

VI.—Formulæ and Scheme of Calculation for the Development of a Function of Two Variables in Spherical Harmonics. By Professor T. BAUSCHINGER, Strassburg. *Translated and communicated by* The General Secretary, Dr C. G. KNOTT.

(MS. received December 7, 1914. Read January 18, 1915.)

(This paper was read at the Napier Tercentenary Celebration on July 27, 1914. The Council, on the suggestion of the Napier Committee, have much pleasure in making it accessible in the Society's *Proceedings*.)

WHEN a function has been expressed as a series of spherical harmonics with constant coefficients, the determination of these coefficients from given values of the function is in the general case one of the most complicated operations which can be set before the calculator.

Since Gauss first carried out these operations in a calculation of this kind,* efforts have not been wanting to simplify them and make their frequent application possible. The most successful of all in this respect was Franz Neumann,† who showed that by a suitable choice of the argument the calculation could be materially shortened.

For the application of Neumann's method H. Seeliger‡ arranged the constant coefficients in tables, and thereby made the calculations so easy and so obvious that even a non-scientific calculator can carry it out. I would now show that some further steps may be taken in this direction, with the advantage that in addition to a further shortening of the calculation the whole process can be carried out by one operation on the calculating machine, since only sums of products have to be formed.

[The given values§ of the function $f(\mu, \phi)$, where $\cos^{-1}\mu(=\theta)$ is the polar distance and ϕ the longitude, are supposed distributed over a spherical surface, such as the earth's, and the function itself is expressed as

$$f(\mu, \phi) = \sum_0^p Y^n$$

* Burckhardt, *Oszillierende Funktionen*, pp. 384 ff.

† *Astronomische Nachrichten*, Bd. xvi, p. 313 (1838).

‡ *Sitzungsberichte der Königl. bayer. Akademie der Wissenschaften München: Math.-phys. Klasse*, Band xx, p. 499 (1891).

§ The part in square brackets has been added by the translator, so as to make the notations immediately intelligible to the reader.

IN Professor J. (not "T.") Bauschinger's paper on Calculation by Spherical Harmonics (*Proc. R.S.E.*, vol. xxxv, p. 64), the following correction should be made:

Third equation from bottom on p. 64 should read

$${}^{\text{A}}(\mu\lambda) = \frac{2n+1}{2} \frac{(n-i)!}{(n+i)!} P_n(\mu_\lambda) a_\lambda$$

$a_1 a_2 \dots a_{p+1}$ being given by the solution of the equations

$$a_1 \mu_1^\lambda + a_2 \mu_2^\lambda + \dots + a_{p+1} \mu_{p+1}^\lambda = a_\lambda \quad (\lambda = 0, 1, 2, 3 \dots p),$$

where $a_\lambda = \int_{-1}^{+1} x^\lambda dx$."

obtain

$$\left. \begin{aligned} A_{ni} &= \sum_{\lambda=1}^{\lambda=p+1} \sum_{\nu=0}^{\nu=2p-1} G_{ni} \left(\mu_{\lambda}, \nu \frac{i\pi}{p} \right) \cdot f \left(\mu_{\lambda}, \nu \frac{\pi}{p} \right) \\ B_{ni} &= \sum_{\lambda=1}^{\lambda=p+1} \sum_{\nu=0}^{\nu=2p-1} H_{ni} \left(\mu_{\lambda}, \nu \frac{i\pi}{p} \right) \cdot f \left(\mu_{\lambda}, \nu \frac{\pi}{p} \right) \end{aligned} \right\} \quad (2)$$

In every case the $2p(p+1)$ coefficients G_{ni} and H_{ni} (of the latter $p+1$ are *ab initio* equal to zero) are tabulated for each combination n, i , and the operation to be carried out with the calculating machine is then continued quite simply so that each of the given $2p(p+1)$ values of the function $f \left(\mu_{\lambda}, \nu \frac{\pi}{p} \right)$ is multiplied with the corresponding G and H respectively, and the sums of all products taken.

After determination of the A_{ni} and B_{ni} , the interpolation formula for the function $f(\mu, \phi)$ becomes

$$\left. \begin{aligned} f(\mu, \phi) &= (P_{00}A_{00} + P_{10}A_{10} + \dots + P_{p0}A_{p0}) \\ &+ (P_{11}A_{11} + P_{21}A_{21} + \dots + P_{p1}A_{p1}) \cos \phi \\ &+ (P_{11}B_{11} + P_{21}B_{21} + \dots + P_{p1}B_{p1}) \sin \phi \\ &+ (P_{22}A_{22} + \dots + P_{p2}A_{p2}) \cos 2\phi \\ &+ (P_{22}B_{22} + \dots + P_{p2}B_{p2}) \sin 2\phi \\ &+ \dots \dots \dots \\ &+ P_{pv}A_{pv} \cos p\phi \end{aligned} \right\} \quad (3)$$

where the associated spherical harmonics P are functions of the powers of $\sin \delta$ and $\cos \delta (= \mu)$. For convenience of application, the expressions within the brackets in (3) require to be changed into rows which are developed in sines and cosines of multiples of θ ; that is, the arrangement takes the form

$$\begin{aligned} f(\theta, \phi) &= (a_{00} + a_{10} \cos \theta + a_{20} \cos 2\theta + \dots + a_{p0} \cos p\theta) \\ &+ (a_{11} \sin \theta + a_{21} \sin 2\theta + \dots + a_{p1} \sin p\theta) \cos \phi \\ &+ (\beta_{11} \sin \theta + \beta_{21} \sin 2\theta + \dots + \beta_{p1} \sin p\theta) \sin \phi \\ &+ (a_{02} + a_{12} \cos \theta + a_{22} \cos 2\theta + \dots + a_{p2} \cos p\theta) \cos 2\phi \\ &+ (\beta_{02} + \beta_{12} \cos \theta + \beta_{22} \cos 2\theta + \dots + \beta_{p2} \cos p\theta) \sin 2\phi \\ &+ \dots \dots \dots \end{aligned}$$

The *second* step to be made in preparing once for all for the carrying out of the calculations is that, instead of the above-named tables for the G_{ni} and H_{ni} , similar tables may be immediately constructed for the calculation of the α_{ni} and β_{ni} . This is easily possible, since the α_{ni}, β_{ni} are simple known functions of A_{ni} and B_{ni} .

The α_{ni} and β_{ni} are then obtained as sums of the $2p(p+1)$ products, of which the one factor is $f\left(\mu_\lambda, \nu \frac{\pi}{p}\right)$ and the other factor stands in the table.

Such a table possesses the advantage that a glance enables us to recognise and calculate the influence of a change of a given value of the function $f\left(\mu_\lambda, \nu \frac{\pi}{p}\right)$ upon the coefficients α and β ; for if $g_{ni\lambda\nu}$ is the value in the table for α_{ni} which corresponds to λ, ν , and $\Delta f_{\lambda\nu}$ the known change, then will $g_{ni\lambda\nu} \Delta f_{\lambda\nu}$ be the corresponding change of α_{ni} , and

$$g_{ni\lambda\nu} \Delta f_{\lambda\nu} \begin{cases} \cos n\theta \\ \sin n\theta \end{cases} \cos i\phi \begin{cases} i \text{ even} \\ i \text{ odd} \end{cases}$$

the change of the function $f(\theta, \phi)$.

In practical work the direct use of the table is not to be recommended, for, although the mode of calculation is indeed very clear, the number of products to be formed is great. It is possible also to supply a much simpler procedure, since the tabulated values of each α_{ni} are for the greater part zero, or equal, or equal and opposite.

We have now left a few of the different coefficients which are to be multiplied by constantly recurring combinations of $f\left(\mu_\lambda, \nu \frac{\pi}{p}\right)$. These latter are made up solely out of the sums and differences of the $f\left(\mu_\lambda, \nu \frac{\pi}{p}\right)$ without factors, and are quickly formed by calculation with the hand; all further working is best done with the machine.

In practice the calculator will mostly be concerned with developments up to the fourth order of spherical harmonics, and only in exceptional cases will be compelled to go as far as the sixth order. I here restrict myself therefore to the communication of the formulæ and numbers for the case $p=4$; their deduction may be left out, as it is quite simple.

In accordance with the theory, we must take as given values of the function the points of section of the meridian

$$\phi = 0^\circ, 45^\circ, 90^\circ, \dots \dots 315^\circ$$

with the parallels whose polar distances are

$$\begin{aligned} \theta_1 &= 154^\circ 58' 57'' \cdot 6 \\ \theta_2 &= 122^\circ 34' 46'' \cdot 2 \\ \theta_3 &= 90^\circ 0' 0'' \cdot 0 \\ \theta_4 &= 57^\circ 25' 13'' \cdot 8 \\ \theta_5 &= 25^\circ 1' 2'' \cdot 4, \end{aligned}$$

forty values in all.

I represent them shortly in the following way:

$$f(\theta_\lambda, \nu 45^\circ) = \lambda \nu$$

$$\lambda = 1, 2, 3, 4, 5$$

$$\nu = 0, 1, 2, 3, 4, 5, 6, 7$$

The coefficients α, β are then expressed by the interpolation formula

$$\begin{aligned} f(\theta, \phi) = & \alpha_{00} + \alpha_{10} \cos \theta + \alpha_{20} \cos 2\theta + \alpha_{30} \cos 3\theta + \alpha_{40} \cos 4\theta \\ & + (\alpha_{11} \sin \theta + \alpha_{21} \sin 2\theta + \alpha_{31} \sin 3\theta + \alpha_{41} \sin 4\theta) \cos \phi \\ & + (\beta_{11} \sin \theta + \beta_{21} \sin 2\theta + \beta_{31} \sin 3\theta + \beta_{41} \sin 4\theta) \sin \phi \\ & + (\alpha_{02} + \alpha_{12} \cos \theta + \alpha_{22} \cos 2\theta + \alpha_{32} \cos 3\theta + \alpha_{42} \cos 4\theta) \cos 2\phi \\ & + (\beta_{02} + \beta_{12} \cos \theta + \beta_{22} \cos 2\theta + \beta_{32} \cos 3\theta + \beta_{42} \sin 4\theta) \sin 2\phi \\ & + (\alpha_{13} \sin \theta + \alpha_{23} \sin 2\theta + \alpha_{33} \sin 3\theta + \alpha_{43} \sin 4\theta) \cos 3\phi \\ & + (\beta_{13} \sin \theta + \beta_{23} \sin 2\theta + \beta_{33} \sin 3\theta + \beta_{43} \sin 4\theta) \sin 3\phi \\ & + (\alpha_{04} + \alpha_{24} \cos 2\theta + \alpha_{44} \cos 4\theta) \cos 4\phi \end{aligned}$$

In the first place, all the combinations of the given values of the function are to be calculated. They are shown in Table A, being represented by the symbols

$$\begin{array}{ll} [1]_1, [2]_1, [3]_1 & \dots \dots \dots [15]_1 \\ [1]_2, [2]_2, [3]_2 & \dots \dots \dots [15]_2 \\ [1]_3, [2]_3, [3]_3 & \dots \dots \dots [15]_3 \end{array}$$

This table gives at the same time an appropriate scheme for carrying out the calculation. The first column contains the forty values of the function arranged in the most convenient order; the other columns explain themselves. It will be seen that numbers which are to be added or subtracted stand directly under one another. For these simple operations controls are hardly necessary, and are indeed furnished by the mode of their summation. The same process repeats itself constantly so as to become strongly impressed on the memory.

The α and β follow as sums of products, with constant factors, of the numbers just determined. These products are given in Table B. It is there evident that for the finding of the thirty coefficients α, β (six of which are immediately expressible in terms of the others), ninety-two products are necessary. Since these can be immediately formed and summed by means of the calculating machine, a further control other than is furnished in the usual way by the working of the machine is superfluous.

In the second table, for simplification of the numbers the first factors with their tenfold totals are set down; compensation is effected most simply by dividing the values of the function by ten before using them, whereby as a rule a desirable homogeneity in the whole set of numbers is brought about.

TABLE A.

Function.	First Sum.	Second Difference.	Successive Sums and Differences.		Successive Differences and Sums.	
10 14 12 16 50 54 52 56	10+14 12+16 50+54 52+56	10-14 12-16 50-54 52-56	$\begin{aligned} \{10+14\} \\ + \{12+16\} \end{aligned} = a_1$	$\begin{aligned} \{10+14\} \\ - \{12+16\} \end{aligned} = b_1$	$(10-14)=c_1$	$(12-16)=d_1$
11 15 13 17 51 55 53 57	11+15 13+17 51+55 53+57	11-15 13-17 51-55 53-57	$\begin{aligned} \{11+15\} \\ + \{13+17\} \end{aligned} = e_1$	$\begin{aligned} \{11+15\} \\ - \{13+17\} \end{aligned} = f_1$	$\begin{aligned} \{11-15\} \\ + \{13-17\} \end{aligned} = g_1$	$\begin{aligned} \{11-15\} \\ - \{13-17\} \end{aligned} = h_1$
20 24 22 26 40 44 42 46	20+24 22+26 40+44 42+46	20-24 22-26 40-44 42-46	$\begin{aligned} \{20+24\} \\ + \{22+26\} \end{aligned} = a_2$	$\begin{aligned} \{20+24\} \\ - \{22+26\} \end{aligned} = b_2$	$(20-24)=c_2$	$(22-26)=d_2$
21 25 23 27 41 45 43 47	21+25 23+27 41+45 43+47	21-25 23-27 41-45 43-47	$\begin{aligned} \{21+25\} \\ + \{23+27\} \end{aligned} = e_2$	$\begin{aligned} \{21+25\} \\ - \{23+27\} \end{aligned} = f_2$	$\begin{aligned} \{21-25\} \\ + \{23-27\} \end{aligned} = g_2$	$\begin{aligned} \{21-25\} \\ - \{23-27\} \end{aligned} = h_2$
30 34 32 36 31 35 33 37	30+34 32+36 31+35 33+37	30-34 32-36 31-35 33-37	$\begin{aligned} \{30+34\} \\ + \{32+36\} \end{aligned} = a_3$	$\begin{aligned} \{30+34\} \\ - \{32+36\} \end{aligned} = b_3$	$(30-34)=c_3$	$(32-36)=d_3$
			$\begin{aligned} \{31+35\} \\ + \{33+37\} \end{aligned} = e_3$	$\begin{aligned} \{31+35\} \\ - \{33+37\} \end{aligned} = f_3$	$\begin{aligned} \{31-35\} \\ + \{33-37\} \end{aligned} = g_3$	$\begin{aligned} \{31-35\} \\ - \{33-37\} \end{aligned} = h_3$

TABLE B.

(The first factors are set down with their tenfold totals.)

$\alpha_{00} = +0.3296[13]_1$ + 0.1444[13] ₂ + 0.3021[13] ₃	$\alpha_{10} = -0.5973[14]_1$ - 0.1555[14] ₂	$\alpha_{20} = +0.5087[13]_1$ - 0.3628[13] ₂ - 0.2917[13] ₃
$\alpha_{30} = -0.3246[14]_1$ + 0.5462[14] ₂	$\alpha_{40} = +0.1791[13]_1$ - 0.5072[13] ₂ + 0.6562[13] ₃	
$\alpha_{11} = +0.3155[3]_1 + 0.2231[11]_1$ + 0.8306[3] ₂ + 0.5873[11] ₂ + 0.8333[3] ₃ + 0.5893[11] ₃	$\beta_{11} = +0.3155[5]_1 + 0.2231[9]_1$ + 0.8306[5] ₂ + 0.5873[9] ₂ + 0.8333[5] ₃ + 0.5893[9] ₃	
$\alpha_{21} = -0.6449[4]_1 - 0.4560[12]_1$ - 0.9328[4] ₂ - 0.5887[12] ₂	$\beta_{21} = -0.6449[6]_1 - 0.4560[10]_1$ - 0.9328[6] ₂ - 0.5887[10] ₂	
$\alpha_{31} = +0.6382[3]_1 + 0.4513[11]_1$ + 0.3720[3] ₂ + 0.2630[11] ₂ - 1.1667[3] ₃ - 0.8250[11] ₃	$\beta_{31} = +0.6382[5]_1 + 0.4513[9]_1$ + 0.3720[5] ₂ + 0.2630[9] ₂ - 1.1667[5] ₃ - 0.8250[9] ₃	
$\alpha_{41} = -0.7675[4]_1 - 0.5427[12]_1$ + 0.6482[4] ₂ + 0.4584[12] ₂	$\beta_{41} = -0.7675[6]_1 - 0.5427[10]_1$ + 0.6482[6] ₂ + 0.4584[10] ₂	
$\alpha_{02} = +0.1823[1]_1$ + 0.6289[1] ₂ + 0.2917[1] ₃	$\beta_{02} = +0.1823[7]_1$ + 0.6289[7] ₂ + 0.2917[7] ₃	
$\alpha_{12} = -0.1575[2]_1$ - 0.7506[2] ₂	$\beta_{12} = -0.1575[8]_1$ - 0.7506[8] ₂	
$\alpha_{22} = +0.1272[1]_1$ - 0.0907[1] ₂ - 1.1667[1] ₃	$\beta_{22} = +0.1272[7]_1$ - 0.0907[7] ₂ - 1.1667[7] ₃	
$\alpha_{32} = -\alpha_{12}$ $\alpha_{42} = -(\alpha_{02} + \alpha_{22})$	$\beta_{32} = -\beta_{12}$ $\beta_{42} = -(\beta_{02} + \beta_{22})$	
$\alpha_{13} = +0.0368[3]_1 - 0.0260[11]_1$ + 0.5872[3] ₂ - 0.4152[11] ₂ + 1.1667[3] ₃ - 0.8250[11] ₃	$\beta_{13} = -0.0368[5]_1 + 0.0260[9]_1$ - 0.5872[5] ₂ + 0.4152[9] ₂ - 1.1667[5] ₃ + 0.8250[9] ₃	
$\alpha_{23} = -0.0997[4]_1 + 0.0706[12]_1$ - 0.9487[4] ₂ + 0.6710[12] ₂	$\beta_{23} = +0.0997[6]_1 - 0.0706[10]_1$ + 0.9487[6] ₂ - 0.6710[10] ₂	
$\alpha_{33} = -\frac{1}{3}\alpha_{13}$ $\alpha_{43} = -\frac{1}{2}\alpha_{23}$	$\beta_{33} = -\frac{1}{3}\beta_{13}$ $\beta_{43} = -\frac{1}{2}\beta_{23}$	
$\alpha_{04} = +0.0044[15]_1$ + 0.1392[15] ₂ + 0.3281[15] ₃	$\alpha_{24} = -\frac{4}{3} \cdot \alpha_{04}$ $\alpha_{44} = +\frac{1}{3} \cdot \alpha_{04}$	

(Issued separately March 16, 1915.)