paper I prove the theorem in its generality, and base my proof on the auxiliary function-theoretic theorem whose demonstration is found in the immediately preceding paper in these Mathematische Annalen.

Suppose that we are given a linear differential system

$$\frac{dy_i}{dx} = \sum_{j=1}^{n} p_{ij}(x)\, y_j \qquad (i = 1, 2, \cdots, n) \tag{1}$$

where the functions $p_{ij}(x)$ are analytic functions of the independent variable x. Any finite point $x = a$ will be an *ordinary point* of this system if the functions $p_{ij}(x)$ are all analytic at $x = a$; the point $x = \infty$ will be an ordinary point if all the functions $p_{ij}(x)$ vanish to the second order at least at $x = \infty$. Any point which is not an ordinary point is termed a singular point of the linear differential system; we restrict ourselves to singular points at which the functions $p_{ij}(x)$ are rational in character i e. are analytic or have a pole. It is no restriction to assume that the singular point under consideration lies at $x = \infty$, and this we shall do.

In the vicinity of $x = \infty$ each coefficient $p_{ij}(x)$ may be expanded in a series of descending integral powers of x. If q is the greatest exponent of the leading power of x in any of these series, then $q + 1$ is the

*) Trans. Am. Math. Soc. 10, p. 436—470.

**) Loc. cit. p. 446 and p. 453.

rank of the singular point $x = \infty$.*) The case of rank 0 ($q = -1$) is the *regular* singular point.

Any linear transformation of the dependent variables in (1)

$$(2) \qquad y_i = \sum_{j=1}^{n} a_{ij}(x)\, \bar{y}_j$$

in which the functions $a_{ij}(x)$ are analytic at $x = \infty$ and such that the determinant $|a_{ij}(x)|$ is not zero at $x = \infty$ takes the given linear differential system (1) into a transformed system

$$(3) \qquad \frac{d\bar{y}_i}{dx} = \sum_{j=1}^{n} \bar{p}_{ij}(x)\, \bar{y}_j \qquad (i = 1, 2, \cdots, n)$$

of the same form. The explicit expression for $\bar{p}_{ij}(x)$ is given by the formula

$$(4) \qquad \bar{p}_{ij}(x) = \sum_{k,l=1}^{n} \bar{a}_{ik}(x)\, p_{kl}(x)\, a_{lj}(x) - \sum_{k=1}^{n} \bar{a}_{ik}(x) \frac{d}{dx} a_{kj}(x)$$
$$(i, j = 1, \cdots, n)$$

where $(\bar{a}_{ij}(x))$ is the matrix of functions inverse to $(a_{ij}(x))$.**) The equations (3) may be found by direct substitution.

If two linear differential systems (1) and (3) are related by a transformation (2) in which the function $a_{ij}(x)$ are not only analytic at $x = \infty$ but reduce to δ_{ij} at $x = \infty$ ($\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$) we will say that the two systems have an *equivalent singular point* at $x = \infty$.***) Since the transformation inverse to (2) is clearly of the same form, this relation of equivalence is a symmetric one. Likewise, since the composition of two transformations like (2) is another of the same form, the relation of equivalence is transitive.

It is obvious from the form (4) of the coefficients $\bar{p}_{ij}(x)$ that the rank of neither one of two systems having an equivalent singular point at $x = \infty$ can exceed that of the other, and hence that the rank of all such systems is the same.

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*) For these definitions see Schlesinger, *Vorlesungen über lineare Differentialgleichungen*, p. 181.

**) For an exposition of the elementary properties of matrices used in the present paper see Schlesinger, loc. cit., p. 18—19.

***) This definition is more satisfactory than the one used in my earlier paper (Trans. Am. Math. Soc., loc. cit.) where the functions $a_{ij}(x)$ were merely restricted to be rational in character at $x = \infty$.

The importance of the notion of equivalent singular points lies in the fact that the solutions of linear differential systems (1) and (3) with equivalent singular points $x=\infty$ must be of essentially the same nature in the vicinity of $x=\infty$ since the coefficients $a_{ij}(x)$ in the transformation (2) are analytic at $x=\infty$. I have made application of the notion of equivalence in my paper in the Transactions of the American Mathematical Society.

We now proceed to prove the theorem: *Every linear differential system* (1) *with a singular point of rank* $q+1$ *at* $x=\infty$ *is equivalent at* $x=\infty$ *to a canonical system of the form*

$$x\frac{dY_i}{dx}=\sum_{j=1}^{n}P_{ij}(x)\,Y_j \qquad (i=1,2,\cdots,n) \tag{5}$$

in which $P_{ij}(x)$ *are polynomials of degree not greater than* $q+1$.

Proof. Outside of a certain circle $|x|=R$ in the complex plane, all of the points $x=a$ are ordinary points of (1). According to the fundamental existence theorem for a linear system (1) we infer then that there exists a set of n linearly independent solutions of (1),

$$(y_{11},\cdots,y_{n1}),\ (y_{12},\cdots,y_{n2}),\cdots,(y_{1n},\cdots,y_{nn}), \tag{6}$$

each element of any one of which is analytic for $|x|\geqq R$; the general solution may be expressed as a linear combination of these particular solutions. The elements of these solutions are not in general single-valued since, when the independent variable x makes a positive circuit of $x=\infty$, these solutions alter respectively to a new set

$$(\tilde{y}_{11},\cdots,\tilde{y}_{n1}),\ (\tilde{y}_{12},\cdots,\tilde{y}_{n2}),\cdots,(\tilde{y}_{1n},\cdots,\tilde{y}_{nn})$$

which are of course linearly independent. Each one of this set of solutions is linearly dependent on the first set of solutions; this relation may be indicated by the matrix formula

$$(\tilde{y}_{ij})=(y_{ij})\,(c_{ij}) \tag{7}$$

where (c_{ij}) is a matrix of constants of determinant not zero. Thus the set of solutions undergoes a linear transformation when x makes a positive circuit of $x=\infty$.

Now according to the well-known theory of such transformations it will be possible, except in special cases, to choose an initial set (6) of solutions so that the matrix (c_{ij}) takes the simple normal form $(\delta_{ij}\varrho_j)$, i. e. there will exist n linearly independent solutions (6) such that each element of the j^{th} one of these is merely multiplied by a factor ϱ_j when x makes a positive circuit of $x=\infty$. For the moment we will assume that (6) is a set of solutions with this property. Later it will be seen that only a slight alteration is necessary to treat the special cases.*)

Now define the quantities $\lambda_1, \cdots, \lambda_n$ up to an additive integer by the equations

$$\lambda_j = \frac{-1}{2\pi\sqrt{-1}} \log \varrho_j \qquad (j = 1, 2, \cdots, n),$$

and then define the functions $l_{ij}(x)$ by the equations

$$(8) \qquad y_{ij}(x) = l_{ij}(x) x^{\lambda_j}.$$

These functions are single-valued and analytic for $|x| \geqq R$ since x^{λ_j} is multiplied by ϱ_j when x makes a positive circuit of $x = \infty$. Moreover the determinant

$$|l_{ij}(x)| = x^{-\lambda_1-\lambda_2-\cdots-\lambda_n} |y_{ij}| = c x^{-\lambda_1-\lambda_2-\cdots-\lambda_n} e^{\int (p_{11}(x)+\cdots+p_{nn}(x))\,dx}$$

is not zero for $|x| \geqq R$**). Hence, by the auxiliary theorem above referred to, we may decompose $(l_{ij}(x))$ into a product of matrices

$$(l_{ij}(x)) = (a_{ij}(x)) \left(e_{ij}(x) x^{k_j}\right)$$

where $(e_{ij}(x))$ is a matrix of functions analytic at $x = \infty$ reducing to the unit matrix (δ_{ij}) at $x = \infty$, where $(e_{ij}(x))$ is a matrix of entire functions of determinant nowhere zero in the finite plane, and where $k_1, \cdots, k_n$ are integers.

Let us choose the functions $a_{ij}(x)$ thus obtained as the functions $a_{ij}(x)$ in the transformation (2). From the equations (8) and the last equation we obtain

$$(y_{ij}) = (a_{ij}(x)) (Y_{ij})$$

where

$$(9) \qquad Y_{ij} = e_{ij}(x) x^{k_j + \lambda_j} = e_{ij}(x) x^{\mu_j}.$$

Hence the n sets of functions $(Y_{1j}, \cdots, Y_{nj})$ where $j = 1, \cdots, n$ form a set of solutions of the transformed equation (3).

If we substitute each of these solutions in the equations (3) in succession we may combine the n^2 resulting equations into the single matrix equation

$$\left(\frac{dY_{ij}}{dx}\right) = (\bar{p}_{ij}(x)) (Y_{ij})$$

whence we find

$$(10) \qquad (\bar{p}_{ij}(x)) = \left(\frac{dY_{ij}}{dx}\right) (Y_{ij})^{-1}.$$

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*) For a complete discussion of the facts outlined here see Schlesinger, loc. cit., p. 90—104.

**) Schlesinger, loc. cit., p. 21.

Now we have

$$(Y_{ij}) = (e_{ij}(x) x^{\mu_j}) = (e_{ij}(x))\,(\delta_{ij} x^{\mu_j})$$

and also

$$\left(\frac{d\,Y_{ij}}{dx}\right) = \left(x^{\mu_j} \frac{d}{dx} e_{ij}(x) + \mu_j x^{\mu_j - 1} e_{ij}(x)\right) = (f_{ij}(x) x^{\mu_j - 1}) = (f_{ij}(x))\,(\delta_{ij} x^{\mu_j - 1})$$

where the functions $f_{ij}(x)$ are entire functions. From (10) we have then

$$(\bar{p}_{ij}(x)) = (f_{ij}(x))\,(\delta_{ij} x^{\mu_j - 1})\,(\delta_{ij} x^{-\mu_j})\,(e_{ij}(x))^{-1} = \frac{1}{x}\,(f_{ij}(x))\,(e_{ij}(x))^{-1}.$$

Hence the matrix $(\bar{p}_{ij}(x))$ is made up of functions single-valued and analytic in the finite plane except for a possible pole of the first order at $x = 0$.

Moreover, since the rank of the singular point $x = \infty$ of (3) is $q+1$, the functions $\bar{p}_{ij}(x)$ are clearly rational in character at $x = \infty$, and their expression in descending powers of x begins with a term in the q^{th} power of x or in a lower power.

Consequently the functions $x\,\bar{p}_{ij}(x)$ are analytic everywhere in the finite plane with a pole of at most the $(q+1)^{\text{th}}$ order at $x = \infty$, and must be polynomials $P_{ij}(x)$ of degree $q+1$ at most. It follows at once that (3) has the form stated in the theorem.

Thus the theorem is proved in the case where all the multipliers $\varrho_1, \cdots, \varrho_n$ are distinct (or more exactly, when the elementary divisors of the matrix $(a_{ij} - \delta_{ij}\lambda)$ are distinct).

It remains to account for the degenerate case when two or more of these multipliers become equal. We consider merely the simplest case when two values of ϱ become equal, say ϱ_1 and ϱ_2, as well as the corresponding elementary divisors. In this case n linearly independent solutions

$$(y_{11}, \cdots, y_{1n}), \cdots, (y_{n1}, \cdots, y_{nn})$$

may be found such that when x makes a positive circuit of infinity these become respectively

$$(\varrho_1 y_{11}, \cdots, \varrho_1 y_{n1})\,(\varrho_1 y_{12} + y_{11}, \cdots, \varrho_1 y_{n2} + y_{11}) \cdots (\varrho_n y_{1n}, \cdots, \varrho_n y_{nn}).$$

In this case we define $\lambda_1, \lambda_3, \cdots, \lambda_n$ as before and write

$$y_{i1} = l_{i1}(x) x^{\lambda_1},\ y_{i2} = \left(\frac{l_{i1}(x) \log x}{2\pi \varrho_1 \sqrt{-1}} + l_{i2}(x)\right) x^{\lambda_1},\ y_{i3} = l_{i3}(x) x^{\lambda_3}, \cdots,$$

thus defining a matrix $(l_{ij}(x))$ of functions which again are single-valued and analytic for $|x| \geqq R$, and of determinant

$$x^{-2\lambda_1 - \lambda_3 - \cdots - \lambda_n}\,|y_{ij}(x)| \neq 0 \quad \text{for} \quad |x| \geqq R.$$

This matrix therefore satisfies the conditions prescribed in the auxiliary theorem.

Now, proceeding essentially as before, we infer that the functions $x\overline{p}_{ij}(x)$ are polynomial in x of degree at most $q+1$.

The formulas (4) show that in all cases the coefficient of x^q in

$$\overline{p}_{ij}(x) = \frac{1}{x} P_{ij}(x)$$

is the same as the corresponding coefficient in the function $p_{ij}(x)$.

In the simplest case of a regular singular point ($q = -1$) the canonical system is of the simple soluble form

$$x\frac{dy_i}{dx} = \sum_{j=i}^{n} p_{ij} Y_j \qquad (i = 1, 2, \cdots, n),$$

where the constants p_{ij} are the coefficients of $\frac{1}{x}$ in the expansion of $p_{ij}(x)$ in descending powers of x.

From this fact the fundamental existence theorem for a regular singular point may be at once derived.