# THE FOUNDATIONS OF THE ELLIPTIC FUNCTIONS.

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1. The following is a modification of a theorem which is found at the end of WEIERSTRASS's paper on Abelian Functions [1]).

THEOREM. — *If $F(u)$ is a given one-valued function of the variable $u$, which has no essential singularity in a finite region $\mathfrak{R}$, say, of the plane, and if $\alpha$ is any polar singularity of $F(u)$ within $\mathfrak{R}$, such that*

$$F(u) = \frac{L}{(u-\alpha)^{\lambda}} + P(u-\alpha),$$

*where there must be only one negative power on the right-hand side, $P(u-\alpha)$ denoting as usual a power series in integral positive powers of $u-\alpha$; while $\lambda$ must be the same for every pole of $F(u)$ in the region $\mathfrak{R}$ in question; if further*

$$\frac{L}{(u-\alpha)^{\lambda}} = l\frac{d^{\lambda}\log(u-\alpha)}{du^{\lambda}},$$

*where $l$ must in all cases be a positive integer, then the general integral $z = f(u)$, say, of the differential equation*

$$(A) \qquad \frac{d^{\lambda}\log z}{du^{\lambda}} = F(u),$$

*is a power series that is uniformly convergent in $\mathfrak{R}$.*

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[1]) See C. WEIERSTRASS, *Theorie der* ABEL'*schen Functionen* [Journal für die reine und angewandte Mathematik, Bd. LII (1856), pp. 285-380]; see also WEIERSTRASS, *Mathematische Werke*, (Berlin, Mayer und Müller), Bd. I, p. 349.

For, let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the polar singularities other than the origin of $F(u)$ within $\mathfrak{R}$, and introduce the auxiliary function $F_1(u)$ defined through the relation

$$F_1(u) = F(u) - l\frac{d^\lambda \log u}{du^\lambda} - l_1\frac{d^\lambda \log(u-\alpha_1)}{du^\lambda} - l_2\frac{d^\lambda \log(u-\alpha_2)}{du^\lambda} - \ldots - l_n\frac{d^\lambda \log(u-\alpha_n)}{du^\lambda},$$

where $l = 0$, if the origin is *not* a polar singularity of $F(u)$. It is seen that $F_1(u)$ has no singularity whatever in $\mathfrak{R}$ and has in this region a finite value that changes in a continuous manner with $u$, so that by the TAYLOR-CAUCHY Theorem, $F_1(u)$ may be represented through a convergent series

$$F_1(u) = \sum_{m=0}^{m=\infty} c_m u^m,$$

which is true for all values of $u$ within $\mathfrak{R}$. The given differential equation may be written in the form

$$(B) \qquad \frac{d^\lambda \log z}{du^\lambda} = F_1(u) + \frac{d^\lambda}{du^\lambda}\log \Pi(u),$$

where

$$\Pi(u) = u^l\left(1 - \frac{u}{\alpha_1}\right)^{l_1}\left(1 - \frac{u}{\alpha_2}\right)^{l_2} \ldots \left(1 - \frac{u}{\alpha_n}\right)^{l_n}$$

multiplied by the constant $\alpha_1^{l_1}\alpha_2^{l_2} \ldots \alpha_n^{l_n}$, which constant may be neglected as it occurs in the derivative of a logarithmic term; and consequently for all values of $u$ within $\mathfrak{R}$, $z$ may be expressed through a convergent power series of the form [particular integral of $(A)$], $z = f_1(u)$, say, where

$$z = f_1(u) = \Pi(u)\, e^{\sum_{m=0}^{m=\infty} \frac{c_m u^{m+\lambda}}{(m+1)(m+2)\ldots(m+\lambda)}},$$

while the general integral is

$$(1) \qquad z = f(u) = e^{\sum_{m=0}^{m=\infty} \frac{c_m u^{m+\lambda}}{(m+1)(m+2)\ldots(m+\lambda)}}\, \Pi(u)\, e^{A_0 + A_1 u + \cdots + A_{\lambda-1}u^{\lambda-1}},$$

where $A_0, A_1, \ldots, A_{\lambda-1}$ are the constants of integration.

If we consider the differential equation $(A)$ within a smaller region $\mathfrak{R}^1$ which includes no pole except possibly the origin, this equation may be written

$$(C) \qquad \frac{d^\lambda \log z}{du^\lambda} = l\frac{d^\lambda \log u}{du^\lambda} + P(u),$$

where $l = 1$, if the origin is *not* a polar singularity of $F(u)$, and where $P(u)$ is the convergent (within $\mathfrak{R}^1$) power series

$$P(u) = \sum_{m=0}^{m=\infty} \gamma_m u^m \ \ldots \ (i).$$

It is seen that the general integral (within $\mathfrak{R}'$) of the equation $(C)$ is

$$(2) \qquad z = f(u) = u^l e^{\sum_{m=0}^{m=\infty} \frac{\Upsilon_m u^{m+\lambda}}{(m+1)(m+2)\ldots(m+\lambda)} + B_0 + B_1 u + \cdots + B_{\lambda-1} u^{\lambda-1}},$$

where $B_0, B_1, \ldots, B_{\lambda-1}$ are the constants of integration. By a comparison of the power series (1) and (2) it is seen that the coefficients of like povers of $u$ must be identical (in $\mathfrak{R}'$, and therefore also in $\mathfrak{R}$), and it follows that the series (2) must be uniformly convergent in $\mathfrak{R}$, the original realm, although this is not necessarily true of the series ($i$) whose coefficients enter in the formation of those of (2).

**2.** The differential equation, which $z = \operatorname{sn} u$ satisfies, is

$$(D) \qquad \left(\frac{dz}{du}\right)^2 = (1 - z^2)(1 - k^2 z^2).$$

When differentiated with regard to u, this equation becomes

$$\frac{d^2 z}{du^2} = -(1 + k^2) z + 2k^2 z^3.$$

It follows that

$$\frac{d^2 \log z}{du^2} = k^2 z^2 - \frac{1}{z^2},$$

or

$$(\mathrm{I}) \qquad \begin{cases} \dfrac{d^2 \log \operatorname{sn} u}{du^2} = k^2 \operatorname{sn}^2 u - \dfrac{1}{\operatorname{sn}^2 u}. & \text{Similarly it is seen that} \\[2ex] \dfrac{d^2 \log \operatorname{cn} u}{du^2} = k^2 \operatorname{sn}^2 u - \dfrac{\operatorname{dn}^2 u}{\operatorname{cn}^2 u}, & \\[2ex] \dfrac{d^2 \log \operatorname{dn} u}{du^2} = k^2 \operatorname{sn}^2 u - \dfrac{k^2 \operatorname{cn}^2 u}{\operatorname{dn}^2 u}. & \end{cases}$$

WEIERSTRASS [2]) writes:

« If $x = \operatorname{sn} u$, we have

$$\frac{d^2 \log x}{du^2} = k^2 x^2 - \frac{1}{x^2}.$$

If next we put $x = \frac{p_1}{p_2}$, it follows that

$$\frac{d^2 \log p_1}{du^2} - \frac{d^2 \log p}{du^2} = k^2 \frac{p_1^2}{p^2} - \frac{p^2}{p_1^2}.$$

This equation becomes two equations, if we put

$$\frac{d^2 \log p_1}{du^2} = -\frac{p^2}{p_1^2}, \qquad \frac{d^2 \log p}{du^2} = -k^2 \frac{p_1^2}{p^2}.$$

[2]) *Mathematische Werke,* vol. I, l. c. [1]), p. 140.

After it has been shown that sn $u$, when the absolute value of $u$ is not greater than an arbitrary limit, may be expressed as the quotient of two integral power series which are convergent for values of $u$ that are less than this limit, then it may be proved by rigorous methods that the two functions $p$, $p_1$, that are defined by the two differential equations above, may be developed in integral positive powers of $u$ in uniformly convergent series. And if the four arbitrary constants which arise, are so determined that for $u = 0$,

$$p = 1, \qquad \frac{dp}{du} = 0, \qquad p_1 = 0, \qquad \frac{dp_1}{du} = 1,$$

then it is seen that in fact $\operatorname{sn} u = \frac{p_1}{p}$. Similar results are true for cn $u$ and dn $u$ and we come in this manner to the representation of the elliptic functions, which Abel in a letter to Legendre [3]) mentions without showing however how this is done and without indicating the method of procedure ».

**3.** In the present paper it is shown that Abel's assertion may be proved so simply that the method of procedure needs no emphasis and at the same time it is made clear that the methods of procedure employed by Weierstrass in his paper *Ueber die Entwicklung der Modular-Functionen* [4]) are not direct and are of little importance. It is also seen that the auxiliary Al-functions introduced by him in this connection, and upon which much emphasis has often been placed, are without value in the development of the elliptic functions. Briot and Bouquet, for example, in the second edition of their *Théorie des Functions elliptiques,* (Gauthier-Villars, Paris, 1875) devote pages 465-475 to this subject.

**4.** Let us first consider the functions $k^2 \operatorname{sn}^2 u$, $\frac{1}{\operatorname{sn}^2 u}$, $\frac{\operatorname{dn}^2 u}{\operatorname{cn}^2 u}$, $\frac{k^2 \operatorname{cn}^2 u}{\operatorname{dn}^2 u}$, which are found on the right hand side in the formulas (I), and in particular note their expansions in the neighborhood of their infinities. To this end consider the nature of the function $z = \operatorname{sn} u$ in the neighborhood of an infinity, say $u = \alpha$.

Writing

$$z = \operatorname{sn} u = c_{-n}(u - \alpha)^{-n} + c_{-n+1}(u - \alpha)^{-n+1} + \cdots,$$

it is seen that

$$\frac{dz}{du} = - c_{-n} n (u - \alpha)^{-n-1} + c_{-n+1}(- n + 1)(u - \alpha)^{-n} + \cdots.$$

3) N. H. Abel, *Précis d'une théorie des fonctions elliptiques* [Journal für die reine und angewandte Mathematik, B. IV (1829), pp. 236-277], p. 244. See also *Fernere mathematische Bruchstücke aus Herrn* N. H. Abel's *Briefen; Schreiben des Herrn* N. H. Abel *an Herrn* Legendre *zu Paris* [Journal für die reine und angewandte Mathematik, Bd. VI (1830), pp. 73-80], p. 76.

4) *Mathematische Werke,* Bd. I, l. c. [1]), pp. 1-49.

If these expressions are substituted in ($D$), it is observed that the lowest exponent on either side of the resulting expression must be equal, and consequently

$$-2n-2=-4n, \qquad \text{or} \qquad n=1.$$

It follows that

$$z = c_{-1}(u-\alpha)^{-1} + c_0 + c_1(u-\alpha) + c_2(u-\alpha)^2 + \cdots,$$

and

$$\frac{dz}{du} = -c_{-1}(u-\alpha)^{-2} + c_1 + 2c_2(u-\alpha) + \cdots.$$

Further, if these values are substituted in the equation ($D$), it is found that

$$c_{-1}^2 = k^2 c_{-1}^4, \qquad \text{or} \qquad c_{-1} = \frac{1}{k}; \; c_{-1}c_0k^2 = 0, \qquad \text{or} \qquad c_0 = 0.$$

It follows that in the neighborhood of a pole $\alpha$ the expansion of $z$ is

$$z = \operatorname{sn} u = \frac{1}{k}\frac{1}{u-\alpha} + c_1(u-\alpha) + \cdots.$$

In the neighborhood of a zero of $\operatorname{sn} u$, write

$$z = (u-\beta)^m[b_0 + b_1(u-\beta) + \cdots],$$

so that

$$\frac{dz}{du} = m(u-\beta)^{m-1}b_0 + \cdots.$$

These series substituted in the differential equation ($D$) show that $2m-2=0$, or $m=1$.

It follows that

$$z = b_0(u-\beta) + b_1(u-\beta)^2 + \cdots,$$

and

$$\frac{dz}{du} = b_0 + 2b_1(u-\beta) + \cdots.$$

Again writing these values in the differential equation, it is seen that

$$b_0 = 1, \qquad b_1 = 0, \; \ldots .$$

Hence in the neighborhood of a zero, say $\beta$, the function $z=\operatorname{sn} u$ may be expanded in the form

$$z = u - \beta + b_2(u-\beta)^3 + \cdots.$$

**5.** Returning to the formulas (I) write

$$\frac{d^2 \log \operatorname{sn} u}{du^2} = k^2 \operatorname{sn}^2 u - \frac{1}{\operatorname{sn}^2 u} = F_2(u) - F_1(u),$$

where

$$F_1(u) = -k^2 \operatorname{sn}^2 u$$

and

$$F_2(u) = -\frac{1}{\operatorname{sn}^2 u}.$$

It follows from the series derived in the preceding article that in the neighborhood of its infinities the expansion of $F_1(u)$ is of the form

$$F_1(u) = -\frac{1}{(u-\alpha)^2} + P(u-\alpha) = \frac{d^2}{du^2}\log(u-\alpha) + P(u-\alpha),$$

$P$ being the usual symbol for an infinite power series in positive powers of the argument.

Likewise in the neighborhood of its infinities the expansion of $F_2(u)$ is of the form

$$F_2(u) = \frac{d^2}{du^2}\log(u-\beta) + P_1(u-\beta),$$

$P_1$ denoting *a* second positive power series.

If we assume certain known properties of the sn-function these results may be established at once. For the expansion of $\operatorname{sn} u$ by the MACLAURIN Theorem is

$$\operatorname{sn} u = u - (1+k^2)\frac{u^3}{3!} + [(u^5)].$$

We further have

$$\operatorname{sn}(v+iK') = \frac{1}{k}\frac{1}{\operatorname{sn} v}; \qquad \text{or, writing} \qquad v = u - iK',$$

$$\operatorname{sn} u = \frac{1}{k}\frac{1}{\operatorname{sn}(u+iK')},$$

so that the expansion of $F_1(u)$ in the neighborhood of an infinity $iK'$ is

$$F_1(u) = \frac{d^2}{du^2}\log(u-iK') + P(u-iK').$$

Writing

$$F_3(v) = -\frac{\operatorname{dn}^2 v}{\operatorname{cn}^2 v},$$

we have

$$F_3(v+K) = -\frac{\operatorname{dn}^2(v+K)}{\operatorname{cn}^2(v+K)} = -\frac{1}{\operatorname{sn}^2 v}.$$

Thus in the neighborhood of the infinity $K$, we have

$$F_3(u) = -\frac{1}{\operatorname{sn}^2(u-K)} = \frac{d^2}{du^2}\log(u-K) + P_3(u-K).$$

Finally, if we put

$$F_4(v) = -k^2\frac{\operatorname{cn}^2 v}{\operatorname{dn}^2 v},$$

it is seen that

$$F_4(u) = -\frac{1}{\operatorname{sn}^2(u - K - iK')},$$

so that in the neighborhood of the infinity $K + iK'$ the expansion of $F_4(u)$ is

$$F_4(u) = \frac{d^2}{du^2}\log(u - K - iK') + P_4(u - K - iK').$$

But it is to be noted that the expansions in these forms for the functions $F_3(u)$ and $F_4(u)$ may be made quite independently of the assumed properties of the sn-, cn-, and dn-functions. We have only to use the differential equations (I) that are satisfied by the cn- and dn- functions and proceed as was done above for the functions $\operatorname{sn} u$.

**6.** It follows directly from the theorem of Article 1 that the differential equation

$$\frac{d^2 \log z}{du^2} = F_1(u) = -k^2 \operatorname{sn}^2 u$$

is satisfied by a uniformly convergent power series, say $g(u)$, so that

$$\text{(II)} \quad \begin{cases} \dfrac{d^2 \log g(u)}{du^2} = -k^2 \operatorname{sn}^2 u; \qquad \text{and similarly} \\ \dfrac{d^2 \log g_1(u)}{du^2} = F_2(u) = -\dfrac{1}{\operatorname{sn}^2 u}, \\ \dfrac{d^2 \log g_2(u)}{du^2} = F_3(u) = -\dfrac{\operatorname{dn}^2 u}{\operatorname{cn}^2 u}, \\ \dfrac{d^2 \log g_3(u)}{du^2} = F_4(u) = -k^2\dfrac{\operatorname{cn}^2 u}{\operatorname{dn}^2 u}, \end{cases}$$

where $g_1(u)$, $g_2(u)$, $g_3(u)$ are likewise uniformly convergent power series.

**7.** The equations (I) may be written in the form

$$\frac{d^2 \log \operatorname{sn} u}{du^2} = F_2(u) - F_1(u) = \frac{d^2 \log g_1(u)}{du^2} - \frac{d^2 \log g(u)}{du^2},$$

$$\frac{d^2 \log \operatorname{cn} u}{du^2} = F_3(u) - F_1(u) = \frac{d^2 \log g_2(u)}{du^2} - \frac{d^2 \log g(u)}{du^2},$$

$$\frac{d^2 \log \operatorname{dn} u}{du^2} = F_4(u) - F_1(u) = \frac{d^2 \log g_3(u)}{du^2} - \frac{d^2 \log g(u)}{du^2}.$$

It follows that

$$\text{(III)} \quad \begin{cases} \operatorname{sn} u = \dfrac{g_1(u)}{g(u)} e^{A_0 + A_1 u}, \\ \operatorname{cn} u = \dfrac{g_2(u)}{g(u)} e^{B_0 + B_1 u}, \\ \operatorname{dn} u = \dfrac{g_3(u)}{g(u)} e^{C_0 + C_1 u}, \end{cases}$$

where $A_0$, $A_1$; $B_0$, $B_1$; $C_0$, $C_1$ are the constants of integration.

**8.** From (II) it is seen that

$$\frac{d \log u}{d u} = - \int_0^u k^2 \operatorname{sn} u \, d u.$$

In my *Elliptic Functions*, Vol. I, p. 288 [5]) [see also HERMITE [6]) in SERRET's *Calculus*, Vol. 2, p. 829] it is shown that

$$- \int_0^u k^2 \operatorname{sn} u \, d u = \frac{\Theta'(u)}{\Theta(u)} - C u,$$

where the constant $C$ is determinate $\left(= \frac{J}{K} = \frac{\Theta''(0)}{\Theta(0)}\right)$.

It follows that

$$\frac{d \log g(u)}{d u} = \frac{d}{d u} \log \Theta(u) - C u,$$

and consequently

$g(u) = \Theta(u) e^{C^I - C \frac{u^2}{2}}$, where $C^I$ is the constant of integration. Similarly [7]) it is evident that

$$g_3 = \Theta_1(u) e^{C^{IV} - C \frac{u^2}{2}},$$

and in like manner

$$g_1(u) = \mathrm{H}(u) e^{C^{II} - C \frac{u^2}{2}}, \qquad g_2(u) = \mathrm{H}_1(u) e^{C^{III} - C \frac{u^2}{2}} \quad ^{8)}.$$

**9.** From (III) it follows, if we make use of the results just written, that

$$\operatorname{sn} u = \frac{\mathrm{H}(u)}{\Theta(u)} e^{A'_0 + A'_1 u}.$$

Since $\operatorname{sn} u$ is an odd function as is also $\mathrm{H}(u)$ while $\Theta(u)$ is an even function, it follows by writing—$u$ in the place of $u$ in this formula that $A'_1 = 0$. We thus have

$$\operatorname{sn} u = C \frac{\mathrm{H}(u)}{\Theta(u)},$$

where the constant $C$ is [9]) found to be $= \frac{1}{\sqrt{k}}$. Thus it is seen that

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5) H. HANCOCK, *Lectures on the theory of elliptic functions*, vol. I (New York, J. Wiley and Sons, 1910).

6) J. A. SERRET, *Cours de calcul différentiel et intégral*, 5e edition (Paris, Gauthier-Villars, 1900).

7) See my *Elliptic Functions*, vol. I, l. c. 5), p. 296.

8) Compare these results with those of BRIOT and BOUQUET, *Théorie des fonctions elliptiques*, 2e édition (Paris, Gauthier-Villars, 1875), p. 465.

9) l. c. 5), p. 241.

$$\text{(IV)}\qquad\begin{cases}\operatorname{sn} u = \dfrac{1}{\sqrt{k}}\,\dfrac{\mathrm{H}(u)}{\Theta(u)}; & \text{and similarly}\\[2ex] \operatorname{cn} u = \sqrt{\dfrac{k'}{k}}\,\dfrac{\mathrm{H}_1(u)}{\Theta(u)}, & \sqrt{k} = \mathrm{H}(K) = \mathrm{H}_1(0),\\[2ex] \operatorname{dn} u = \sqrt{k'}\,\dfrac{\Theta_1(u)}{\Theta(u)}, & \sqrt{k'} = \dfrac{\Theta(0)}{\Theta_1(0)}.\end{cases}$$

The properties of the Elliptic functions may be *then* derived directly from those of the Theta-functions. The Theta-functions being elemental in the whole theory, it is more scientific to establish their properties without making use of the properties of the elementary elliptic functions sn, cn and dn. This may be done through the introduction of the so called « Intermediary Functions » of HERMITE. In Chapter V of my *Elliptic functions* is given the method of procedure of that great mathematical genius with numerous references to his works.

It may be noted in passing that from the present paper we also get some insight into the real nature of the « doubly periodic functions of the second and third sorts (espéce) » that were introduced by HERMITE.

Thus without going into an extended discussion of the properties of the auxiliary functions $g(u)$, $g_1(u)$, ... etc., we are able in a rather simple manner to express the elliptic functions as quotients of Theta-functions. It is seen that the functions $g(u)$, etc. are nothing other that the Al-functions introduced by WEIERSTRASS. It is also evident that the characteristic properties that WEIERSTRASS derived for these functions, — which it may be observed that he did by making use of the properties of the elementary elliptic functions, — are merely properties that were already known for the Theta-functions as from the above it is evident that they must be. It is difficult to see why WEIERSTRASS should ascribe such importance to these Al-functions as to use them as an introduction to his collected works. (See statement made by him in this connection in his *Mathematische Werke,* Vol. 1, p. 50). It is perfectly evident that these Al-functions add nothing either new or in themselves interesting to the theory of Elliptic Functions.

**10.** We may next consider a second form of the differential equation through which the elemental elliptic functions may be defined,

$$(E)\qquad \left(\frac{dz}{du}\right)^2 = 4z^3 - g_2 z - g_3 = S(z),$$

a form due to HERMITE and CAYLEY [10]).

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[10]) See A. CAYLEY, *Note sur les covariants d'une fonction quadratique, cubique, ou biquadratique à deux indéterminées* [Journal für die reine und angewandte Mathematik, Bd. L. (1855), pp. 285-287], p. 287; F. BRIOSCHI, *Sur une formule de* M. CAYLEY [Ibidem, Bd. LIII (1857), pp. 377-378].

If $z$ considered as a function of $u$ is written in the WEIERSTRASSian notation $z = \wp u$, it follows as in Article 4 that in the neighborhood of an infinity, $\alpha$ say, the expansion of $\wp u$ takes the form

$$z = \wp u = \frac{1}{(u-\alpha)^2} + P(u-\alpha).$$

Where $P$ denotes an integral power series.

Similarly in the neighborhood of a zero $u = \beta$, say, it is seen that

$$z = \wp u = b_1(u-\beta) + b_2(u-\beta)^2 + \cdots,$$

where

$$b_1 = \sqrt{-g_3} \quad \text{and} \quad b_2 = -\frac{g_2}{4}.$$

It follows at once from the differential equation $(E)$ that

$$\frac{d^2 z}{du^2} = 6z^2 - \frac{1}{2}g_2.$$

Then from the formula

$$\frac{d^2 \log z}{du^2} = \frac{1}{z}\frac{d^2 z}{du^2} - \frac{1}{z^2}\left(\frac{dz}{du}\right)^2$$

we have as a form of the differential equation through which $z = \wp u$ is defined

$$(F) \qquad \frac{d^2 \log z}{du^2} = 2z + \frac{1}{2}\frac{g_2}{z} + \frac{g_3}{z^2}.$$

Writing the above differential equation in the form

$$(F') \qquad \frac{d^2 \log z}{du^2} = F_2(u) - F_1(u),$$

where

$$F_2(u) = \frac{1}{2}\frac{g_2}{z} + \frac{g_3}{z^2} = \frac{1}{2}\frac{g_2}{\wp u} + \frac{g_3}{(\wp u)^2},$$

and

$$F_1(u) = -2z = -2\wp u,$$

we must observe first whether $F_2(u)$ and $F_1(u)$ satisfy the postulates made in the theorem of Article 1. It is seen at once, if we observe the expansion of the $\wp$-function given in the preceding article that in the neighborhood of an infinitiy $z = \alpha$,

$$F_1(u) = 2\frac{d^2 \log(u-\alpha)}{du^2} + P_1(u-\alpha).$$

In the neighborhood of its infinities we also have

$$F_2(u) = \frac{1}{2}\frac{g_2}{b_1}\frac{1}{u-\beta} - \frac{2b_2}{b_1^2}g_3\frac{1}{u-\beta} + \frac{g_3}{b_1^2}\frac{1}{(u-\beta)^2} + P_2(u-\beta),$$

where $b_1 = \sqrt{-b_3}$, $b_2 = -\frac{1}{4} g_2$ and consequently

$$F_2(u) = -\frac{1}{(u-\beta)^2} + P_2(u-\beta) = \frac{d^2}{du^2}\log(u-\beta) + P_2(u-\beta).$$

The postulates of Article 1 are thus shown to be satisfied for the functions $F_1(u)$ and $F_2(u)$.

**12.** If we write

$$\text{(V)} \qquad \begin{cases} \dfrac{d^2 \log w}{du^2} = F_1(u), \\ \dfrac{d^2 \log w}{du^2} = F_2(u), \end{cases}$$

it follows that these equations are satisfied, say, by $w = f_1(u)$ and $w = \rho(u)$, where $f_1(u)$ and $\rho(u)$ are uniformly convergent power series.

If we put $\frac{d^2 \log w}{du^2} = -\wp u$, it is seen that this differential equation is also satisfied by a uniformly convergent power series which denote by $f(u)$.

Thus from (V) we have

$$\frac{d^2 \log f_1(u)}{du^2} = F_1(u) = -2\wp u = 2\frac{d^2}{du^2}\log f(u);$$

or

$$f_1(u) = f(u)^2 e^{a_0 + a_1 u},$$

where $a_0$ and $a_1$ are the constants of integration.

The function $f(u)$ is defined through the power series

$$\text{(G)} \qquad \frac{d^2 \log f(u)}{du^2} = -\wp u.$$

Expanding $\wp u$, it is seen that

$$\frac{d^2 \log f(u)}{du^2} = -\frac{1}{u^2} - c_2 u^2 - \cdots = \frac{d^2}{du^2}\log u - c_2 u^2 - \cdots.$$

It follows that

$$\log f(u) = \log u - \frac{c_2}{3 \cdot 4} u^4 - \cdots + A_0 + A_1 u,$$

where $A_0$ and $A_1$ are the constants of integration.

If the constants $A_0$, $A_1$ are made zero, we have here the function introduced by WEIERSTRASS, namely $\sigma(u) = u\left[1 - \frac{1 \cdot 2}{4!} c_2 u^4 - \cdots\right]$. The further expansion is derived on p. 327 of my *Elliptic Functions*, Vol. I [11]). Returning to the formula ($F'$) we note that it may be written

$$\frac{d^2 \log z}{d u^2} = \frac{d^2 \log \rho(u)}{d u^2} - \frac{d^2 \log f_1(u)}{d u^2}.$$

Through integration we have

$$(H) \qquad z = \wp u = \frac{\rho(u)}{(\sigma u)^2} e^{\alpha_0 + \alpha_1 u},$$

where $\alpha_0$ and $\alpha_1$ are arbitrary constants, and where $\rho(u)$ may be regarded as the particular integral of the equation

$$(I) \qquad \frac{d^2 \log w}{d u^2} = \frac{1}{2} \frac{g_2}{\wp u} + \frac{g_3}{(\wp u)^2}.$$

It may be observed here that unless one introduces a somewhat different theory, the function $\wp u$ unlike the function $\operatorname{sn} u$ above is *not* expressed in the usual WEIERSTRASSian theory through the quotient of two functions that are analogous to the Theta-functions. Instead $\wp u$ is defined in the WEIERSTRASSian theory through the equation ($G$), which is in fact

$$\wp u = - \frac{d^2}{d u^2} \log \sigma u.$$

In this connection note the formulas (II) which correspond to the formula just written defining $\wp u$ and then compare with the formulas (III).

From the differential equation ($D$) it follows that

$$\frac{d z}{d u} = \sqrt{1 - z^2} . \sqrt{1 - k^2 z^2};$$

and, if $z$ is defined as $\operatorname{sn} u$, the two expressions to the right, namely $\sqrt{1 - z^2}$ and $\sqrt{1 - k^2 z^2}$ are the functions $\operatorname{cn} u$ and $\operatorname{dn} u$. We expressed above all three of these functions as quotients of Theta-functions.

**13.** In like manner write the differential equation ($E$) in the form

$$\frac{d z}{d u} = \sqrt{4 z^3 - g_2 z - g_3} = 2 \sqrt{z - e_1} . \sqrt{z - e_2} . \sqrt{z - e_3},$$

where $e_1$, $e_2$ and $e_3$ are the roots of the cubic $4 z^3 - g_2 z - g_3 = 0$. If $z$ is defined through the function $\wp u$, we may define $\sqrt{z - e_1}$ through $\wp_1 u$, $\sqrt{z - e_2}$ through $\wp_2 u$ and $\sqrt{z - e_3}$ through $\wp_3 u$. It may be shown that these functions may be expressed as quotients of power series that are uniformly convergent.

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[11]) l. c. [5]).

For

$$\frac{d}{du}\wp_1(u) = \frac{1}{2}\frac{\frac{dz}{du}}{\sqrt{z-e_1}} = \sqrt{(z-e_2)(z-e_3)} = \wp_2 u \wp_3 u$$

and

$$\frac{d \log \wp_1 u}{du} = \frac{1}{2}\frac{1}{\sqrt{z-e_1}}\frac{dz}{du},$$

$$\frac{d^2 \log \wp_1 u}{du^2} = \frac{1}{2}\frac{1}{z-e_1}\frac{d^2 z}{du^2} - \frac{1}{2}\frac{1}{(z-e_1)^2}\left(\frac{dz}{du}\right)^2.$$

Noting that

$$\frac{d^2 z}{du^2} = 6z^2 - \frac{1}{2}g_2 = \frac{1}{2}S'(z),$$

we observe that

$$\frac{d^2 \log \wp_1(u)}{du^2} = \frac{1}{4}\frac{S'(z)}{z-e_1} - \frac{1}{2}\frac{S(z)}{(z-e_1)^2}.$$

It follows that

$$\frac{d^2 \log(\wp_1 u)^2}{du^2} = \frac{1}{2}\frac{S'(z)}{z-e_1} - \frac{S(z)}{(z-e_1)^2}$$

(which expanded in powers of $z - e_1$ is)

$$= \frac{\frac{1}{2}S'(e_1) + \frac{1}{2}S''(e_1)(z-e_1) + \frac{1}{4}S'''(e_1)(z-e_1)^2}{z-e_1}$$

$$- \frac{S(e_1) + S'(e_1)(z-e_1) + \frac{1}{2}S''(e_1)(z-e_1)^2 + \frac{1}{6}S'''(e_1)(z-e_1)^3}{(z-e_1)^2};$$

or, since $S(e_1) = 0$ and $S'''(e_1) = 24$, we have

(VI) $$\frac{d^2 \log(\wp_1 u)^2}{du^2} = -\frac{1}{2}\frac{S'(e_1)}{z-e_1} + 2(z-e_1).$$

We may consequently write

$$\frac{d^2 \log(\wp_1 u)^2}{du^2} = 2z - \left(\frac{1}{2}\frac{S'(e_1)}{z-e_1} + 2e_1\right) = F_2(u) - F_1(u),$$

where

$$F_1(u) = -2z = -2\wp u = \frac{d^2}{du^2}\log(\sigma u)^2,$$

as shown above; and where

$$F_2(u) = -\left(\frac{1}{2}\frac{S'(e_1)}{z-e_1} + 2e_1\right).$$

**14.** It remains to consider the nature of the function $F_2(u)$ in the neighborhood of its infinity $z = e_1$.

From the definition $\wp_1(u) = \sqrt{z - e_1}$, it is seen that for a zero $u = \alpha$ of $\wp_1 u$ we have $z = e_1$.

Writing

$$\wp_1 u = a_1(u - \alpha) + a_2(u - \alpha)^2 + \cdots,$$

we have

$$\frac{d\wp u_1}{du} = a_1 + 2a_2(u - \alpha) + \cdots.$$

From the relation

$$\left(\frac{d\wp_1 u}{du}\right)^2 = (z - e_2)(z - e_3) = [(\wp_1 u)^2 + e_1 - e_2][(\wp_1 u)^2 + e_1 - e_3],$$

it follows that

$$a_1^2 = (e_1 - e_2)(e_1 - e_3) = \tfrac{1}{4}S'(e_1),$$

$$4a_1 a_2 = 0, \qquad \text{or} \qquad a_2 = 0,$$

$$6a_1 a_3 = (e_1 - e_2)a_1^2 + (e_1 - e_3)a_1^2, \qquad \text{or} \qquad a_3 = \tfrac{1}{96}\sqrt{S'(e_1)}.S''(e_1),$$

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We thus have the expansion

$$\frac{1}{(\wp_1 u)^2} = \frac{1}{z - e_1} = \frac{1}{\frac{1}{4}S'(e_1)} \cdot \frac{1}{(u - \alpha)^2} \cdot \frac{1}{1 + P(u - \alpha)};$$

and consequently

$$F_2(u) = -2\frac{1}{(u - \alpha)^2} + P_2(u - \alpha),$$

where $P_2(u - \alpha)$ is a positive integral power series.

It is thus shown that $F_2(u)$ satisfies the postulates of the theorem stated in Art. 1. It follows that if we write $t = f_2(u)$ where $t$ is defined through the differential equation

$$\text{(VII)} \qquad \frac{d^2}{du^2}\log t = -\frac{1}{4}\frac{S'(e_1)}{\wp u - e_1} - e_1,$$

then $f_2(u)$ is *a* uniformly convergent power series.

**15.** It is seen at once that formula (VI) takes the form

$$\frac{d^2}{du^2}\log(\wp_1 u)^2 = \frac{d^2}{du^2}\log f_2(u)^2 - \frac{d^2}{du^2}\log(\sigma u)^2,$$

or

$$\text{(VIII)} \qquad \wp_1(u) = \sqrt{\wp u - e_1} = \frac{f_2(u)}{\sigma u}e^{A_1 + B_1 u},$$

where $A_1$, $B_1$ are arbitrary constants.

We observe that $f_2(u)$ is an even function, being a particular integral of the differential equation (VII), and consequently changing $u$ to $-u$ in the formula (VIII) it is seen that $B_1 = 0$, and consequently

$$\wp_1 u = \sqrt{\wp u - e_1} = C\frac{f_2(u)}{\sigma u}$$

with similar expressions for $\wp_2 u$ and $\wp_3 u$.

If we put $Cf_2(u) = \sigma_1(u)$, it is seen that the expansion of $\sigma_1 u$ in integral powers of $u$ may be effected as on p. 394 of the first volume of my *Elliptic Functions*, l. c. [5]).

It would be better, however, first to evolve the theory of sigma-functions as was implied (Art. 9) for the Theta-functions and then identify the function $\sigma_1 u$ with the integral of the differential equation (VII).

This again is unnecessary; for, from the differential equations already introduced it will be seen in the next article that the sigma-functions are nothing other than Theta-functions except as to the constants of integration.

**16.** In Art. 8, we saw that

$$k^2 \operatorname{sn} u = \frac{J}{K} - \frac{d^2}{du^2}\log \Theta(u),$$

or, writing $u + iK'$ for $u$, this formula becomes $1/\operatorname{sn}^2 u = \frac{J}{K} - \frac{d^2}{du^2}\log H(u)$. On p. 278 of my *Elliptic Functions,* Vol. I, l. c. [5]), it is proved that

$$\wp(v\sqrt{\varepsilon}) = e_3 + \frac{1}{\varepsilon \operatorname{sn}^2 v}, \qquad \text{where} \qquad \varepsilon = \frac{1}{e_1 - e_3}.$$

It follows that

$$\wp(v\sqrt{\varepsilon}) = e_3 + \frac{1}{\varepsilon}\frac{J}{K} - \frac{1}{\varepsilon}\frac{d^2}{dv^2}\log \mathrm{H}(v).$$

The differential equation which defines $\sigma u$ is (Art. 12)

$$-\wp u = \frac{d^2}{du^2}\log \sigma u.$$

From this it is seen (as on p. 304 of my *Elliptic Functions*) that

$$\sigma u = \beta e^{2\eta\omega v^2}\mathrm{H}(2Kv),$$
$$\sigma_1 u = \beta_1 e^{2\eta\omega v^2}\mathrm{H}_1(2Kv),$$
$$\cdots\cdots\cdots\cdots\cdots$$

where the constant $\eta$ has the value $-\left(e_3 + \frac{1}{\varepsilon}\frac{J}{K}\right)$, and where $\beta, \beta_1, \ldots,$ are determinate constants [loc. cit. [5]), p. 408].

**17.** Thus it has been shown, that after the auxiliary theta-functions have been introduced, formed and defined through the simple « intermediary functions » of HERMITE, the whole theory of the Elliptic Functions, whether it be expressed in the JACOBI notation or in the WEIERSTRASSian notation or in the generalized notation indicated in Article 12, is made dependent upon the solution of the differential equations

$$\frac{d^2 \log t}{d u^2} = - k^2 \operatorname{sn}^2 u,$$

$$\frac{d^2 \log t}{d u^2} = - \frac{1}{\operatorname{sn}^2 u},$$

$$\frac{d^2 \log t}{d u^2} = - \frac{\operatorname{dn}^2 u}{\operatorname{cn}^2 u},$$

$$\frac{d^2 \log t}{d u^2} = - k^2 \frac{\operatorname{cn}^2 u}{\operatorname{dn}^2 u},$$

equations which we have seen to be integrable (Art. 8) in terms of Theta-functions.

**18.** Similar results may be derived, if we write the original differential equation $(D)$ in the form

$$\left(\frac{dt}{du}\right)^2 = 4t(1 - t)(1 - \lambda t) \qquad \text{(RIEMANN)}$$

or in the form

$$\left(\frac{dt}{du}\right)^2 = t(1 - \rho t + t^2) \qquad \text{(KRONECKER)}.$$

In either of these cases we may introduce functions corresponding to the Theta-functions or to the sigma-functions and we may derive for these new functions the analogous properties of the functions just mentioned. One may then in a general and in a more direct manner obtain the results that have been indicated by RIEMANN and KRONECKER. These new functions are merely other forms of Theta-functions. This is, of course, due to the fact that the differential equations that define the functions which are expressible as quotients of these new functions, may be transformed into the differential equation $(D)$. It follows also that they like the Al-functions of WEIERSTRASS add nothing new to the general theory.

Finally it may be observed that the results of this paper may be obtained without employig the theorem given in the first article; however, by making use of that theorem one knows à priori that such results as occur, must exist.

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