

INDETERMINATE MULTIPLIERS FOR THE CONTINUOUS GIRDER.

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Mr. Merriman's system of indeterminate multipliers for the continuous girder has long been recognized as the most convenient method of solving this problem. I propose in the following article to develop that theory into a formula by which any set of equations may be solved which have the form of the Three Moment Equations, in which the unknown terms correspond to the pier moments, but *all other terms are entirely arbitrary and may have any value whatever, either real or imaginary.* The problem will however be discussed as if it were a problem of obtaining pier moments.

Let M be any pier moment;

l be the length of a span;

s be the number of spans;

y be the arbitrary term at the end of the equations.

The quantities M , l , and y , are to be defined by a subscript denoting which one is meant. The " y " quantities include all terms denoting the state of loading and height of supports, and need not be further defined than to show where they appear in the "Three moment" equations.

These equations will then be:

$$\begin{aligned}
 2 M_2 (l_1 + l_2) + M_3 l_2 - y_2 &= 0 \\
 M_2 l_2 + 2 M_3 (l_2 + l_3) + M_4 l_3 - y_3 &= 0 \\
 \&c. \qquad \&c. \qquad \&c. \qquad \&c. \qquad \&c. \qquad \&c. & (1) \\
 M_{s-2} l_{s-2} + 2 M_{s-1} (l_{s-2} + l_{s-1}) + M_s l_{s-1} - y_{s-1} &= 0 \\
 M_{s-1} l_{s-1} + 2 M_s (l_{s-1} + l_s) - y_s &= 0
 \end{aligned}$$

There being $(s-1)$ equations.

Now suppose these equations multiplied by indeter-

minate multipliers, the first by D_2 , the second by D_3 , &c. &c., and the equations thus multiplied be added together, and arranged according to the moments, then we would have the following equation :

$$\left\{ \begin{array}{l} M_2 (2 D_2 (l_1 + l_2) + D_3 l_2) \dots D_2 y_2 \\ + M_3 (D_2 l_2 + 2 D_3 (l_2 + l_3) + D_4 l_3) \dots D_3 y_3 \\ + \text{&c.} \quad \text{&c.} \quad \text{&c.} \quad \text{&c.} \quad \text{&c.} \quad \text{&c.} \quad \text{&c.} \\ + M_{s-1} (D_{s-2} l_{s-2} + 2 D_{s-1} (l_{s-2} + l_{s-1}) + \\ \qquad \qquad \qquad + D_s l_{s-1}) \dots D_{s-1} y_{s-1} \\ + M_s (D_{s-1} l_{s-1} + 2 D_s (l_{s-1} + l_s)) \dots D_s y_s \end{array} \right\} = 0 \quad (2)$$

Now the problem is to find a set of multipliers so that *any pier moment at will*, that we may select, that we will designate by M_q , shall have the co-efficient Z_q , while at the same time the coefficients of all the other pier moments become = 0, so that we can write

$$M_q = \frac{D_2 y_2 + D_3 y_3 + \text{&c.} + D_{s-1} y_{s-1} + D_s y_s}{Z_q} \quad (3)$$

and we must find a short and easy method of obtaining Z_q , and the value of each separate indeterminate multiplier and prove that the "Three Moment" equations are fully satisfied.

Assume the following set of equations :

$$\begin{array}{l} D_1 l_1 + 2 D_2 (l_1 + l_2) + D_3 l_2 = Z_2 \\ D_2 l_2 + 2 D_3 (l_2 + l_3) + D_4 l_3 = Z_3 \\ \text{&c.} \qquad \qquad \text{&c.} \qquad \qquad \text{&c.} \quad \text{&c.} \\ D_{s-2} l_{s-2} + 2 D_{s-1} (l_{s-2} + l_{s-1}) + D_s l_{s-1} = Z_{s-1} \\ D_{s-1} l_{s-1} + 2 D_s (l_{s-1} + l_s) + D_{s+1} l_s = Z_s \end{array} \quad (4)$$

There being also $(s-1)$ of these equations.

(1) Write c for D in these equations, assuming that $c_1 = 0$, $c_2 = 1$, and that $c_{s+1} l_s = -Z_s$, and substitute 0 for the Z^s on the right. These will be the c multipliers of Mr. Merriman.

(2) Write d for D in these equations, assuming that

$d_s = 1$, $d_{s+1} = 0$, and substitute 0 for the Z^s on the right, as before. These will be the d multipliers of Mr. Merriman, but their subscripts are different because they are numbered the other way. These subscripts become the same as his by changing their sign and adding $(s + 2)$.

(3) Let any general multiplier be denoted by the sign $[q]_n$ instead of D in equations (3) and (4), and let the enclosed q be the subscript of the pier moment that is to be found, and let its own subscript n denote the place of that multiplier in its own series. There being as many series as there are pier moments. Also assume that $[q]_1 = 0$, $[q]_2 = 1$, $[q]_{s+1} = 0$, and that all the Z^s in equation (4) = 0 except Z_q , then we have

$$[q]_{q-1} l_{q-1} + 2 [q]_q (l_{q-1} + l_q) + [q]_{q+1} l_q = Z_q \quad (5)$$

These multipliers in this form must be transformed, because this is altogether too complex a formula.

Now the equations for the determination of $[q]_n$, where n is equal to or less than q , are the same as those for c_2 , c_3 , &c. Hence

$$[q]_n = c_n \text{ for } n = \text{or } < q \quad (6)$$

Also the equations for determining the relation between $[q]_n$ and $[q]_s$ are the same as those for the relation between d_n and d_s , and, therefore, we have

$$\frac{[q]_n}{[q]_s} = \frac{d_n}{d_s} = d_n \text{ since } d_s = 1 \quad (7)$$

for $n = \text{or } > q$

and making $n = q$ Eq. (6).

$$[q]_s = \frac{c_q}{d_q} \quad (8)$$

and therefore

$$[q]_n = \frac{c_q}{d_q} d_n, \text{ for } n = \text{or } > q \quad (9)$$

and by substitution in Eq. (5), we have

$$c_{q-1} l_{q-1} + 2 c_q (l_{q-1} + l_q) + \frac{c_q}{d_q} d_{q+1} l_q = Z_q \quad (10)$$

This, however, may be simplified still more by a little

reduction. The corresponding equations for the c and d multipliers are :

$$c_{q-1} l_{q-1} + 2 c_q (l_{q-1} + l_q) + c_{q+1} l_q = 0 \quad (11)$$

$$d_{q-1} l_{q-1} + 2 d_q (l_{q-1} + l_q) + d_{q+1} l_q = 0 \quad (12)$$

and from Eq. (11), we get

$$c_q d_{q+1} - c_{q+1} d_q = \frac{Z_q d_q}{l_q} \quad (13)$$

Multiply Eq. (11) by d_q and Eq. (12) by c_q and subtract, and we have

$$(c_{q-1} d_q - c_q d_{q-1}) l_{q-1} - (c_q d_{q+1} - c_{q+1} d_q) l_q = 0 \quad (14)$$

whence by comparison with Eq. (13)

$$Z_{q-1} d_{q-1} = Z_q d_q$$

and since q may have any of the successive values from 2 to s , inclusive, we have

$$d_2 Z_2 = Z_q d_q = d_s Z_s = Z_s = c_{s-1} l_{s-1} + 2 c_s (l_{s-1} + l_s) \quad (15)$$

because $d_s = 1$ and from Eq. (10) making $q = 2$

$$2 d_2 (l_1 + l_2) + d_3 l_2 = d_2 Z_2 = Z_s \quad (16)$$

If we use Mr. Merriman's notation this would be

$$2 d_s (l_1 + l_2) + d_{s-1} l_2 = d_s Z_2 = Z_s$$

The identity of this expression with Z_s is apparent as soon as we find the numerical values of the multipliers, but *seems to have been treated hitherto as a mere coincidence* by most authors, including Mr. Merriman himself. It simplifies formulas and also furnishes a valuable check on the calculations.

By the aid of Eqs. (6), (9) and (15), the formula for the pier moment becomes apparent, and is

$$\begin{aligned} M_q &= \frac{d_q}{Z_s} (c_2 y_2 + c_3 y_3 + \&c. + c_{q-1} y_{q-1} + c_q y_q) + \\ &+ \frac{c_q}{Z_s} (d_{q+1} y_{q+1} + d_{q+2} y_{q+2} + \&c. + d_{s-1} y_{s-1} + \\ &+ d_s y_s) \end{aligned} \quad (17)$$

and the rule for writing this formula is very simple. Thus

let y_n be any final term. Then if n be equal to or less than q its numerator coefficient is $d_q c_n$, but if n be either equal to or greater than q it becomes $c_q d_n$. For $n = q$, the coefficient is $c_q d_q$.

Hence,

$$M_1 = M_{s+1} = 0$$

by this formula also.

Now since the values of the y^s have not entered the discussion, we may assume *any arbitrary values whatever* for them. If, however, we use the usual values as given in works on the continuous girder, then for supports all on a level and only one span loaded, this formula is the same as Mr. Merriman's, since only two of the y^s appear in the formula, all the rest being put $= 0$. It must be remembered, however, that in this formula the subscripts of the d multipliers have been changed. This agreement with Mr. Merriman's formula is, of course, the necessary result of using his system of multipliers. There is, moreover, no special reason why the quantities $l_1, l_2, \&c.$, should be real and positive except the nature of the problem usually solved in this manner, viz: the continuous girder. Hence, if necessary, these may either be negative or imaginary also.

In order to test this formula by the "Three Moment" Equation, we must write the formulas for M_{q-1} and M_{q+1} which are developed the same way:

$$M_{q-1} = \frac{d_{q-1}}{Z_s} (c_2 y_2 + c_3 y_3 + \&c. + c_{q-1} y_{q-1}) + \frac{c_{q-1}}{Z_s} (d_q y_q + d_{q+1} y_{q+1} + \&c. + d_{s-1} y_{s-1} + d_s y_s) \quad (18)$$

$$M_{q+1} = \frac{d_{q+1}}{Z_s} (c_2 y_2 + c_3 y_3 + \&c. + c_{q-1} y_{q-1} + c_q y_q + c_{q+1} y_{q+1}) + \frac{c_{q+1}}{Z_s} (d_{q+2} y_{q+2} + \&c. + d_{s-1} y_{s-1} + d_s y_s) \quad (19)$$

Multiply Eq. (18) by l_{q-1} ; Eq. (17) by $2(l_{q-1} + l_q)$, and Eq. (19) by l_q , and add the products and arrange as coefficients of $y_2 y_3 \&c. \dots y_s$. The quantities $c_2 y_2, c_3 y_3 \&c.$, up to and

inclusive of $c_{q-1} y_{q-1}$, will have a common numerator coefficient.

$$= d_{q-1} l_{q-1} + 2 d_q (l_{q-1} + l_q) + d_{q+1} l_q = 0 \text{ [Eq. (12)]}$$

The numerator coefficient of y_q will be

$$d_q c_{q-1} l_{q-1} + 2 c_q d_q (l_{q-1} + l_q) + c_q d_{q+1} l_q = Z_q d_q = Z_s \text{ [Eq. (10)]}$$

The numerator coefficient of $d_{q+1} y_{q+1}$, $d_{q+2} y_{q+2}$, &c., up to and inclusive of $d_s y_s$, will be

$$c_{q-1} l_{q-1} + 2 c_q (l_{q-1} + l_q) + c_{q+1} l_q = 0 \text{ [Eq. (11)]}$$

and the equation of "Three Moments,"

$$M_{q-1} l_{q-1} + 2 M_q (l_{q-1} + l_q) + M_{q+1} l_q - y_q = 0$$

is fully satisfied, *which was to be proved.*

I have thus demonstrated that there is no theoretical necessity either for the hypothesis of single spans in succession loaded, or for the use of b multipliers dependent on the cubes of the spans or any hypothesis as to heights of supports, but that all can be solved by the aid of Mr. Merri-
man's c and d multipliers alone by this method.

Moreover, while I have written of pier moments and spans, the argument would have been fully as valid if I had called them simply unknown quantities and their coefficients, if the equations followed the same law. It is, therefore, a solution of any set of equations relating to any subject whatever if these equations have the form of those of the "Three Moments."