278. Notation of Binomial Coefficients

Author(s): W. F. Sheppard

Source: *The Mathematical Gazette*, Vol. 4, No. 74 (Oct., 1908), pp. 328-330
Published by: Mathematical Association
Stable URL: http://www.jstor.org/stable/3603118
Accessed: 03-11-2015 10:39 UTC

For instance,
$$ax(ax^2+c)-b(bx+c) \equiv P.(ax-b),$$
$$bx^2(ax+b)-c(ax^2+c) \equiv P.(bx-c),$$
$$ax^3(ax+b)-c(bx+c) \equiv P.(ax^2-c),$$
$$a(ax^2+c)^2+b^2(bx+c) \equiv P.\{a^2x^2-abx+ac+b^2\},$$
and so on.

The same principle applies to equations of higher degrees.

Thus $(x^3+q)^3-p^3(px+q)$ is divisible by x^3+px+q, ...(i)

$(px^2+q)^2(x+p)+qx^4$ „ „ x^3+px^2+q, ...(ii)

$(x^m+q)^2+p^2x^{2n-m}(px+q)$ „ „ x^m+px^n+q), ...(iii)

the quotients being

(i) $x^6-px^4+2qx^3+p^2x^2-pqx+q^2-p^3$,

(ii) $p^2x^2+qx+pq$,

(iii) $x^m-px^n+p^2x^{2n-m}+q$.

R. F. Davis.

277. [M[4]. c. α.]. *On the equation of a certain spiral.*

Some years ago a Portuguese writer published an article on a certain "new" spiral to which he gave the name of "the binomial spiral of the first degree," defined by the equation

$$r=(a-a_0)\frac{\theta}{\pi}+r_0=a\theta+b. \qquad \text{.............................(1)}$$

With Messrs. Brocard, V. Jamet, and the late Prof. Longchamps, I hold that this equation may be reduced to the form $r=a\theta$, under which it is at once recognised as the spiral of Archimedes.

At first sight the curve (1) appears to be a conchoid of the curve $r=a\theta$, and in that case it is likely to be a different curve, for in most cases the conchoid of a curve is a curve of a higher degree. But in this instance the conchoid does not differ essentially from the original since (1) reduces to the form

$$r=a(\theta+k), \qquad \text{.................................(2)}$$

where k is a constant. It is, in fact, the spiral of Archimedes turned through a constant angle.

Rodolphe Guimarães, Capt. R.E.

Elves, Portugal.

278. [V. a.]. *Notation of Binomial Coefficients.*

It is desirable that there should be a uniform and consistent notation for the binomial coefficients, and also for other expressions which occur in the binomial theorem, Vandermonde's theorem, Taylor's theorem, the ordinary interpolation formula, etc.

The most important expression is the binomial coefficient $\frac{n(n-1)\ldots\{r\}}{1.2\ldots\{r\}}$, where $\{m\}$ denotes the presence of m factors. In English works this is usually represented by a C, with n and r associated; *e.g.*, by C_r^n. A fundamental objection to this notation is that C_r^n properly represents, not the number given above, but the number of combinations of n things r together; it is true that the two are equal, but to use them as identical involves a confusion of thought. A further objection is that the n in C_r^n suggests the index of a power. This last difficulty is avoided by using ^nC_r or $_nC_r$; but these again are open to the objection that they take up more space, and further that the n, in printed matter, is liable to be attributed to something immediately preceding.

(The latter is one of the typographical faults of Chrystal's *Algebra.*)

Before choosing a symbol, we should see what are the various analogous expressions for which symbols are required. We have first the binomial theorem

$$(x+y)(x+y)\ldots\{n\}=x\,.\,x\ldots\{n\}+\ldots$$
$$+\frac{n(n-1)\ldots\{r\}}{1\,.\,2\ldots\{r\}}\,.\,x\,.\,x\ldots\{n-r\}\,.\,y\,.\,y\ldots\{r\}+\ldots(1).$$

If, however, we compare this with Taylor's theorem

$$f(x+y)=f(x)+\frac{y}{1}\,.\,D_x\,.\,f(x)+\ldots+\frac{y\,.\,y\ldots\{r\}}{1\,.\,2\ldots\{r\}}\,.\,D_x\,.\,D_x\ldots\{r\}\,.\,f(x)+\ldots(2),$$

it will be seen that the $n(n-1)\ldots\{r\}$ really belongs to $x\,.\,x\ldots\{n-r\}$, and the $1\,.\,2\ldots\{r\}$ to $y\,.\,y\ldots\{r\}$.

We may in fact, when n is a positive integer, write the binomial theorem in the form

$$\frac{(x+y)(x+y)\ldots\{n\}}{1\,.\,2\ldots\{n\}}=\frac{x\,.\,x\ldots\{n\}}{1\,.\,2\ldots\{n\}}+\ldots+\frac{x\,.\,x\ldots\{n-r\}}{1\,.\,2\ldots\{n-r\}}\,.\,\frac{y\,.\,y\ldots\{r\}}{1\,.\,2\ldots\{r\}}+\ldots(3).$$

Again, we have Vandermonde's theorem

$$(x+y)(x+y-1)\ldots\{n\}=x(x-1)\ldots\{n\}+\ldots$$
$$+\frac{n(n-1)\ldots\{r\}}{1\,.\,2\ldots\{r\}}\,.\,x(x-1)\ldots\{n-r\}\,.\,y(y-1)\ldots\{r\}+\ldots(4),$$

which is allied to the interpolation-formula

$$f(x+y)=f(x)+\frac{y}{1}\,.\,\Delta_x\,.\,f(x)+\ldots+\frac{y(y-1)\ldots\{r\}}{1\,.\,2\ldots\{r\}}\Delta_x\,.\,\Delta_x\ldots\{r\}\,.\,f(x)+\ldots(5),$$

and may more conveniently be written

$$\frac{(x+y)(x+y-1)\ldots\{n\}}{1\,.\,2\ldots\{n\}}=\frac{x(x-1)\ldots\{n\}}{1\,.\,2\ldots\{n\}}+\ldots+\frac{x(x-1)\ldots\{n-r\}}{1\,.\,2\ldots\{n-r\}}$$
$$.\,\frac{y(y-1)\ldots\{r\}}{1\,.\,2\ldots\{r\}}+\ldots(6).$$

Corresponding to (4) and (6), we have also

$$(x+y)(x+y+1)\ldots\{n\}=x(x+1)\ldots\{n\}+\ldots$$
$$+\frac{n(n-1)\ldots\{r\}}{1\,.\,2\ldots\{r\}}\,.\,x(x+1)\ldots\{n-r\}\,.\,y(y+1)\ldots\{r\}+\ldots(7),$$

$$\frac{(x+y)(x+y+1)\ldots\{n\}}{1\,.\,2\ldots\{n\}}=\frac{x(x+1)\ldots\{n\}}{1\,.\,2\ldots\{n\}}+\ldots+\frac{x(x+1)\ldots\{n-r\}}{1\,.\,2\ldots\{n-r\}}$$
$$.\,\frac{y(y+1)\ldots\{r\}}{1\,.\,2\ldots\{r\}}+\ldots(8).$$

Thus the expressions for which symbols are required are

(*a*) $x\,.\,x\ldots\{r\}$, (*b*) $n(n-1)\ldots\{r\}$, (*c*) $n(n+1)\ldots\{r\}$,

(*d*) $\dfrac{x\,.\,x\ldots\{r\}}{1\,.\,2\ldots\{r\}}$, (*e*) $\dfrac{n(n-1)\ldots\{r\}}{1\,.\,2\ldots\{r\}}$, (*f*) $\dfrac{n(n+1)\ldots\{r\}}{1\,.\,2\ldots\{r\}}$.

It is only for (*a*) that we have a settled notation, viz. x^r. For (*b*) $n^{(r)}$ is sometimes used; this gives parallel relations

$$D_x\,.\,x^r=r\,.\,x^{r-1},\qquad \Delta_n\,.\,n^{(r)}=r\,.\,n^{(r-1)}.$$

For (*e*) there are various notations, in addition to those mentioned above. H. Weber uses $B_r^{(n)}$. Other German writers sometimes use $\binom{n}{r}$ or $(n)_r$; in the second of these the brackets seem unnecessary. H. B. Fine uses n_r; but C. Smith uses n_r to denote (*b*), which, as seen above, may conveniently have an affix above the line. On the other hand, Sir A. G. Greenhill uses x_r to denote (*d*)

It certainly seems best that (*b*) and (*c*), like (*a*), should have affixes above the line, and that (*d*), (*e*), and (*f*), should have affixes below the line. The most suggestive notation would be

$$(a)\ \overset{(r)}{x},\qquad (b)\ {}^{(r)}n,\qquad (c)\ n^{(r)},$$
$$(d)\ \underset{(r)}{x},\qquad (e)\ {}_{(r)}n,\qquad (f)\ n_{(r)},$$

but this of course is impossible, and (*b*) and (*e*) would be open to the objection mentioned in the second paragraph above.

On the whole, since (*e*) is, after (*a*), the most important, I would suggest alternative symbols for it, with corresponding alternatives for (*f*). The following seem best :

$$(a)\ x^r \equiv x\,.\,x\ldots\{r\},\quad (b)\ n^{(r)} \equiv n(n-1)\ldots\{r\},\quad (c)\ n^{[r]} = n(n+1)\ldots\{r\},$$

$$(d)\ x_r \equiv \frac{x\,.\,x\ldots\{r\}}{1\,.\,2\ldots\{r\}},\quad (e)\ n_{(r)} \equiv \binom{n}{r} \equiv \frac{n(n-1)\ldots\{r\}}{1\,.\,2\ldots\{r\}},\quad (f)\ n_{[r]} \equiv \begin{bmatrix} n \\ r \end{bmatrix} \equiv \frac{n(n+1)\ldots\{r\}}{1\,.\,2\ldots\{r\}}.$$

Thus the formulae (1)-(6) above become

$$(1)\ (x+y)^n = x^n + \binom{n}{1}x^{n-1}y^1 + \ldots + \binom{n}{r}x^{n-r}y^r + \ldots,$$

$$(2)\ f(x+y) = f(x) + y_1\,.\,D_x f(x) + \ldots + y_r\,.\,D_x{}^r f(x) + \ldots,$$

$$(3)\ (x+y)_n = x_n + x_{n-1}y_1 + \ldots + x_{n-r}y_r + \ldots,$$

$$(4)\ (x+y)^{(n)} = x^{(n)} + \binom{n}{1}x^{(n-1)}y^{(1)} + \ldots + \binom{n}{r}x^{(n-r)}y^{(r)} + \ldots,$$

$$(5)\ f(x+y) = f(x) + \binom{y}{1}\Delta_x f(x) + \ldots + \binom{y}{r}\Delta_x{}^r f(x) + \ldots,$$

$$(6)\ (x+y)_{(n)} = x_{(n)} + x_{(n-1)}y_{(1)} + \ldots + x_{(n-r)}y_{(r)} + \ldots,$$

or

$$\binom{x+y}{n} = \binom{x}{n} + \binom{x}{n-1}\binom{y}{1} + \ldots + \binom{x}{n-r}\binom{y}{r} + \ldots,$$

with corresponding formulae in place of (7) and (8). W. F. SHEPPARD.

279. [**K. 21.**]. *Two approximate geometrical constructions for inscribing a Nonagon in a circle.*

1. Col. Weldon's Construction :

Let $\alpha\beta\gamma$ be the inscribed circle, centre I, of the equilateral triangle ABC cutting IA, IB, IC at α, β, γ respectively. With centre β and radius $\beta\gamma$ describe a circle cutting the circle whose centre is A and radius AB, at X. Then BX is a side of the nonagon.

2. Mr. J. Houghton Spencer's Construction :

Let PQ, QR be two sides of a hexagon escribed to a circle, touching the circle at B and C. Bisect QC at a. Join Pa cutting the circle at X. Then BX is a side of the nonagon.

Error. The angle $BAX = 40°\ 5'\ 57{\cdot}6''$ showing an error of approximately $6'$ equivalent to $\frac{1}{30}$ part of an inch on an arc of 20 inch radius.

The remarkable part of the two constructions given above, and discovered independently, is that they are in reality identical as is seen from the following geometrical proof.

Let QR be one side of an escribed hexagon touching ⊙ at C.

Bisect QC at a. Join Pa.