STRUTS AND TIE-RODS IN MOTION.

By H. MAWSON, B.Sc., of the University of Liverpool.

[*Selected for Publication only.*]

In the present Paper it is proposed to show how the stresses in a rod which is in motion and subjected to an endlong force may be calculated; and also to show that the formulæ which have been derived for stresses in stationary struts are only special cases of those obtained for rods in motion.

Take first the case of a uniform circular shaft of weight w per unit length rotating at a radians per second in bearings which do not constrain it in any way, and also let it be subjected to an endlong compressive force F.

The rod will be slightly deflected, Fig. 1 (page 462), and in consequence of this deflection y at a distance x from an axis taken through O the centre of the rod, there will be an additional load at this point of $\frac{wa^2y}{g}$ per unit length.

Treating the rod as a strut with only an endlong force F applied to it, we have $\frac{d^2y}{dx^2} = -\frac{\mathrm{F}y}{\mathrm{EI}}$. As a beam $\frac{d^2y}{dx^2} = \frac{\mathrm{M}}{\mathrm{EI}}$.

As a strut laterally loaded or as a combination of strut and beam

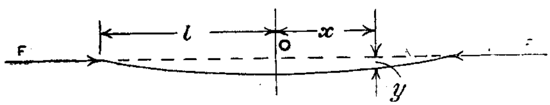
$$\frac{d^2y}{dx^2}=\frac{\mathrm{M}}{\mathrm{EI}}-\frac{\mathrm{F}y}{\mathrm{EI}}$$

$$\frac{d^3y}{dx^3}=\frac{\mathrm{S}}{\mathrm{EI}}-\frac{\mathrm{F}}{\mathrm{EI}}\frac{dy}{dx}$$ where S is the shearing force

$$\frac{d^4y}{dx^4}=\frac{w_1}{\mathrm{EI}}-\frac{\mathrm{F}}{\mathrm{EI}}\frac{d^2y}{dx^2},$$

w_1, denoting the loading per unit length.

Fig. 1.



Taking the worst case when the centrifugal force and weight act together

$$w_1 = w + \frac{w}{g}\alpha^2 y,$$

$$\frac{d^4y}{dx^4}+\frac{\mathrm{F}}{\mathrm{EI}}\frac{d^2y}{dx^2}-\frac{w}{g}\frac{\alpha^2 y}{\mathrm{EI}}=\frac{w}{\mathrm{EI}} \quad . \quad . \quad . \quad (1)$$

The solution of this equation is given by

$$y = \mathrm{A}_1 \cos\phi x + \mathrm{A}_2 \sin\phi x + \mathrm{A}_3 e^{\theta x} + \mathrm{A}_4 e^{-\theta x} - \frac{g}{\alpha^2} \quad . \quad (2)$$

where

$$\phi^2 = \frac{\mathrm{F}}{2\mathrm{EI}} + \sqrt{\frac{\mathrm{F}'^2}{4\mathrm{E}^2\mathrm{I}^2}+\frac{w\alpha^2}{g\mathrm{EI}}},$$

and

$$\theta^2 = -\frac{\mathrm{F}}{2\mathrm{EI}} + \sqrt{\frac{\mathrm{F}^2}{4\mathrm{E}^2\mathrm{I}^2}+\frac{w\alpha^2}{g\mathrm{EI}}}.$$

The constants A_1, A_2, A_3 and A_4 can be calculated from the conditions that y has the same values for equal positive and negative values of x, and also when $x = l$ both y and $\frac{d^2y}{dx^2}$ become zero.

From these conditions it will be found that

$$A_1 = 0, \quad A_2 = \frac{\theta^2 g}{\alpha^2 (\theta^2 + \phi^2) \cos \phi l},$$

$$A_3 = A_4 = \frac{\phi^2 g}{2\alpha^2 (\theta^2 + \phi^2) \cosh \theta l}.$$

Placing these values in equation (2), a relation for y is obtained, and remembering that the B.M. $= \mathrm{EI}\,\frac{d^2y}{dx^2}$, the B.M. at any point can be found. The B.M. is a maximum when $x =$ zero, and will be found to be

$$M_{\max} = \frac{\mathrm{EI}\theta^2\phi^2 g}{\alpha^2 (\theta^2 + \phi^2)} \left(\frac{1}{\cosh \theta l} - \frac{1}{\cos \phi l}\right)$$

$$= \frac{w}{2\sqrt{\frac{F^2}{4E^2I^2} + \frac{w\alpha^2}{g\mathrm{EI}}}} \left\{\frac{1}{\cosh \theta l} - \frac{1}{\cos \phi l}\right\} \quad . \quad . \quad (3)$$

The maximum stress in the material is given by

$$f_{\max} = \frac{w}{2Z\sqrt{\frac{F^2}{4E^2I^2} + \frac{w\alpha^2}{g\mathrm{EI}}}} \left\{\frac{1}{\cosh \theta l} - \frac{1}{\cos \phi l}\right\} + \frac{F}{A} \quad . \quad (4)$$

A being the sectional area of the shaft, and Z the modulus of section.

Both the stress in the material and the B.M. become infinite when $\cos \phi l = 0$, i.e., when

$$\sqrt{\frac{F}{2\mathrm{EI}}} + \sqrt{\frac{F^2}{4E^2I^2} + \frac{w\alpha^2}{g\mathrm{EI}}} = \frac{\pi}{2l},$$

$$F = \frac{\pi^2 \mathrm{EI}}{4l^2} - \frac{4w\alpha^2 l^2}{g\pi^2} \quad . \quad . \quad . \quad . \quad (5)$$

$$\text{or} \quad \alpha = \frac{\pi}{2l}\sqrt{\frac{g}{w}\left(\frac{\pi^2\mathrm{EI}}{4l^2} - F\right)} . \quad . \quad . \quad (6)$$

Equation (5) shows that as α increases, the critical endlong force rapidly diminishes. If also in equation (5) α be put zero, we have a stationary strut loaded with w per unit length, and the critical endlong force is found to be Euler's Load. Again, if α be placed equal to zero, $\phi = \sqrt{\frac{F}{\mathrm{EI}}}$ and $\theta =$ zero and the maximum stress in equation (4) reduces to

$$f_{\max} = \frac{w\mathrm{EI}}{\mathrm{FZ}}\left[1 - \sec l\sqrt{\frac{\mathrm{F}}{\mathrm{EI}}}\right] + \frac{\mathrm{F}}{\mathrm{A}} \quad . \quad . \quad (7)$$

which agrees with that obtained for a stationary strut laterally loaded.*

Equation (6) gives an expression for the critical speed of the shaft which is found to decrease as F increases; moreover, when F is zero, we have an expression for the whirling speed of a shaft rotating in bearings and not subjected to any endlong force.

As a second example, take the case of a rotating shaft subjected to an endlong tensile force F. Treating in exactly the same manner as before, we arrive at the equation

$$\frac{d^4y}{dx^4} - \frac{\mathrm{F}}{\mathrm{EI}}\frac{d^2y}{dx^2} - \frac{wa^2}{g\mathrm{EI}} = \frac{w}{\mathrm{EI}} \quad . \quad . \quad . \quad (8)$$

The solution of this equation is given by

$$y = \mathrm{A}_1 \cos \theta x + \mathrm{A}_2 \sin \theta x + \mathrm{A}_3 e^{\phi x} + \mathrm{A}_4 e^{-\phi x} - \frac{g}{a^2}$$

where θ and ϕ have the same values as before. The same conditions hold as in the first place, and hence

$$\mathrm{A}_1 = 0, \quad \mathrm{A}_2 = \frac{\phi^2 g}{a^2(\theta^2 + \phi^2)\cos \theta l},$$

$$\mathrm{A}_3 = \mathrm{A}_4 = \frac{\theta^2 g}{2a^2(\theta^2 + \phi^2)\cosh \phi l}.$$

The maximum B.M. will be given by

$$\mathrm{M}_{\max} = \frac{\mathrm{EI}\theta^2\phi^2 g}{a^2(\theta^2 + \phi^2)}\left\{\frac{1}{\cosh \phi l} - \frac{1}{\cos \theta l}\right\}$$

$$= \frac{w}{2\sqrt{\frac{\mathrm{F}^2}{4\mathrm{E}^2\mathrm{I}^2} + \frac{wa^2}{g\mathrm{EI}}}}\left\{\frac{1}{\cosh \phi l} - \frac{1}{\cos \theta l}\right\} \quad . \quad . \quad (9)$$

$$f_{\max} = \frac{w}{2\mathrm{Z}\sqrt{\frac{\mathrm{F}^2}{4\mathrm{E}^2\mathrm{I}^2} + \frac{wa^2}{g\mathrm{EI}}}}\left\{\frac{1}{\cosh \phi l} - \frac{1}{\cos \theta l}\right\} + \frac{\mathrm{F}}{\mathrm{A}} \quad . \quad (10)$$

The B.M. and the stress are now infinite when $\cos \theta l$ is zero, that is, when

* *See* Morley's "Strength of Materials."

$$\sqrt{-\frac{F}{2EI}+\sqrt{\frac{F^2}{4E^2I^2}+\frac{w\alpha^2}{gEI}}}=\frac{\pi}{2l}$$

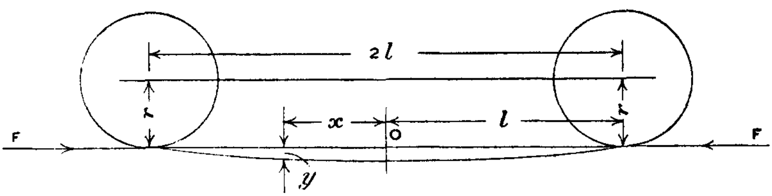
$$F=\frac{4w\alpha^2l^2}{g\pi^2}-\frac{\pi^2EI}{4l^2} \quad . \quad . \quad . \quad . \quad (11)$$

or

$$\alpha=\frac{\pi}{2l}\sqrt{\frac{g}{w}\left(\frac{\pi^2EI}{4l^2}+F\right)} \quad . \quad . \quad (12)$$

Equation (11) agrees exactly with equation (5), for F in this case is $-F$ of the previous case, $F=0$ and $\alpha=0$ give the same relations in equation (11) as they do in (5).

FIG. 2.



From (12), however, the greater the value of F the greater the angular velocity has to be before whirling takes place, and hence whirling might be prevented by applying an endlong tensile force to the rotating shaft.

Placing $\alpha=0$, ϕ becomes $\sqrt{\frac{F}{EI}}$ and θ becomes zero as before, and the maximum stress in equation (10) is given by

$$f_{\max}=\frac{wEI}{FZ}\left\{\operatorname{sech} l\sqrt{\frac{F}{EI}}-1\right\}+\frac{F}{A},$$

which agrees with the maximum stress for a stationary tie loaded with w per unit length.*

Now consider a uniform rod, every portion of which describes a vertical circle of radius r, Fig. 2, to be subjected to an endlong compressive force F.

Taking the rod in its extreme vertical positions, the equation can again be written

$$\frac{d^4y}{dx^4}=\frac{\text{load per unit length}}{EI}-\frac{F}{EI}\frac{d^2y}{dx^2}.$$

* *See* Morley's "Strength of Materials."

Three cases may arise :—

(a) In which the rod is in its lowest position and the weight and centrifugal force act together. Then the load per unit length at any point distant x from the axis is given by

$$\frac{w}{g} \alpha^2 (r + y) + w.$$

(b) The rod in its highest position, the weight and centrifugal force opposing each other, the centrifugal force being the greater, then

$$\text{Load per unit length} = \frac{w}{g} \alpha^2 (r + y) - w.$$

(c) Same as case (b), only the weight the greater.

$$\text{Load per unit length} = w - \frac{w}{g} \alpha^2 (r + y).$$

Taking case (a) we have

$$\frac{d^4y}{dx^4} + \frac{\mathrm{F}}{\mathrm{EI}} \frac{d^2y}{dx^2} - \frac{w\alpha^2}{g\mathrm{EI}} y = \frac{w + \frac{w}{g} \alpha^2 r}{\mathrm{EI}} \quad . \quad . \quad (13)$$

an equation only differing from (1) in the constant term on the right-hand side.

The solution of this equation is

$$y = \mathrm{A}_1 \cos \phi x + \mathrm{A}_2 \sin \phi x + \mathrm{A}_3 e^{\theta x} + \mathrm{A}_4 e^{-\theta x} - \frac{g}{\alpha^2} - r \quad . \quad (14)$$

where θ and ϕ have the same meaning as before.

The same conditions to determine A_1, A_2, A_3, and A_4 may be applied, and the maximum B.M. is now given by

$$\mathrm{M}_{\max} = \frac{\mathrm{EI}\left(\frac{g}{\alpha^2} + r\right) \theta^2 \phi^2}{\theta^2 + \phi^2} \left\{\frac{1}{\cosh \theta l} - \frac{1}{\cos \phi l}\right\}$$

$$= \frac{w + \frac{w}{g} \alpha^2 r}{2\sqrt{\frac{\mathrm{F}^2}{4\mathrm{E}^2\mathrm{I}^2} + \frac{w\alpha^2}{g\mathrm{EI}}}} \left\{\frac{1}{\cosh \theta l} - \frac{1}{\cos \phi l}\right\} \quad . \quad (15)$$

$$f_{\max} = \frac{w + \frac{w}{g} \alpha^2 r}{2\mathrm{Z}\sqrt{\frac{\mathrm{F}^2}{4\mathrm{E}^2\mathrm{I}^2} + \frac{w\alpha^2}{g\mathrm{EI}}}} \left\{\frac{1}{\cosh \theta l} - \frac{1}{\cos \phi l}\right\} + \frac{\mathrm{F}}{\mathrm{A}} \quad (16)$$

This again becomes infinite when $\cos \phi l$ is zero. Hence we obtain relations for F and α as given by equations (5) and (6), and the deductions from (5) and (6) apply here also. We have therefore a proof of an interesting fact that a circular rod rotated as a coupling-rod, and subjected to an endlong compressive force, will whirl at the same angular velocity as it would do if rotated as a shaft and subjected to the same endlong force, no matter what the radius r may be, since the moment of inertia about all axes of bending is constant.

Again, if in equation (16) α be made zero, equation (7) is obtained for the maximum stress in a stationary strut carrying a uniformly distributed load of w per unit length.

Treating a tie-rod having the motion of a coupling-rod in a similar manner, we have, corresponding to case (a), the equation

$$\frac{d^4y}{dx^4} - \frac{\mathrm{F}}{\mathrm{EI}}\frac{d^2y}{dx^2} - \frac{w\alpha^2 y}{g\mathrm{EI}} = \frac{w + \frac{w}{g}\alpha^2 r}{\mathrm{EI}} \quad . \quad . \quad (17)$$

The solution of which is given by

$$y = \mathrm{A}_1 \cos \theta x + \mathrm{A}_2 \sin \theta x + \mathrm{A}_3 e^{\phi x} \times \mathrm{A}_4 e^{-\phi x} - \frac{g}{\alpha^2} - r$$

$$\mathrm{M}_{\max} = \frac{\mathrm{EI}\left(\frac{g}{\alpha^2} + r\right)\theta^2\phi^2}{(\theta^2 + \phi^2)}\left\{\frac{1}{\cosh \phi l} - \frac{1}{\cos \theta l}\right\}$$

$$= \frac{w + \frac{w}{g}\alpha^2 r}{2\sqrt{\frac{\mathrm{F}^2}{4\mathrm{E}^2\mathrm{I}^2} + \frac{w\alpha^2}{g\mathrm{EI}}}}\left\{\frac{1}{\cosh \phi l} - \frac{1}{\cos \theta l}\right\} \quad . \quad . \quad (18)$$

$$f_{\max} = \frac{w + \frac{w}{g}\alpha^2 r}{2\mathrm{Z}\sqrt{\frac{\mathrm{F}^2}{4\mathrm{E}^2\mathrm{I}^2} + \frac{w\alpha^2}{g\mathrm{EI}}}}\left\{\frac{1}{\cosh \phi l} - \frac{1}{\cos \theta l}\right\} + \frac{\mathrm{F}}{\mathrm{A}} \quad . \quad (19)$$

The B.M. and stress are again infinite when $\cos \theta l$ is zero, and therefore the values of F and α as given by equations (11) and (12) apply here also. The value $\alpha = 0$ in equation (19) gives the maximum stress in a stationary tie laterally loaded.

Equation (16) has been applied to a particular coupling-rod, which broke whilst in service, and the results are tabulated in Table 1.

TABLE 1.

Speed of Engine in miles per hour.	Steam-Pressure. 160 lb. per sq. in.	Steam-Pressure. 200 lb. per sq. in.
	Stress in rod. Tons per sq. in.	Stress in rod. Tons per sq. in.
10	3·2	3·95
20	3·78	4·60
30	4·70	5·25
40	6·18	6·8
50	7·70	8·7
60	9·82	10·8
70	12·30	13·3
80	15·2	16·4

The particulars and dimensions relating to this rod, which was of I section, were kindly supplied by Mr. George Hughes, Chief Mechanical Engineer to the Lancashire and Yorkshire Railway Co., and are as follows:—

Diameter of cylinder, 18 inches; working steam-pressure, 160 lb. per sq. in.; radius at which rod acts, 12 inches; distance between centres, 8 ft. 11 in.; diameter of wheels to which rod was attached, 6 feet; mean sectional area 6·53 sq. in.; mean moment of inertia I, about the axes of bending, 12·5 in.^{4} units; mean weight per inch run, 1·85 lb. Calculations have been made for steam-pressures of 160 and 200 lb. per square inch at different speeds of the engine, the direct stress $\frac{F}{A}$ being taken as the full steam-pressure on the piston divided by the sectional area of the coupling-rod.

It will be noticed that the stress increases very rapidly as the speed of the engine increases, and that at the high speeds an, increase in steam-pressure does not appear to have as disastrous an effect as an increase in speed. It is also seen that at 60 miles per hour, with the ordinary working pressure of 160 lb. per square inch, the stress in the rod is 9·82 tons per square inch, which is rather high considering that the stresses are of an alternating character, first compressive, and then tensile. Should the wheels slip upon the rail, it is quite possible that they may revolve at a speed equivalent to 80 or more miles per hour, and the stresses may then exceed the elastic limit. Rods have been known to break in cases of derailment owing to the high speed with which they rotate in such cases.

Professor Perry, by considering a coupling-rod to be uniformly loaded and to be bent in a cosine curve, deduced the equation

$$\frac{d^2y}{dx^2} + \frac{\mathrm{F}}{\mathrm{EI}}y + \frac{w_1 l^2}{2\mathrm{EI}}\cos\frac{\pi}{2l}x = 0$$

from which he obtained the stress

$$f_{\max} = \frac{w_1 l^2}{2\mathrm{Z}}\left(\frac{\frac{\pi^2\mathrm{EI}}{4l^2}}{\frac{\pi^2\mathrm{EI}}{4l^2} - \mathrm{F}}\right) + \frac{\mathrm{F}}{\mathrm{A}}.$$

The results obtained by applying his formula to this particular rod are shown in column 7, Table 2 (page 471), and Fig. 3 (page 470), and are found to agree very closely with the more mathematical determination obtained from equation (16) and tabulated in column 9 of the same Table.

Professor Unwin in his work on "Machine Design" gives the formula

$$f = \frac{wv^2l^2}{gr8\mathrm{Z}}\ \text{lb. per square inch.}$$

where w is the weight per inch run in pounds,

v the linear velocity of the rod in feet per second,

g the acceleration of gravity in feet per second, per second,

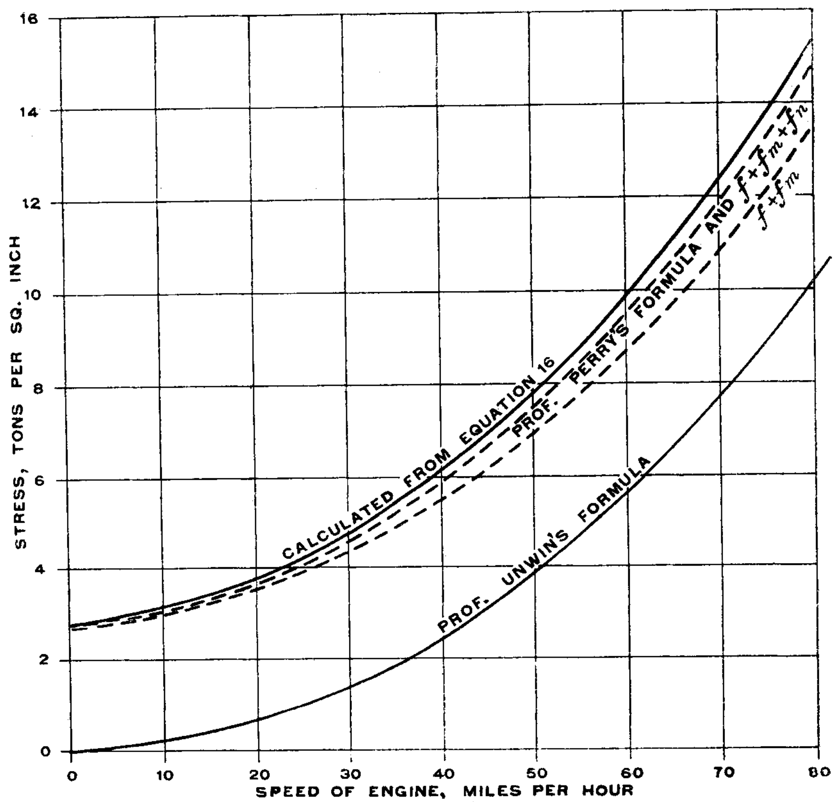
r the radius at which the rod acts in feet,

l the distance between the centres in inches, and

Z the modulus of section in inch^4 units.

Column 8, Table 2, gives the results obtained by applying this formula to the same rod, and it is seen that the stresses obtained are considerably below those in column 9.

Fig. 3.—*Stresses in Coupling-Rod at Different Speeds of the Engine, Calculated by Different Formulæ.*
Steam-Pressure 160 lb. per square inch.



To obtain the stress from equation (16) is laborious, and the Author would suggest the following method, which will be found to give results in very close agreement with those in column 9, Table 2, and can be easily followed.

TABLE 2.

Steam-Pressure, 160 *lb. per square inch.*

1	2	3	4	5	6	7	8	9
Speed of Engine. Miles per hour.	Direct Stress, f tons per sq. in.	Stress due to uniform Load, f_m tons per sq. in.	Stress due to eccentricity of F, f_n tons per sq. in.	$f + f_m$ tons per sq. in.	Total Stress $f + f_m + f_n$ tons per sq. in.	Stress calculated from Prof. Perry's formula. Tons per sq. in.	Stress calculated from Prof. Unwin's formula. Tons per sq. in.	Stress calculated from equation (16). Tons per sq. in.
10	2·79	0·35	0·05	3·14	3·19	3·19	0·16	3·20
20	2·79	0·84	0·11	3·63	3·74	3·73	0·63	3·78
30	2·79	1·63	0·22	4·42	4·64	4·64	1·42	4·70
40	2·79	2·73	0·37	5·52	5·89	5·81	2·53	6·18
50	2·79	4·11	0·55	6·90	7·45	7·53	3·95	7·70
60	2·79	5·91	0·78	8·70	9·48	9·51	5·70	9·82
70	2·79	7·91	1·03	10·70	11·73	11·76	7·75	12·30
80	2·79	10·41	1·38	13·20	14·58	14·61	10·10	15·20

Consider the coupling-rod to be subjected to a uniform load of $w_1 = w + \frac{w}{g}a^2r$ lb. per unit run, where w is the weight of unit length of the rod.

The B.M. due to this

$$= \frac{w_1 l^2}{8}.$$

The stress due to this B.M.

$$= \frac{w_1 l^2}{8Z} = f_m.$$

The deflection at the centre due to this uniform load is

$$\frac{5}{384} \frac{w_1 l^4}{EI} = \delta.$$

The stress due to the pressure F acting at a distance δ from the axis

$$= \frac{F\delta}{Z} = f_n.$$

The direct stress

$$= \frac{F}{A} = f.$$

Total stress

$$= f + f_m + f_n.$$

Treating the rod in this manner, the results have been obtained for different speeds of the engine when the steam-pressure is 160 lb. per square inch. The results, as shown in column 6, Table 2, agree very closely with those calculated by Professor Perry's formula, and are very near to those obtained in column 9. Hence, either this method or the one suggested by Professor Perry of assuming the rod to bend in a cosine curve can be adopted without any serious errors.

The Paper is illustrated by 3 Figs. in the letterpress.