V.—The "Geometria Organica" of Colin Maclaurin: A Historical and Critical Survey.

By Charles Tweedie, M.A., B.Sc., Lecturer in Mathematics, Edinburgh University.

(MS. received October 15, 1915. Read December 6, 1915.)

Introduction.

Colin Maclaurin, the celebrated mathematician, was born in 1698 at Kilmodan in Argyllshire, where his father was minister of the parish. In 1709 he entered Glasgow University, where his mathematical talent rapidly developed under the fostering care of Professor Robert Simson. In 1717 he successfully competed for the Chair of Mathematics in the Marischal College of Aberdeen University. In 1719 he came directly under the personal influence of Newton, when on a visit to London, bearing with him the manuscript of the *Geometria Organica*, published in quarto in 1720. The publication of this work immediately brought him into prominence in the scientific world. In 1725 he was, on the recommendation of Newton, elected to the Chair of Mathematics in Edinburgh University, which he occupied until his death in 1746.

As a lecturer Maclaurin was a conspicuous success. He took great pains to make his subject as clear and attractive as possible, so much so that he made mathematics "a fashionable study." The labour of teaching his numerous students seriously curtailed the time he could spare for original research. In quantity his works do not bulk largely, but what he did produce was, in the main, of superlative quality, presented clearly and concisely. The *Geometria Organica* and the Geometrical Appendix to his *Treatise on Algebra* give him a place in the first rank of great geometers, forming as they do the basis of the theory of the Higher Plane Curves; while his *Treatise of Fluxions* (1742) furnished an unassailable bulwark and text-book for the study of the Calculus.

In a sense he may be regarded as a founder of the Royal Society of Edinburgh, for it was at his instigation that a Medical Society in Edinburgh was encouraged to broaden its field of research and develop into the Philosophical Society, which gave rise in its turn to the Royal Society of Edinburgh in 1783.

§ 1. During comparatively recent years the study of geometrical science has been enriched by a number of publications dealing with the history of particular curves, and the general development of the theory. Prominent among such works may be mentioned Loria's *Ebene Kurven*, Wieleitner's *Spezielle Kurven*, in German; and Teixeira's *Courbes algebriques*, in French or Spanish. These works, compiled with great care, are indispensable to the geometer in the study of his subject, but a perusal of the early rare treatise of Maclaurin on the *Geometria Organica* reveals the fact that the claims of the latter geometer have frequently been entirely overlooked.

For example, Teixeira himself, in a note on the Researches of Maclaurin on Circular Cubics (*Proc. Edinburgh Math. Soc.*, 1912), points out that many of the classic properties connected with these curves are due to Maclaurin, although his name does not even appear in the list of writings on the Strophoid published by Tortolini and Günther.

Again, the whole theory of Pedals, and more particularly the Pedals of the Conic Section, is given in the *Geometria Organica*—a theory to be rediscovered and named in the nineteenth century, more than a hundred years after the publication of Maclaurin's work. In this connection it may be pointed out that Maclaurin invented the term, the Radial Equation of a Curve (for its $p-r$ equation), long before the term Radial came to be applied to another purpose by Tucker.

These two examples sufficiently illustrate my contention that Maclaurin's treatise has been strangely overlooked. It is the business of the present note to indicate others, to point out how fully he has in many cases anticipated writers of comparatively recent times, and to vindicate his claims to a far more careful consideration than has of late been the fashion. It may here be remarked that Poncelet in his magistral *Traité* gives full credit to the importance of Maclaurin's two geometrical treatises, the *Geometria Organica* and the *Proprietates Linearum Curvarum*, published as an appendix to his posthumous *Treatise on Algebra*. In fact, the French school generally does more justice to the Scottish geometers of the eighteenth century than do English writers in the sister kingdom.

§ 2. The *Geometria Organica* is the first great treatise of Maclaurin, and appeared in London, in 1720, under the royal imprimatur (1719) of Newton, to whom the work is dedicated. At the time the youthful Maclaurin (for he was only twenty-one years of age) held the Chair of Mathematics in the New College * in Aberdeen. The work expands and develops two earlier memoirs published in the *Philosophical Transactions of the Royal Society*:—

(i) *Tractatus de Curvarum Constructione et Mensura, etc.*, 1718, giving the Theory of Pedals: (ii) *Nova Methodus Universalis Curvas Omnes*

* *i.e.* Marischal College.

cujus-cumque Ordinis Mechanice describendi sola datorum Angulorum et Rectarum Ope, 1719.

Maclaurin's imagination had been fired by Newton's classic *Enumeratio Linearum Curvarum Tertii Ordinis*, and by the organic description of the Conic given in the *Principia*; and in his attempt to generalise the latter so as to obtain curves of all possible degrees by a mechanical description he was led to write the *Geometria Organica*.

It will appear in the sequel how remarkably successful he was in obtaining nearly all the particular curves known in his time (which he is careful to ascribe to their inventors), besides a whole host of new curves never before discussed, and which have since been named and investigated with but scant acknowledgment of their true inventor. His method, however, does not furnish all curves, though it may furnish curves of all degrees; and it will be the business of this note to point out some of the limitations of the method applied, as well as the rare mistakes Maclaurin makes regarding the double points of the curves investigated,—a weakness more pronounced in the earlier memoirs.

In establishing his theorems he frequently employs the method of analysis furnished by the Cartesian geometry. The Cartesian geometry was then in its infancy, and Maclaurin's use of it seems to us nowadays somewhat cumbersome and certainly tedious. But when Maclaurin reasons "more veterum," he handles geometry with consummate skill; and the impression gains upon the reader that, however imperishable his reputation in analysis may be, he was greater as a geometer than as an analyst. He occasionally makes petty errors in his analytical demonstrations which somewhat mar the interest in his work, but the beauty of his synthetic reasoning is untarnished by any such blemish.

In any analysis that follows, the demonstrations he gives are frequently replaced by others that are more in touch with modern methods, but this does not apply to the geometrical reasoning proper, which is as fresh to-day as when written. The treatise is divided into two parts. In the first part the only loci admitted are straight lines along which the vertices of constant angles are made to move. In the second part the curves so found in the first part are added to the loci to obtain curves of higher order. It contains, in particular, the theory of pedals and the epicycloidal generation of curves by rolling one curve upon a congruent curve. A section is devoted to the application to mechanics; and the last section contains some general theorems in curves forming the foundation of the theory of Higher Plane Curves. It also contains what is erroneously termed Cramer's Paradox, the paradox being really Maclaurin's, for

Cramer in his *Courbes algébriques* expressly quotes the *Geometria Organica* as his authority.

For the sake of brevity I have, in what follows, restricted my attention to what would seem of modern interest, and have on this account omitted entirely the discussion of asymptotes to curves and the exhaustive enumeration of cubic curves as based on Newton's work. The nomenclature is also modern, save where the curves were already familiar to mathematicians in Maclaurin's day. Maclaurin rarely attempts to give names to the hosts of new curves generated by his methods. A remarkable feature of interest lies in the fact that many of the methods employed only require an obvious generalisation to furnish standard methods of generating unicursal cubics and quartics supposed to have been invented during the latter half of the nineteenth century. It will be my special object to indicate these at the proper time and place. In order to emphasise Maclaurin's own work I have followed the order of his Propositions, and the numbers attached to these are taken from the *Geometria Organica*.

It has been found necessary, however, to use a more convenient notation for the figures.

Universal Description of Geometrical Lines.

Part I.

Wherein, by a Universal Method, Lines of all Orders are described by the sole use of constant given Angles and Straight Lines.

SECTION I.

The Conic.

§ 3. This section gives an analytical demonstration of Newton's Organic Description of Conics (*Principia,* Bk. I; and *Arithmetica Universalis*). It is the generalisation of this method that gives rise to Maclaurin's treatise.

Prop. I.

O and O′ are fixed points: $\angle$ POQ $=\alpha$, and $\angle$ PO′Q $=\beta$, two angles of constant magnitude that can be rotated round O and O′ respectively. If the intersection P of OP and O′P is conducted along a straight line l, the point Q in general traces out a conic section through O and O′.

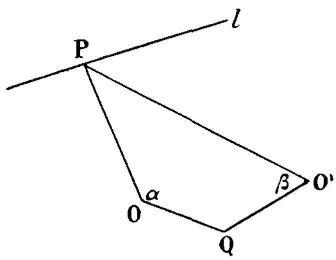

Fig. 1.

To get the conic, therefore, one straight-line locus and two given angles are required. In modern terms, if OP and O′P are in perspective correspondence, Q generates a conic.

For any point P on l may be supposed to have the co-ordinates

$$x = at + b$$
$$y = ct + d$$

where t is a variable parameter: and the ordinary calculations give the equation to OQ in the form

$$L_1 + tL_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and to O′Q in the form

$$M_1 + tM_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

so that the locus of Q is given by

$$L_1M_2 - L_2M_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and is therefore a conic through O

$$(L_1 = 0\,;\ L_2 = 0),$$

and through O′

$$(M_1 = 0\,;\ M_2 = 0).$$

Cor. 2. By assuming the converse theorem (proved later) Maclaurin deduces that if P, instead of lying on a straight line, moves on a conic through O and O′, Q still generates a conic through O and O′.

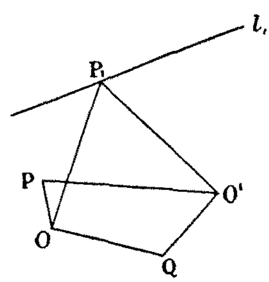

Fig. 2.

Dem.

For a straight line l_1 can then be found, and a point P_1 moving on it, so that

$$\angle P_1OP = \alpha'$$
$$\angle P_1O'P = \beta'$$

are constant angles.

Hence P_1OQ and $P_1O'Q$ are constant angles; and so, when P_1 traces out l_1, Q generates a conic through O and O′. (There is, in fact, a $1-1$ correspondence between OP and O′P, and ∴ between OQ and O′Q. ∴ etc.)

§ 4. *Prop. II*

determines the species and asymptotes of the conic.

On OO′ describe a segment of a circle OKO′ containing an angle γ so that $\alpha+\beta+\gamma$ = a multiple of two right angles. Let it cut l in A and B. When P coincides with either A or B the angle at Q in POQO′ is zero, *i.e.* Q is at infinity on the curve, and OQ (or O′Q) is parallel to an asymptote. The angle AOB (=AO′B) measures the angle between the asymptotes.

The conic is a hyperbola, a parabola, or an ellipse, according as A and B are real and distinct, coincident, or imaginary.

Cor. 4. The curve cannot be a circle when l is not the line at infinity.

Cor. 6. When $\alpha+\beta=\pi$ the curve is a hyperbola in general, but a parabola when l is parallel to OO′.

Cor. 7. If l passes through the centre of the circle OKO′, the angle AOB is a right angle and the hyperbola is equilateral.

§ 5. *Prop. III.*

But if for any position of P on l, OQ and O′Q coincide simultaneously with OO′, the conic degenerates into a straight line (along with OO′).

The proof given is analytical.

Cor. 2. This may happen, for example, when $\alpha+\beta=\pi$, and l is inclined at angle α to OO′ (viz. when P is at infinity on l).

Cor. 3. In particular this is so when $\alpha=\beta=\pi/2$, and l perpendicular to OO′, when Q is on another line perpendicular to OO′, which is the image of l in the mid-point of OO′.

Cor. 4 contains an important statement.

Find I on l such that $\angle$ OO′I $=\beta$, and let IOO′ $=\alpha'$.

Let P trace out l, and let Q′ be taken in quadrilateral POQ′O′ for angles α' and β. Then Q′ traces out a straight line. The angle QOQ′ $=\alpha'-\alpha$ and is therefore constant, and O′, Q, Q′ are collinear. Hence we

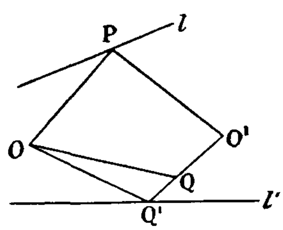

FIG. 3.

may trace the conic locus of Q by making Q′ lie in l' and taking QOQ′ $=\alpha'-\alpha$; *i.e.* in the preceding constructions we may replace an angle by a straight line rotating round one of the fixed points.

§ 6. *Prop. IV*

proves the converse of Prop. I by solving the problem :—*To describe a conic through five given points.*

Let the points be A, B, C, D, E. Form ΔCAB, and let $\angle$ CAB $=\alpha$, $\angle$ CBA $=\beta$. Rotate angles α and β round A and B respectively, and let the intersection of two arms be in D and then in E, while the intersection of the other arms comes to be at D′ and E′.

Let the line D′E′ be taken for l; then if P traces out l, Q generates a conic which passes through D and E and also through C, A, and B, *i.e.* Q generates the conic through the five given points.

If four points only are given, an infinity of conics can be described through them. Thus there are two parabolas through the four points, or two hyperbolas whose asymptotes intersect at a given angle. For example, let the parabolas through A, B, C, D be sought. Proceed as before and find D′. On AB describe a segment of a circle containing an angle γ such that

$$\alpha + \beta + \gamma = \pi \text{ (or } 2\pi).$$

Either tangent from D′ to this circle will furnish the line l for the parabola.

"The method employed will furnish the complete system of conic sections which were the objects of research of the older geometers. Newton was the first to attack the problem to enumerate and classify Curves of the Third Order, and thereby added a fresh triumph to his genius. We now proceed to delineate curves of this order."

§ 7. *Newton's Organic Description as a Cremona Transformation.*

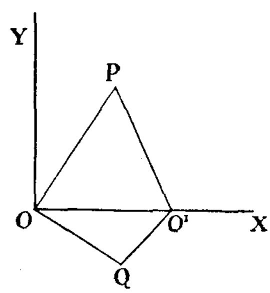

Fig. 4.

Let O be the origin, O′ the point $(a, 0)$, P any point (ξ, η). Then O′P is given by

$$y = \frac{\eta}{\xi - a}(x - a) = \mu(x - a), \text{ say} \quad . \quad . \quad . \quad . \quad (1)$$

and O′Q is given by

$$y = m(x - a)$$

where

$$\frac{m - \mu}{1 = m\mu} = \tan\beta,$$

i.e.

$$m = (\mu + \tan\beta)/(1 - \mu\tan\beta)$$

$$= \frac{\eta + (\xi - a)\tan\beta}{\xi - a - \eta\tan\beta} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

so that O′Q has the equation

$$y(\xi - a - \eta \tan\beta) = (x-a)(\eta + \overline{\xi - a}\tan\beta)$$

or

$$\xi(y - \overline{x-a}\tan\beta) - \eta(y\tan\beta + x - a) - a(y - \overline{x-a}\tan\beta) = 0 \quad . \quad . \quad (3)$$

Also OQ has the equation

$$\xi(y - x\tan\alpha) - \eta(y\tan\alpha + x) = 0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

So that if P traces out the line

$$A\xi + B\eta + C = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Q traces out the conic

$$\begin{vmatrix} A & -B & C \\ y-(x-a)\tan\beta & y\tan\beta + x - a & -a(y-\overline{x-a})\tan\beta \\ y - x\tan\alpha & y\tan\alpha + x & 0 \end{vmatrix} = 0 \quad . \quad (6)$$

passing through the fixed points (0, 0), $(a, 0)$; and

$$\left(-\frac{a\tan\beta}{\tan\alpha - \tan\beta},\ -\frac{a\tan\alpha\tan\beta}{\tan\alpha - \tan\beta}\right).$$

Denote the last point by O″.

The three points are the singular points of the transformation, and are as in the figure.

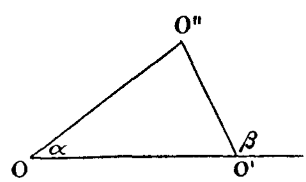

FIG. 5.

When $\alpha = \beta$, O″ is at infinity. The curve cannot be an ellipse, and is a parabola when l is parallel to OO′. When l passes through one of the points O, O′, O″, the conic reduces to a straight line.

SECTION II.

DESCRIPTION OF LINES OF THE THIRD ORDER HAVING A DOUBLE POINT.

§ 8. Maclaurin's researches on these curves will well stand comparison with the modern theory of these curves, which he may fairly be described as anticipating.

A cubic possessing a double point is a unicursal or rational curve, whose freedom equations may be written in the form

$$\left.\begin{aligned} x &= A(t)/C(t) \\ y &= B(t)/C(t) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where A, B, C are integral cubic functions of t.

They may also be considered as generated by the point common to the two straight lines

$$\left.\begin{aligned} L_0 + tL_1 &= 0 \\ M_0 + tM_1 + t^2M_2 &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $L_1 \ldots M_2$ are linear functions of x and y. (*Vide, e.g.*, Tweedie, "Courbes Unicursales," *L'Enseignement Mathématique*, 1912.)

The equation

$$L_0 + tL_1 = 0$$

represents a pencil of lines.

The equation

$$M_0 + tM_1 + t^2M_2 = 0$$

represents a system of straight lines whose envelope is the conic

$$M_1^2 - 4M_0M_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the corresponding rays of the pencil and the tangents to the conic are in projective correspondence.

§ 9. In this and the next section Maclaurin makes frequent use of a constant angle OPQ, where O is a fixed point while P is any point on a line l.

In such a case PQ envelops a parabola. For let OP in any position be given by the equation

$$y - tx = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The co-ordinates of P on l are then of the form

$$\left(\frac{at+b}{ct+d}, \frac{pt+q}{ct+d}\right).$$

The gradient of PQ is also rational and linear in t, so that PQ has an equation of the form

$$L_0 + 2tL_1 + t^2L_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

whose envelope is the conic

$$L_1^2 - L_0L_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in this case a parabola, since, when P is at infinity on l, PQ lies entirely at infinity.

§ 10. *Prop. V.*

O_1 and O_2 are two fixed points. The vertex P of the constant angle $O_1PQ(=\alpha)$ lies on a given line l. QO_2R is an angle of constant

magnitude β. If O_1P and O_2R intersect in R on a line l′, then the point Q traces out a cubic having a double point at O_2.

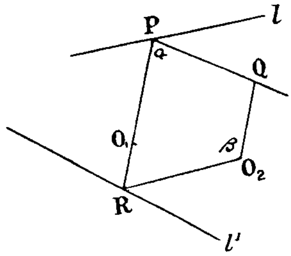

FIG. 6.

Let O_1 be chosen as origin, and let RP have the equation

$$y = tx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Then PQ has an equation of the form

$$L_0 + tL_1 + t^2L_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

while O_2R and therefore O_2Q has an equation of the form

$$M_0 + tM_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The elimination of t from (2) and (3) leads to a cubic with a double point at O_2. Also O_2Q and O_1P cut in a conic. *Cf.* § 18.

A geometrical construction for the tangents at the double point is also given.

[It is at once obvious that Maclaurin's generation of a singular cubic is the simplest case of the standard generation of these curves, which may be stated geometrically as follows:—

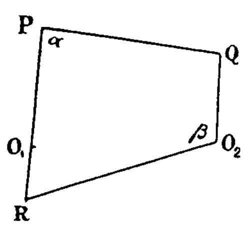

FIG. 7.

In the quadrilateral $RPQO_2$ the angles at P and O_2 are constant, and O_1 and O_2 are fixed points. Let R be on a conic that passes through O_1 and O_2, or on a straight line. Let PQ be constantly tangent to a conic whose focus is at O_1. Then P lies upon a circle (or a straight line, if the conic is a parabola). See the Theory of Pedals in Part II.

There is thus a projective correspondence between the ray O_2Q and the tangent PQ to the conic, and Q generates the singular cubic. For the present he is restricted to the use of straight lines as loci, and of these he uses two.]

§ 11. *Prop. VI*

shows how to determine the asymptotes, and also the species of the cubic, according to Newton's classification of cubics.

The next theorem is Lemma I.

If O is a fixed point, P any point on a given straight line, and OPQ a triangle of given species, then the locus of Q is a straight line.

We need not add the proof.

§ 12. *Prop. VII.*

All the cubics of Prop. V may be obtained by taking $\angle QO_2R = \pi$.

Find K on l' such that $O_1O_2K = \beta$. Let $KO_1O_2 = \gamma$. Draw O_1T so that $PO_1T = \pi - \gamma$, and let it meet QP in T and QO_2 in S. Then, by the lemma, when P moves on l, T generates a straight line, while S also generates a straight line l'' (by Prop. III).

We may thus obtain the locus of Q from the constant angle STQ and the intersection of O_2S with QT.

Cor. 1. Either α or β may be replaced by a right angle or by an angle of any given magnitude.

This is easily deduced by starting from TQS.

The remaining cor. discuss the asymptotes and a variety of particular cases.

E.g. Maclaurin notes that when O_2 goes to infinity, the pencil of lines becomes a system of parallel lines. Special cases arise when l and l' are parallel, or when the rays are inclined to l' at angle α.

§ 13. *Prop. VIII*

considers the reduction of the equation of the cubic to a standard Newtonian form.

Some particular sub-cases are given.

Ex. 1.

Let l and l' be parallel and perpendicular to O_1O_2, and let $O_1PQ = \frac{\pi}{2}$.

Choose the origin at O_2, and let A, B, O_1 be the points (a, o), (b, o), (d, o).

If the equation to O_2R is

$$y - tx = 0 \qquad (1)$$

O_1RP is given by

$$y = \frac{bt}{b-d}(x-d) ;$$

and PQ by

$$y - bt\frac{a-d}{b-d} + \frac{b-d}{bt}(x-a) = 0 . \qquad (2)$$

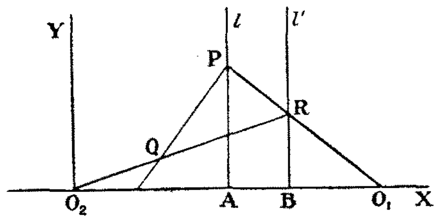

Fig. 8.

The locus of Q is therefore given by

$$xy^2 - y^2b\frac{a-d}{b-d} + \frac{b-d}{b}(x^3 - ax^2) = 0 \qquad (3)$$

In particular:

If l passes through O_2, so that $a = 0$, (3) reduces to the form

$$y^2 = \frac{Kx^3}{(x-d')}.$$

If O_2 is the foot of the perpendicular from O_1 on l, and l' is the line at infinity, the curve is the Cissoid of Diocles.

[The Trisectrix of Maclaurin is also the particular case when l' is the line at infinity, l perpendicular to O_1O_2, and cutting O_1O_2 in A, so that

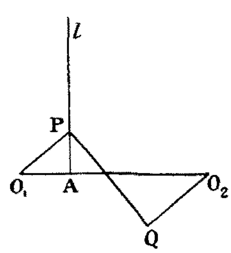

Fig. 9.

$O_1A = \frac{1}{3}AO_2$; but is not here explicitly quoted, occurring with another definition in the *Fluxions*.]

Case X.

Let l be parallel to O_1O_2, l' perpendicular to O_1O_2; $O_1PQ = \frac{\pi}{2}$.

Let $O_2A = a$; $O_2B = b$; $O_2O_1 = d$.

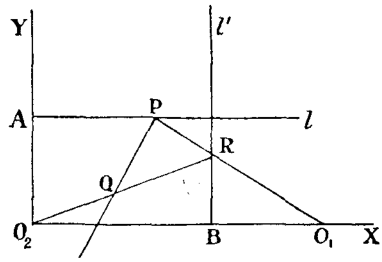

FIG. 10.

Then the equation to the locus of Q is

$$y^3 - ay^2 + \frac{b-d}{b}\left(x^2y - dxy - a\frac{b-d}{b}x^2\right) = 0.$$

Case XVIII.

Let $O_1PQ = QO_2R = \frac{\pi}{2}$ (in Prop. V); l parallel to O_1O_2, l' perpendicular to O_1O_2.

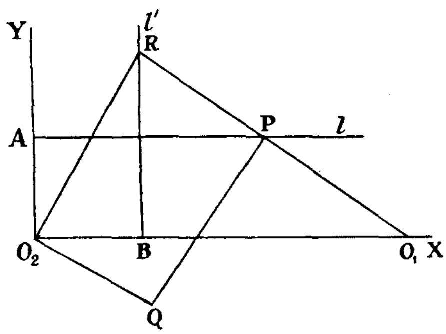

FIG. 11.

If $y = tx$ is the equation to O_2R, R is the point (b, tb), O_1PR has the equation

$$y = bt(x-d)/(b-d) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and P is the point

$$\left(d + \frac{a(b-d)}{bt}, a\right).$$

Hence PQ has the equation

$$y - a = \frac{d-b}{bt}\left(x - d - \frac{a(b-d)}{bt}\right) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

while O_2Q is given by

$$ty + x = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The equation to the locus of Q is therefore

$$dx^2y - abx^2 + d(b-d)xy = a\frac{(b-d)^2}{b}y^2 \quad . \quad . \quad . \quad (4)$$

Case XXI.

l and l' both perpendicular to O_1O_2.

Locus of Q,

$$\frac{d}{b}xy^2 + \frac{b(a-d)}{b-d}x^2 + a\frac{b-d}{b}y^2 = 0.$$

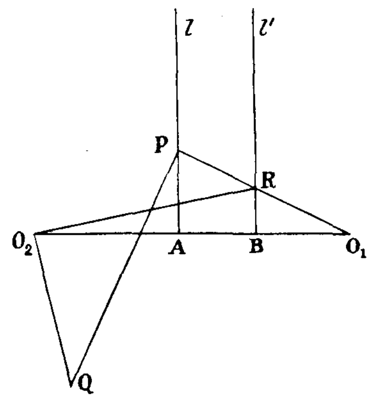

Fig. 12.

Case XXII.

l and l' parallel to O_1O_2; l midway between l' and O_1O_2.

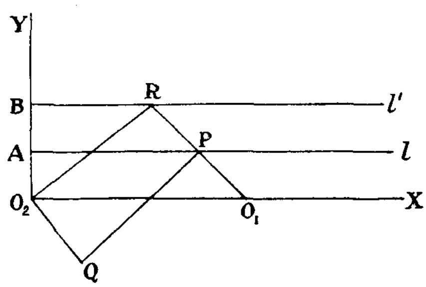

Fig. 13.

Then

$$O_2A = a,$$
$$O_2B = 2a.$$

Let the equation to O_2R be

$$ty = x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Then R is the point $(2at, 2a)$.

The equation to O_1R is

$$\frac{y}{2a} = \frac{x-d}{2at-d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

so that P is the point

$$\left(\frac{2at+d}{2}, a\right).$$

∴ the equation to PQ is

$$y - a = \frac{d-2at}{2a}\left(x - \frac{2at+d}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the equation to the locus of Q is

$$4a^2y^2 = (d^2 - 4a^2)x^2 - 2dx^3 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In particular, when $d = \pm 2a$ (4) becomes

$$y^2 = \mp \frac{1}{a}x^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which is Neil's parabola.

§ 14. In XVII the remark occurs : "*Curvas Omnes pure Hyperbolicas tertii Ordinis quæ punctum duplex habent ad distantiam finitam descripsimus. Restant Curvæ Hyperbolo-Parabolicæ et pure Parabolicæ quarum Descriptiones facillimæ ex methodo ipsius Prop. V deduci possunt.*"

[We proceed to discuss Maclaurin's claim to have found a method for generating all rational cubics, by showing that his method is the simplest for obtaining the standard generation of these curves as given by

$$L_0 + tL_1 + t^2L_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$M_0 + tM_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Maclaurin proves in Part II that the pedal of a conic when the pole is at the focus is a circle for the central conic, and a straight line for the parabola.

The converse also holds, and the analysis shows that the pencil of perpendiculars through the focus is in projective correspondence with the tangents to the conic, *i.e.* corresponding ray and tangent have equations of the form

$$\left.\begin{array}{r} M_0 + tM_1 = 0 \\ L_0 + tL_1 + t^2L_2 = 0 \end{array}\right\}$$

though not the most general of this kind.

Let O_1 be the focus, and let O_2 be another point the rays through which are in 1–1 correspondence with the rays through O_1, so that corresponding rays intersect in R on a conic passing through O_1 and O_2. This latter conic may be replaced by a straight line without loss of generalisation. For let T be any point on it. Let

$$\angle TO_1O_2 = \alpha'$$

$$\angle TO_2O_1 = \beta'.$$

Make

$$\angle \mathrm{RO_1S} = \alpha'$$
$$\angle \mathrm{RO_2S} = \beta'$$

and let R move on the conic. Then S generates a straight line.

Let O_1S cut PQ in P′. Since PO_1P' is constant, and P lies on a straight line, the locus of P′ is, by the lemma, likewise a straight line; and the angle SO_2Q is constant. Hence we obtain a reduction to Prop. V as for the quadrilateral $P'SO_2Q$; and thence to Prop. VII. We may therefore assume that R lies on a straight line, and P on a straight line or circle. It remains to prove that the locus of P may without loss of generality be taken to be a straight line in general, so that we obtain a reduction to Maclaurin's generation of the singular cubic.

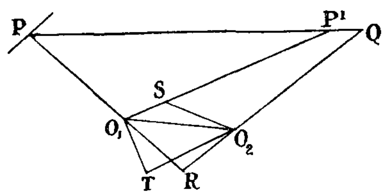

Fig. 14.

Let the cubic be given by the intersection of

$$a_0x + b_0y + c_0 + t(a_1x + b_1y + c_1) + t^2(a_2x + b_2y + c_2) = 0$$

or

$$L_0 + tL_1 + t^2L_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$m_0x + n_0y + p_0 + t(m_1x + n_1y + p_1) = 0$$

or

$$M_0 + tM_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These may be replaced by

$$L_0 + tL_1 + t^2L_2 + (At + B)(M_0 + tM_1) = 0 \quad . \quad . \quad . \quad . \quad (3)$$

and

$$M_0 + tM_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in which A and B are arbitrary.

The equation (3) will envelop a parabola provided a value of t can be found for which (3) is the line at infinity, *i.e.* so that

$$a_0 + ta_1 + t^2a_2 + (At + B)(m_0 + tm_1) = 0 \quad . \quad . \quad . \quad . \quad (5)$$
$$b_0 + tb_1 + t^2b_2 + (At + B)(n_0 + tn_1) = 0 \quad . \quad . \quad . \quad . \quad (6)$$

Hence t must be such that

$$\frac{a_0 + ta_1 + t^2a_2}{b_0 + tb_1 + t^2b_2} = \frac{m_0 + tm_1}{n_0 + tn_1} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This equation leads in general to a cubic in t, with at least one real root; A and B may then be chosen in an infinity of ways.

When $\frac{a_0}{b_0}=\frac{m_0}{n_0}$ one real root is $t=0$; and when $\frac{a_2}{b_2}=\frac{m_1}{n_1}$ a real root is $t=\infty$.

If the numerator and denominator of the left side of (7) have a common factor $t-a$, then, for $t=a$, (1) is already the line at infinity, and its envelope is a parabola.

The case in which the numerators of (7) or the denominators of (7) have a common factor presents no difficulty.

When the numerator and denominator of the right side of (7) have a common factor, the pencil of lines consists of a system of parallel lines with the vertex at infinity.

In such a case a change of parameter and change of axes will enable us to write (1) and (2) as

$$L_0+L_1t+L_2t^2=0 \qquad (8)$$

$$x+t=0 \qquad (9)$$

and the equation (7) as

$$\frac{a_0+a_1t+a_2t^2}{b_0+b_1t+b_2t^2}=\frac{1}{0}.$$

Hence

$$b_0+b_1t+b_2t^2=0 \qquad (10)$$

so that t and therefore A and B may not be real or may be real. Thus, when the double point is at infinity, the parabolic envelope may not or may be real.

In any case, the analysis leads to the conclusion that, when the double point of the cubic is a finite point, Maclaurin's method will furnish a real means of generating it.]

§ 15. *Prop. IX.*

If, when PQ passes through O_2, QO_2 at the same time coincides with O_2P, the locus degenerates into this line and a conic.

Cor. 1. This furnishes a means of describing a conic when it is to pass through one only of the two points O_1, O_2.

§ 16. *Prop. X.*

If O_1PQ and QO_2R are as before, but P and Q are restricted to lie on l and l' respectively, the locus of R is a cubic possessing a double point at O_1.

Let the equation to O_1P be

$$y = tx \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

so that the equation to PQ is of the form

$$L_0 + L_1 t + L_2 t^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let QO_2 have an equation

$$M_0 + \mu M_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

so that O_2R has an equation

$$N_0 + \mu N_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

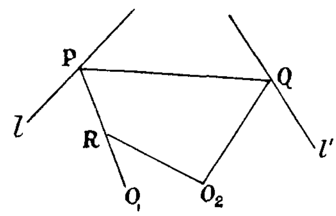

FIG. 15.

The condition that (2), (3), and l' are concurrent leads to a relation

$$f(t, \mu) = 0,$$

of the second degree in t and linear in μ.

$$\therefore \mu = \frac{at^2 + bt + c}{lt^2 + mt + n} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Eliminate t and μ from (1), (4), and (5), when the result follows.

Cor. 3. If S is taken on PQ such that $\angle PO_1S$ is constant, then S by the Lemma describes a straight line and $\angle O_1SQ$ is constant. Hence another way of obtaining such a cubic by using S in place of P, and the intersection of O_1S with O_2R.

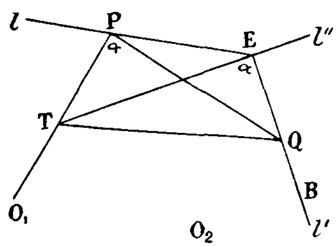

FIG. 16.

Cor. 4. Also thus: Let l and l' cut in E. Draw ET (l'') making an angle $\alpha = O_1PQ$ with l', and cutting O_1P in T. Then $QTO_1 = QEP$ is constant.

Hence we may replace P by T and l by l''.

Cor. 5. If $O_1EB = O_1PQ$ the locus is a conic and not a cubic. For in such a case O_1PEQ is a cyclic quadrilateral and QO_1P is constant, being the supplement of PEQ, so that the locus is that of Prop. I.

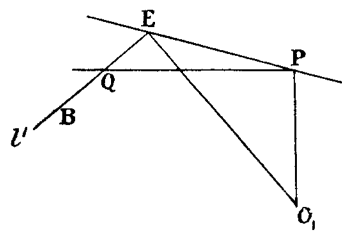

Fig. 17.

§ 17. *Prop. XI.*

If, in Prop. X, O_1P and O_2R simultaneously coincide with O_1O_2, then the curve degenerates into a conic.

Cor. 1. Thus, if

$$O_2AB = \alpha$$
$$AO_2B = \beta,$$

where B is on l', the locus is a conic.

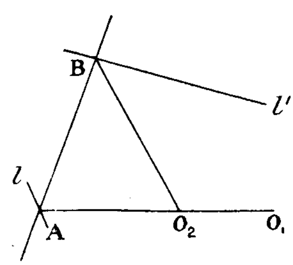

Fig 18.

Cor. 2. In particular, if $\alpha + \beta = \pi$, and l' parallel to AB, the curve is a conic, *e.g.* when $\alpha = \beta = \frac{\pi}{2}$, and $l' \perp^r O_1O_2$.

The Circular Cubic with a Double Point.

§ 18. *Lemma II.*

This lemma, along with the corollaries attached to it by Maclaurin, contains a variety of ways of tracing an important species of cubics to which Teixeira has recently drawn attention (*Proc. Ed. Math. Society*, 1912).

O_1 and O_2 are two fixed points, P any point on a fixed line l. If $O_1PN = \alpha$ is constant and Q is taken on PN so that O_2QN is constant, $= \beta$, then the locus of Q is a cubic.

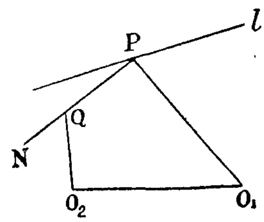

Fig. 19.

[We may note that, if O_1P and O_2Q cut in R, the locus of R is a segment of a circle on O_1O_2. Hence another method of generating the curve.

Of course, Maclaurin is restricted to the use of linear loci only.]

Maclaurin first shows that we may, without loss of generality, suppose $\alpha = \beta = \frac{\pi}{2}$, so that O_1P and O_2Q cut on the line at infinity, and the lemma is a particular case of Prop. VII.

For draw $O_1B \perp^r O_1O_2$ as in fig. 20, and make $O_1O_2B = \frac{\pi}{2} - \beta$, so that B is a fixed point.

Draw O_1R parallel to O_2Q, meeting PQ in R, so that, by Lemma I, R generates a straight line. Draw RS parallel to O_1B, and QS perpendicular

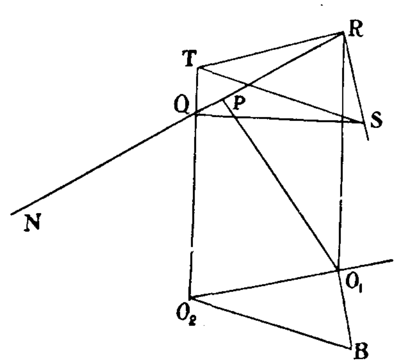

Fig. 20.

to O_2Q; also RT parallel to O_1O_2, cutting O_2Q in T. Then SRTQ is cyclic, and $RTS = RQS = \frac{\pi}{2} - \beta = O_1O_2B$.

But $RT = O_1O_2$. Hence $\Delta\, RTS \simeq \Delta\, O_1O_2B$ and $RS = O_1B$ is constant.

Therefore S generates a straight line l', and BS, parallel to O_2T, is perpendicular to SQ.

But B is fixed, S lies on l', $BSQ = O_2QS = \frac{\pi}{2}$; $\therefore$ etc.

[Since PN in the original construction is always tangent to a parabola, the constant angle β shows that the locus of Q is simply the oblique pedal of a parabola and falls to be discussed in Part II as a pedal.]

We now assume

$$\alpha = \beta = \frac{\pi}{2}.$$

Maclaurin notes when O_2 is a node, a conjugate point, or a cusp. When l passes through O_2 and is $\perp^r$ O_1O_2 the curve is the cissoid.

§ 19. Equation to the Curve.

Choose the origin at O_2.

Let O_1 be the point (a, b); and let the equation to l be

$$y = mx + n \qquad \qquad (1)$$

Let P be the point

$$(\xi, m\xi + n),$$

so that the gradient of O_1P is

$$\frac{m\xi + n - b}{\xi - a},$$

and PQ has the equation

$$y - m\xi - n = \frac{a - \xi}{m\xi + n - b}(x - \xi) \qquad \qquad (2)$$

while O_2Q has the equation

$$y = \frac{m\xi + n - b}{\xi - a}x \qquad \qquad (3)$$

To obtain the locus of Q, eliminate ξ between (2) and (3).

$$\therefore (y - mx)(x^2 + y^2) + x^2(b - n) + xy(bm - a) - y^2(am + n) = 0 \qquad (4)$$

Now any circular cubic with double point at O may be written as

$$(y - \lambda x)(x^2 + y^2) + Ax^2 + Bxy + Cy^2 = 0 \qquad \qquad (5)$$

If (4) and (5) represent the same curve, we must have

$$\left.\begin{aligned} m &= \lambda \\ b - n &= A \\ bm - a &= B \\ -am - n &= C \end{aligned}\right\} \qquad \qquad (6)$$

These determine m, n, a, b uniquely, corresponding to any equation (5). Maclaurin's generation therefore furnishes all the circular cubics.

To the lemma Maclaurin attaches several corollaries of special interest.

Cor. 1. Cissoidal Generation of the Curve.

Draw PT parallel to O_1O_2 cutting O_2Q in T. Describe the semicircle O_1RO_2. Then PO_1RQ is a rectangle, and PO_1O_2T is a parallelogram.

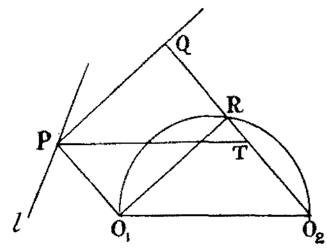

Fig. 21.

Hence

$$QR = O_1P = O_2T,$$

so that

$$TQ = O_2R.$$

Now T traces out a line l'.

Thus, to get the locus of Q, take T any point on l' and let the circle determine the chord O_2R on O_2T. Produce O_2T to Q so that $TQ = O_2R$.

Cor. 2. The same results obtain if on O_1O_2 is described a segment of a circle instead of a semicircle.

Cor. 3 gives another method of generating the curve.

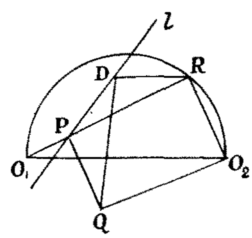

Fig. 22.

On O_1O_2 describe the semicircle O_1RO_2. Draw O_2D at right angles to l.

Then O_2RDPQ lie on a circle whose diameter is O_2P; and PRO_2Q is a rectangle. Also $RDQ = RO_2Q = \frac{\pi}{2}$.

Hence rotate two right angles RDQ and RO_2Q round the two fixed points D and O_2, and let R trace out the semicircle, when Q generates the cubic. *Cf.* § 39.

Cor. 4 contains a generalisation of Cor. 3, as Cor. 2 is of Cor. 1.

Let

$$O_1PQ = \alpha, \ O_2QP = \beta.$$

Describe a circle round O_2QP cutting l in D, and O_1P again in R. D is ∴ a fixed point, and $O_1RO_2 = \pi - \beta$, so that R generates a segment of a circle on O_1O_2.

Also RDQ is constant $=\pi-\alpha$, and $RO_2Q=\alpha$. Hence in Cor. 3 replace the right angle RDQ by $\pi-\alpha$, and RO_2Q by α.

Cor. 5. (The Strophoid.)

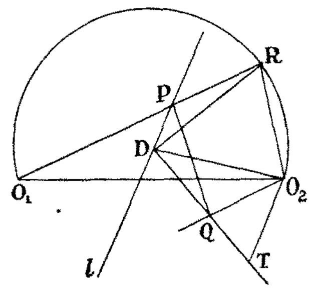

Fig. 23.

Let $\alpha=\beta$, and let D coincide with the centre of the circle O_1RO_2. Draw O_2T parallel to l cutting DQ in T.

Then

$$O_2QT=O_2RD=RO_2D=O_1PD=QO_2T.$$

Hence

$$TQ=TO_2.$$

Hence (Barrow's) generation of the curve:—

D is a fixed point in the plane, and T any point on a fixed line O_2T.

If Q is taken on DT so that $TQ(=TQ')=O_2T$, the locus of Q is the strophoid (oblique, or right when DO_2 is perpendicular to O_2T).

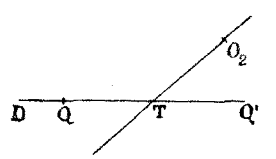

Fig. 24.

Cor. 7. In this corollary Maclaurin generalises the construction of Cor. 5 by taking for the point O_2 any point in the plane, T being still on the line l, while $TQ=Tq=O_2T$.

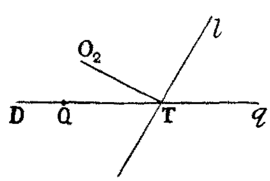

Fig. 25.

When the origin is taken at D, with the y-axis parallel to l, Teixeira gives the equation to the locus of Q as

$$x(x^2+y^2)-2a(x^2+y^2)+(2a\alpha-\alpha^2-\beta^2)x+2\alpha\beta y=0.$$

(O_2 is the point (α, β) and l is the line $x-\alpha=0$.)

Teixeira points out that the identical locus is discussed by Lagrange (*Nouvelles Annales*, 1900), and that the equation represents part of the curves known as Van Rees' Focals, for which the equation may be reduced to

$$x(x^2+y^2)=A(x^2+y^2)+Bx+Cy.$$

Maclaurin shows that the curve has a closed oval and a serpentine branch save (Cor. 8) when O_2 is on the line l, when there is a node.

Cor. 10. If O_2 is on l, and DO_2 perpendicular to l, the curve is that described by De Moivre in No. 345 of the *Philosophical Transactions.*

Cor. 11. The strophoid may also be thus generated:—

D and O_2 are fixed points, and O_2P a fixed line l.

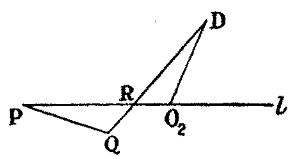

FIG. 26.

PQD is a constant angle$=PO_2D$, and $PQ=DO_2$. Then, as P slides on l, point Q generates the strophoid ($RO_2=RQ$).

Also the mid-point of PQ generates the cissoid of Diocles when $PQD=\frac{\pi}{2}$ (Newton).

[The description of the strophoid as the intersection of two rays rotating round two fixed centres with angular velocities in the ratio 1 : 2 is ascribed to Plateau (1828) by Kohn and Loria in their article on Special Plane Curves in the *Encyk. der. Math. Wiss.* This is historically inaccurate, for Maclaurin gave this generation three-quarters of a century earlier in his *Fluxions* (p. 262).]

§ 20. *Prop. XII*

discusses the asymptotes and also the subvarieties of the curves of Prop. X.

Ex. 1. Let $\alpha=\beta=\frac{\pi}{2}$; $l \perp^r O_1O_2$; $l' \parallel^l O_1O_2$.

If O_1 is the origin, O_2 the point $(d, 0)$; equation to l, $x=a$; equation to l', $y=b$; the locus of R is given by

$$ay^2(x-d)+bdxy=(d-a)x^2(x-d).$$

The case l and l' both parallel to O_1O_2 is discussed in Ex. 4.

Ex. 5. $O_1PQ = \frac{\pi}{2}$; l' and $l \perp^r O_1O_2$: QO_2R three collinear points.

Equation $xy^2(b-d) = (x-d)(ay^2 + \overline{a-b}x^2)$.

§ 21. *Prop. XIII.*

When Q and R move on fixed straight lines l' and l'', then the locus of P is in general a cubic with a double point at O_1.

Maclaurin's proof is analytic.

The geometrical method he would employ later runs thus:—

Leave Q free, and restrict P to lie on a straight line m. Then Q lies on a cubic cutting l' in three points Q_1, Q_2, Q_3, to which correspond P_1, P_2, P_3 on m. Hence, when Q lies on l', P traces out a curve cut by m in three points P_1, P_2, P_3. The curve is therefore in general a cubic.

But it may degenerate.

SECTION III.

On the Description of Lines of the Fourth Order, and those of the Third Order which have no Double Point.

§ 22. "We have described Lines of the Second Order by the rotation of two constant angles round two fixed points; also Lines of the Third Order by the use of as many angles, of which we have supposed one to be rotated round a fixed point, while the other is conducted along a fixed straight line.

"We now proceed to the description of Lines of the Fourth Order by conducting each angle along a straight line." (The quartics obtained have either two or three double points.)

Prop. XIV.

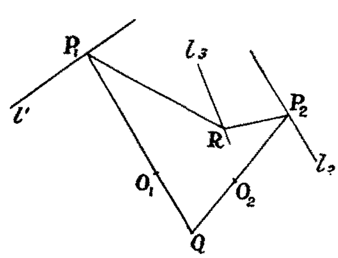

Fig. 27.

Given O_1 and O_2 two fixed points; $\angle O_1P_1R = \alpha$; $\angle O_2P_2R = \beta$, constant angles, where P_1 and P_2 lies on fixed lines l_1 and l_2 respectively.

If R is restricted to lie on a straight line l_3, the intersection Q of O_1P_1 and O_2P_2 in general generates a quartic having double points at O_1 and O_2.

Dem.

Let O_1P_1 have equation

$$L_1 + \lambda L_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and O_2P_2 have equation

$$M_1 + \mu M_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Then P_1R has an equation of the form

$$\lambda^2 N_1 + \lambda N_2 + N_3 = 0,$$

or

$$xA_2(\lambda) + yB_2(\lambda) + C_2(\lambda) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Similarly, P_2R has an equation of the form

$$xf_2(\mu) + y\phi_2(\mu) + \psi_2(\mu) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The condition that R lies on the line l_3, viz. on

$$lx + my + n = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

gives rise to the condition

$$\begin{vmatrix} l & m & n \\ A_2(\lambda) & B_2(\lambda) & C_2(\lambda) \\ f_2(\mu) & \phi_2(\mu) & \psi_2(\mu) \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In (6) substitute $-L_1/L_2$ for λ, and $-M_1/M_2$ for μ, when we obtain a quartic equation for the locus of Q representing a quartic curve having double points at O_1 and O_2.

The biquadratic relation (6) at once indicates the *genre* of the curve.

The existence of the double points is deduced analytically in Cor. 1, geometrically in Cor. 2; and the six possible varieties of these are enumerated in Cor. 4.

§ 23. *Prop. XV.*

If P_1Q and P_2Q coincide simultaneously with O_1O_2, the quartic degenerates into the straight line O_1O_2 and a cubic curve through O_1O_2 devoid of double points.

Cor. 2. This can happen when l_1 and l_2 cut on O_1O_2 and $\alpha + \beta = \pi$.

Cor. 3. Also when $\alpha + \beta = \pi$ and l_3 makes an angle α with O_1O_2.

For, when P_1 comes to lie on O_1O_2, R goes to infinity on l_3, while O_1Q and O_2Q coincide simultaneously with O_1O_2.

Cor. 4. In particular this will happen when $\alpha = \beta = \frac{\pi}{2}$, and either l_1 and l_2 intersect on O_1O_2, or $l_3 \perp^r O_1O_2$.

Cor. 10. It can also happen when l_1, l_2, l_3 are parallel, and α and β are the angles at which they cut O_1O_2.

§ 24. *Prop. XVI.*

Let l_1 and l_3 cut in A. If $O_1Al_3 = \alpha$ the curve is a cubic (with a double point).

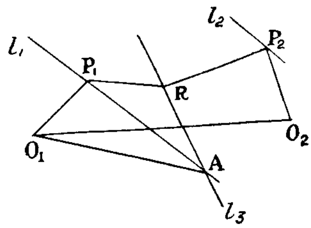

FIG. 28.

For O_1ARP is a cyclic quadrilateral.

Hence $P_1O_1R = P_1AR$ is constant, so that there is a reduction to Prop. V.

Similarly, if l_2 and l_3 cut in A_2 and $O_2Al_3 = \beta$ the curve is a cubic. When both hypotheses hold the curve is a conic, as in Prop. I.

§ 25. *Prop. XVII.*

When in the quadrilateral P_1RP_2Q it is Q and not R that is restricted to lie on a straight line, the locus of R is a quartic curve.

Dem.

Let as before O_1P_1 and O_2P_2 be given by

$$L_1 + \lambda L_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$M_1 + \mu M_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Then the condition that Q lies on l_3 leads to a relation

$$\mu = (a\lambda + b)/(c\lambda + d).$$

Hence the equations to P_1R and P_2R may be written in the form

$$\lambda^2 L_1 + \lambda M_1 + N_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\lambda^2 L_2 + \lambda M_2 + N_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and the elimination of λ from (3) and (4) leads to a quartic equation in x and y.

Cor. 1. The curve does not pass through O_1 or O_2.

[This description of a quartic is of especial interest. Maclaurin does not observe that the curve must possess three double points; for in virtue of (3) and (4) it must be a unicursal curve, and the double points are given by

$$L_1/L_2 = M_1/M_2 = N_1/N_2 \qquad . \quad . \quad . \quad . \quad . \quad (5)$$

(*vide* "Courbes Unicursales," *L'Ens. Math.*, 1912).

The equations (3) and (4) are not the most general of their kind, for the envelope is in each case a parabola. But it may be shown that any unicursal quartic with three double points may be considered generated by the intersection of two lines,

$$L_1\lambda^2 + M_1\lambda + N_1 = 0$$
$$L_2\lambda^2 + M_2\lambda + N_2 = 0,$$

which envelop two conics, and which may be obtained by making a constant angle O_1P_1R move with its vertex P_1 on a circle (or a straight line), and similarly a constant angle O_2P_2R move with its vertex P_2 on another circle (or straight line), while Q lies on a conic through O_1 and O_2. We may show as before that, without loss of generalisation, this conic may be replaced by a straight line. Maclaurin's generation is therefore the simplest of the above, and it is an easy step to proceed from it to the more general one in which circles are employed. It must not be forgotten that in Part I he only makes use of linear loci.]

§ 26. *Prop. XVIII.*

If in Prop. XVII l_1, l_2, l_3 are parallel, the curve is a cubic.

For the parabolic envelopes have in common the tangent line at infinity, so that the quartic reduces to this line and a cubic.

Particular Cases.

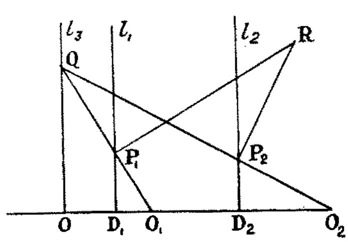

Fig. 29.

(More generally we find a cubic when two corresponding tangents to the parabolas coincide.)

Let $\alpha = \beta = \frac{\pi}{2}$; and let l_1, l_2, l_3 be $\perp^r O_1O_2$.

In the figure choose O_1O_2 as x-axis with origin at O.

Let $OO_1 = a$; $OO_2 = b$; $OD_1 = d$; $OD_2 = \delta$; $OQ = \gamma$.

Then O_1P_1 is given by

$$x/a+y/\gamma=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and P_1 is the point

$$\left(d, \gamma\frac{a-d}{a}\right).$$

P_1R ∴ has the equation

$$y-\gamma\frac{a-d}{a}=\frac{a}{\gamma}(x-d) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or

$$\gamma^2(a-d)-a\gamma y+a^2x-a^2d=0 \quad . \quad . \quad . \quad . \quad (3)$$

The equation to P_2R is

$$\gamma^2(b-\delta)-b\gamma y+b^2x-b^2\delta=0 \quad . \quad . \quad . \quad . \quad (4)$$

Hence, on solving for γ^2 and γ, we have

$$\gamma^2=Ax+B$$

$$\gamma=\frac{lx+m}{y},$$

so that

$$y^2(Ax+B)=(lx+m)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

a cubic with double point at

$$\left(-\frac{m}{l}, 0\right).$$

Cor. 5. l_1, l_2, l_3 parallel to O_1O_2.

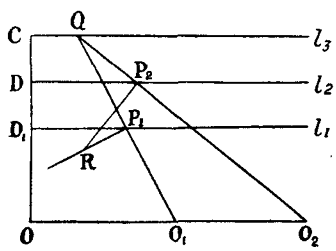

FIG. 30.

Take O any point in O_1O_2 as origin.

Let $OD_1=d$; $OD=\delta$; $OC=c$; $OO_1=a$; $OO_2=b$.

Let Q be the point (ξ, c).

∴ QO_1 has the equation

$$\frac{y}{c}=\frac{x-a}{\xi-a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and P_1 is the point

$$\left(a+\frac{d}{c}(\xi-a), d\right).$$

Thus P_1R has the equation

$$y-d=\frac{a-\xi}{c}\left(x-a-\frac{d}{c}\overline{\xi-a}\right)$$

or

$$\frac{\xi^2}{c^2}d+\xi\left(\frac{a}{c}-\frac{x}{c}-\frac{2ad}{c^2}\right)-y+d+\frac{a}{c}\left(x-a+\frac{ad}{c}\right)=0 \quad . \quad . \quad . \quad (2)$$

Similarly P_2R has the equation

$$\frac{\xi^2}{c^2}\delta+\xi\left(\frac{b}{c}-\frac{x}{c}-\frac{2b\delta}{c^2}\right)-y+\delta+\frac{b}{c}\left(x-b+\frac{b\delta}{c}\right)=0 \quad . \quad . \quad . \quad (3)$$

On solving (2) and (3) for ξ^2 and ξ we obtain

$$\xi^2=\frac{\alpha x^2+\beta xy+\gamma x+\delta y+\epsilon}{Ax+B}$$

$$\xi=\frac{\lambda x+\mu y+\nu}{Ax+B}.$$

$\therefore$ the equation to the locus of R is

$$(\lambda x+\mu y+\nu)^2=(Ax+B)(\alpha x^2+\ .\ .\ .\ +\epsilon) \quad . \quad . \quad . \quad (4)$$

Cor. 7. If, as before, $\alpha=\beta=\frac{\pi}{2}$, and l_1 and l_2 coincide, the curve is a cubic.

For, let l_1 and l_3 cut O_1O_2 in D_1 and D_3 respectively.

Then D_1E $\perp^r$ O_1O_2 forms part of the locus.

§ 27. *Prop. XIX.*

If in the figure of Prop. XIV Q, P_2, and R are restricted to lie on straight lines, the point P_1 generates a quartic with a triple point at O_1.

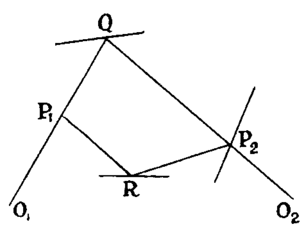

FIG. 31.

Take O_1 as origin.

Let O_1Q have the equation

$$y-\mu x=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and O_2Q have the equation

$$y=m(x-a) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

There is $\therefore$ a $1-1$ correspondence between m and μ.

P_2R has an equation of the form

$$xf_2(\mu)+y\phi_2(\mu)+\psi_2(\mu)=0 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If P_1R is given by

$$y = px + q,$$

then

$$p = (A\mu + B)/(C\mu + D) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

But P_1R and P_2R concur on a fixed line

$$ax + by + c = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence

$$\begin{vmatrix} a & b & c \\ f_2 & \phi_2 & \psi_2 \\ A\mu + B & -(C\mu + D) & q(C\mu + D) \end{vmatrix} = 0 \quad . \quad . \quad (6)$$

and

$$1/q = (C\mu + D)F_2(\mu)/F_3(\mu).$$

Thus the equation to P_1R may be written as

$$y = \frac{A\mu + B}{C\mu + D}x + \frac{F_3(\mu)}{(C\mu + D)F_2(\mu)} \quad . \quad . \quad . \quad . \quad (7)$$

and OP_1 is given by

$$y = \mu x.$$

Put y/x for μ in (7), when we obtain for the locus of P_1 a quartic with a triple point at O_1.

Scholium.

§ 28. In the scholium Maclaurin points out how complicated is the task of furnishing a classification of quartics similar to that given by Newton for cubic curves.

He makes it clear that a quartic cannot have more than three double points. It seems doubtful whether he was aware that the quartics given by Prop. XVII have three double points.

But he shows that if there are three double points they cannot lie on a straight line.

General Corollary.

From Props. XIV, XVII, and XIX we conclude that when, in a quadrilateral QP_1RP_2, the angles at P_1 and P_2 are constant, while QP_1 and QP_2 pass through two fixed points O_1 and O_2, then, if any three of the vertices lie on given straight lines, the remaining vertex in general generates a quartic.

SECTION IV.

Wherein are demonstrated General Theorems regarding the Description of Curves of any Order by the Use only of Linear Loci and Constant Angles.

§ 29. This section takes up the discussion from a more general point of view, and, while Maclaurin's theorems are adhered to in their order, their demonstrations, when analytical, are frequently altered. Before we proceed to these it will be convenient, just as Maclaurin does, to pave the way by some preliminary theorems.

Instead of an ordinary angle he makes use of what may be termed a *serrate angle* consisting of a broken line $OP_1P_2 \ldots P_nP$, in which the component angles at the teeth are of constant magnitude, while the

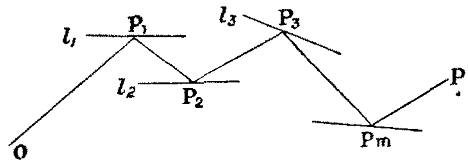

Fig. 32.

segments of the line are freely variable. The vertices $P_1P_2 \ldots P_n$ lie on linear loci $l_1, l_2, \ldots l_n$, and O is a fixed point.

Let the equation to OP_1 depend on a parameter t, and be given by

$$L_0 + tL_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Then the equation to P_1P_2 is of the form

$$M_0 + tM_1 + t^2M_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Similarly, for P_2P_3 we in general find an equation of the form

$$N_0 + tN_1 + t^2N_2 + t^3N_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

etc., etc.

The lines P_1P_2, P_2P_3, etc., envelop unicursal curves of class 2 (parabola), 3, etc, having a special relation to the line at infinity.

Also the co-ordinates of P_n are rational functions of t of degree n.

§ 30. *Prop. XX.*

Let $O_1P_1P_2 \ldots P\ Q$ be a serrate angle (n lines $l_1, l_2, \ldots l_n$), O_2 a second fixed point through which O_2Q is drawn such that O_2QP_n is a constant angle, then the locus of Q is a curve of degree $n+2$.

For P_nQ has an equation of the form

$$L_0 + tL_1 + \ldots + t^{n+1}L_{n+1} = 0 \qquad (4)$$

and O_2Q, which really makes a constant angle with O_1P_1, has an equation of the form

$$M_0 + tM_1 = 0 \qquad (5)$$

The elimination of t between (4) and (5) leads to an $(n+2)$-ic having an $(n+1)$-ple point at O_2.

[We might state the theorem thus. Given O_1 and O_2 fixed points, and the serrate angle

$$O_1P_1P_2 \ldots P_nQO_2,$$

in which $P_1 \ldots P_n$ lie on fixed straight lines, the locus of Q is an $(n+2)$-ic with an $(n+1)$-ple point at O_2. Or, again, the locus of Q is simply a pedal of the envelope of P_nQ.]

§ 31. *Prop. XXI.*

Given the serrate angle $O_1P_1 \ldots P_{n-1}Q$ ($n-1$ lines $l_1, l_2, \ldots l_{n-1}$) and the constant angle $R_1O_2R_2$ which is rotated round O_2. If the intersection R_1 of O_1P_1 and O_2R_1 lies on a fixed line l_n, then the intersection Q of $P_{n-1}Q$ and O_2R_2 generates a curve of degree $n+1$.

For the equation to $P_{n-1}Q$ is of the form

$$L_0 + tL_1 + \ldots - t^nL_n = 0 \qquad (1)$$

In virtue of l_n the parameter of O_2R_1 and $\therefore$ of O_2R_2 is in $1-1$ correspondence with t, so that the equation to O_2R_2 is of the form

$$M_0 + tM_1 = 0 \qquad (2)$$

The elimination of t between (1) and (2) gives rise to an $(n+1)$-ic with an n-ple point at O_2.

Cor. The curve may, of course, degenerate and be of lower order in its component curves.

Cor. 6. The angle $R_1O_2R_2$ may be a straight angle, so that $R_1O_2R_2$ is a straight line rotating round O_2.

Cor. 7. When $n=3$, the curve is a quartic with a triple point at O_2.

§ 32. *Prop. XXII.*

If all the points but one of the $n+1$ points

$$R_1P_1 \ldots P_{n-1}Q$$

are restricted to lie on straight lines, the remaining point generates a curve of degree $n+1$.

The proof is exactly on the lines of Prop. XIII.

§ 33. *Prop. XXIII.*

Let the intersection of R_1O_2 and $P_{r-1}P_r$ lie on the straight line

$$l'x + m'y + n' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and Q will generate a curve of degree $n+r$.

Let R_1O_2 be given by the equation

$$M_0 + \mu M_1 = 0$$

or

$$xA_1(\mu) + yB_1(\mu) + C_1(\mu) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Then $P_{r-1}P_r$ has an equation of the form

$$xf_r(t) + y\phi_r(t) + \psi_r(t) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The condition that (1), (2), (3) be concurrent is

$$\begin{vmatrix} l' & m' & n' \\ A_1(\mu) & B_1(\mu) & C_1(\mu) \\ f_r(t) & \phi_r(t) & \psi_r(t) \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence μ = a rational function of t of degree r in numerator and denominator, and the equation to O_2R_2 may be written in the form

$$N_0 + tN_1 + \ . \ . \ . \ + t^rN_r = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

But $P_{n-1}Q$ has an equation of the form

$$M_0 + tM_1 + \ . \ . \ . \ + t^nM_n = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The equations (5) and (6) therefore give for the locus of Q a unicursal curve of degree $n+r$.

Cor. 1. The line (6) envelops a curve of class n. Hence n lines of the system pass through O_2, so that O_2 is an n–ple point on the locus of Q.

Cor. 3. When of the points $R_1QP_1 \ . \ . \ . \ P_{n-1}$ all but one lie on fixed straight lines, the remaining point generates a curve of degree $n+r$.

Cor. 5. By variation of n and r subject to the condition $n+r$ = constant, we may deduce a variety of ways of drawing curves of degree $n+r$.

Cor. 6 is not correct.

Maclaurin states the following generation of a quartic:—

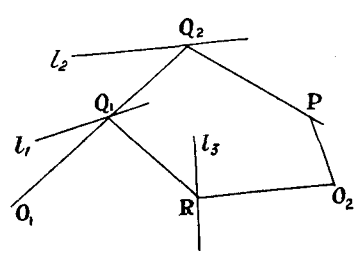

Fig. 33.

O_1 and O_2 are fixed points; Q_1 and Q_2 are two variable points on two linear loci such that Q_1Q_2 passes through O_1. The angles O_1Q_1R and O_1Q_2P are constant, and R is a point on a line l_3. If RO_2P is also an angle of constant magnitude, the locus of P is, according to Maclaurin, a quartic curve with a double point at O_2. But if, when Q_2P passes through O_2, O_2P coincides with it, then the locus degenerates into a cubic devoid of a double point.

In reality the curve is, in general, a unicursal quartic having three double points, while the cubic is also unicursal and therefore possesses a double point.

Dem.

Take the origin of co-ordinates at O_2.

Let O_2P have the equation

$$y - mx = 0 \qquad (1)$$

and $O_1Q_1Q_2$ have equation

$$L_1 + tL_2 = 0 \qquad (2)$$

Then Q_1R has an equation of the form

$$xA_2(t) + yB_2(t) + C_2(t) = 0 \qquad (3)$$

The line O_2R which is in $1-1$ correspondence with O_2P cuts Q_1R on l_3. Hence

$$m = f_2(t)/\phi_2(t) \qquad (4)$$

(1) may $\therefore$ be written

$$y\phi_2(t) - xf_2(t) = 0 \qquad (5)$$

Also the equation to Q_2P is of the form

$$xP_2(t) + yQ_2(t) + R_2(t) = 0 \qquad (6)$$

On solving (5) and (6) for x and y we obtain the unicursal equations to a unicursal quartic, which possesses a double point at O_2, it is true, but also possesses other two double points in general.

Suppose, however, when Q_2P passes through O_2 that O_2P coincides with it, then the curve is a unicursal cubic, in which O_2 is an ordinary point, but the curve necessarily has a double point elsewhere in virtue of its unicursality.

Dem.

Let α be the value of t for which Q_2P passes through O_2.

Let $R_2(t)$ in $(6) = (t-\alpha)(t-\beta)$, and let

$$m_0 = f_2(\alpha)/\phi_2(\alpha) = -P_2(\alpha)/Q_2(\alpha) \qquad (7)$$

Then $R_2(t)$ and

$$\begin{vmatrix} f_2(t) & -\phi_2(t) \\ P_2(t) & Q_2(t) \end{vmatrix}$$

vanish when $t=a$.

Hence on solving (5) and (6) for x and y we find

$$\left.\begin{aligned} x &= f_3(t)(t-a)/\phi_3(t)(t-a) \\ y &= \psi_3(t)(t-a)/\phi_3(t)(t-a) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$\therefore$ etc.

§ 34. *Prop. XXIV.*

Consider two serrate angles

$$O_1P_1P_2 \ldots P_mP$$

and

$$O_2Q_1Q_2 \ldots Q_nQ$$

in which $P_1P_2 \ldots P_m$ *lies on* m *fixed lines*, and $Q_1Q_2 \ldots Q_n$ *on* n *fixed lines.*

If the intersection of P_mP *and* Q_nQ *also lies on a given straight line, the intersection of* O_1P_1 *and* O_2Q_1 *in general generates a curve of degree* $n+m+2$ *possessing an* $(m+1)$*-ple point at* O_1 *and an* $(n+1)$*-ple point at* O_2.

Let O_1P_1 have equation

$$y-\lambda x=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Then P_mP has an equation of the form

$$xA_{m+1}(\lambda)+yB_{m+1}(\lambda)+C_{m+1}(\lambda)=0 \quad . \quad . \quad . \quad . \quad (2)$$

If O_2Q_1 has an equation of the form

$$L_1+tL_2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Q_nQ has an equation of the form

$$xA_{n+1}(t)+yB_{n+1}(t)+C_{n+1}(t)=0 \quad . \quad . \quad . \quad . \quad (4)$$

Let P_mP and Q_nQ intersect on

$$ax+by+c=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence

$$\begin{vmatrix} a & b & c \\ A_{m+1}(\lambda) & B_{m+1}(\lambda) & C_{m+1}(\lambda) \\ A_{n+1}(t) & B_{n+1}(t) & C_{n+1}(t) \end{vmatrix}=0 \quad . \quad . \quad . \quad . \quad (6)$$

Substitute y/x for λ, and $-L_1/L_2$ for t in (6), when the result follows at once.

Cor. 2. If of the points $P_1P_2 \ldots P_mQ_1Q_2 \ldots Q_nRT$ (T being the intersection of O_1P_1 and O_2Q_1, and R of P_mP and Q_nQ), all but one lie on straight lines, the remaining point generates a curve of degree $n+m+2$.

Cor. 4. There is no change in the nature of the curve if, instead of the intersection of O_1P_1 and O_2Q_1, we take the intersection of two lines through O_1 and O_2 making given angles with O_1P_1 and O_2Q_1 (in virtue of the $1-1$ correspondence).

Cor. 5. The number $n+m+2$ for the degree is a maximum, and may not always be attained.

§ 35. *Prop. XXV.*

If the intersection of $P_{s-1}P_s$ and O_2Q_1 is restricted to lie on a straight line, the point of intersection of P_mP and Q_nQ is on a curve of degree $ns+s+m+1$

For $P_{s-1}P_s$ has an equation of the form

$$xA_s(\lambda)+yB_s(\lambda)+C_s(\lambda)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and O_2Q_1 of the form

$$L_1+tL_2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and ∴

$$t=f_s(\lambda)/\phi_s(\lambda), \text{ say} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But P_mP and Q_nQ have equations

$$xA_{m+1}(\lambda)+yB_{m+1}(\lambda)+C_{m+1}(\lambda)=0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$xA_{n+1}(t)+yB_{n+1}(t)+C_{n+1}(t)=0$$

or

$$xA_{ns+s}(\lambda)+yB_{ns+s}(\lambda)+C_{ns+s}(\lambda)=0 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and the desired result follows from (4) and (5).

§ 36. *Prop. XXVI*

If the intersection of P_mP and Q_nQ is on the line

$$ax+by+c=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

then the intersection of $P_{r-1}P_r$ and $Q_{s-1}Q_s$ generates a curve of degree

$$r(n+1)+s(m+1).$$

We have the relation

$$\begin{vmatrix} a & b & c \\ A_{m+1}(\lambda) & B_{m+1}(\lambda) & C_{m+1}(\lambda) \\ A_{n+1}(t) & B_{n+1}(t) & C_{n+1}(t) \end{vmatrix}=0 \quad . \quad . \quad . \quad . \quad (2)$$

while $P_{r-1}P_r$ and $Q_{s-1}Q_s$ have equations of the form

$$XA_r(\lambda)+yB_r(\lambda)+C_r(\lambda)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$XA_s(t)+yB_s(t)+C_s(t)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In how many points can the curve given by the intersection of (3) and (4) be cut by the straight line

$$Ax + By + C = 0 \text{ ?} \qquad \qquad (5)$$

At such points we must have

$$\begin{vmatrix} A & B & C \\ A_r(\lambda) & B_r(\lambda) & C_r(\lambda) \\ A_s(t) & B_s(t) & C_s(t) \end{vmatrix} = 0 \qquad \qquad (6)$$

taken along with (2).

By the theory of equations the t-eliminant of (2) and (6) is of degree

$$(m+1)s + (n+1)r,$$

and this number therefore represents the number of intersections of the curve with a straight line, and so the degree of the curve.

Cor. 2. The theorem may be extended as in Cor. 2 of Prop. XXIV.

§ 37. *Prop. XXVII.*

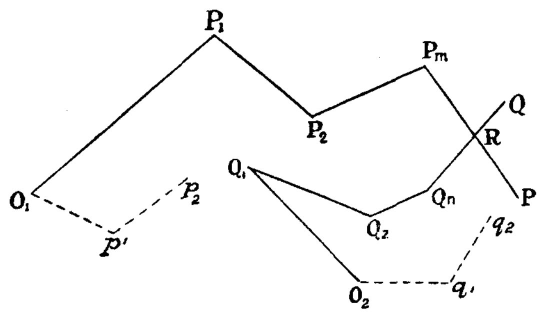

Fig. 34.

Suppose that, in addition to the data of Prop. XXIV, there are given the serrate angles

$$P_1O_1p_1 \;\cdots\; p_{r-1}p$$
$$Q_1O_2q_1 \qquad q_{s-1}q,$$

then the intersection of $p_{r-1}p$ and $q_{s-1}q$ is on a curve of degree $ms+nr+s+r$.

For the datum that P_mP and Q_nQ intersect on a straight line leads to (2) of preceding.

There is a $1-1$ correspondence between P_1O_1 and O_1p_1 so that $p_{r-1}p$ has an equation like (3). Similarly, $q_{s-1}q_s$ has an equation like (4). ∴ etc.

Scholium.

§ 38. In the scholium Maclaurin gives credit to Fermat, Varignon, De la Hire, Nicole for special curves: and to Newton's great work on Cubic

Curves. He points out the desirability of having a general method of generating curves of all degrees. The method employed does not give all curves, but it may serve to pave the way for future perfection of the theory.

In the part just completed only straight lines and constant angles have been employed. In Part II other curve loci are utilised from which to obtain more complicated curves of higher degree.

Part II.

Wherein Curves of all Higher Orders are described by the Use of Curves of Lower Order.

SECTION I.

§ 39. Newton's Organic Description of Curves.

Prop. I.

Round the fixed points O_1 and O_2 are rotated the constant angles $PO_1Q = \alpha$, $PO_2Q = \beta$.

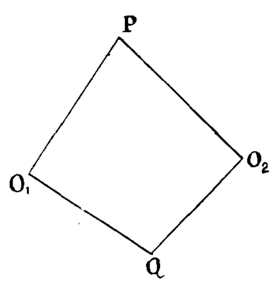

Fig. 35.

If P traces out a conic through O_1, Q generates a cubic having a double point at O_1 and an ordinary point at O_2.

Maclaurin's proof runs thus. Find in how many points a straight line l can cut the curve, *i.e.* how many points Q can lie on l.

Let Q trace out the line l, P being left free. P will generate a conic through O_1 and O_2 cutting the given conic in four points O_1, P_1, P_2, P_3. To P_1, P_2, P_3 correspond three points Q_1, Q_2, Q_3 on l: so that the locus cuts l in three points and is therefore a cubic.

Let $O_1O_2R = \beta$ and let O_2R cut the given conic in R and R′. Then, when P comes to R or to R′, Q comes to O_1, which is thus a double point. Similarly it passes once through O_2.

Cor. 6. If O_1Q and O_2Q coincide simultaneously with O_1O_2, the curve reduces to a conic.

Cor. 8. Particular cases of Newton's organic description as a Cremona transformation when $\alpha = \beta = \frac{\pi}{2}$.

Choose O_1 as origin, and O_2 as $(a, 0)$.

Then, if P is (ξ, η) and Q (ξ', η'),

$$\xi' = a - \xi \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\eta' = \xi(\xi - a)/\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Thus:

(I.) To

$$lx + my + n = 0$$

corresponds

$$l(a - x)y - mx(a - x) + ny = 0,$$

a conic through

$$(0, 0); \ (a, 0); \ (a, \infty).$$

(II.) To the parabola

$$y^2 - mx = 0$$

corresponds

$$x^2(x - a)^2 + m(x - a)y^2 = 0,$$

or

$$my^2 = x^2(a - x).$$

(III.) To the rectangular hyperbola

$$x^2 - y^2 = mx$$

corresponds

$$y^2(a - m - x) = x^2(a - x).$$

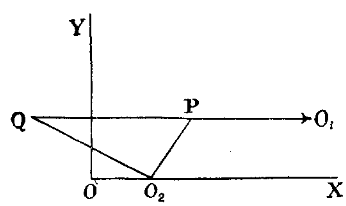

FIG. 36.

(IV.) O_1 at infinity on the x-axis; O_2, as before, $(a, 0)$; $PO_2Q = \frac{\pi}{2}$; O_1PQ collinear points; $xy = p$ the locus of P.

Let P be the point $(p/\eta, \eta)$. $\therefore$ O_2P has equation

$$y/\eta = \eta(x - a)/(p - a\eta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The equation to O_2Q is

$$y = \frac{a\eta - p}{\eta^2}(x - a) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the locus of Q is given by

$$y^3=(ay-p)(x-a) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(V.) Let P lie on $xy=p$.

If O_1 is the point at infinity on the y-axis, O_1PQ parallel to OY, and $\angle PO_2Q=\pi/2$, the locus of Q is given by

$$y=-x(a=x)^2/p.$$

Cor. 9. In this corollary Maclaurin proves the generality of this construction for a singular cubic by using it to describe a cubic having a double point at O_1, and passing through other six points $O_2P_1P_2 \ldots P_5$.

Construct $\Delta P_1O_1O_2$, and let $\hat{O}_1=\alpha$, $\hat{O}_2=\beta$. Take a quadrilateral PO_1QO_2, in which $\hat{O}_1=\alpha$, $\hat{O}_2=\beta$, and place P in coincidence with P_2, P_3, P_4, P_5, when Q will take up positions Q_2, Q_3, Q_4, Q_5. Construct the conic through O_1, Q_2, Q_3, Q_4, Q_5, and restrict the vertex Q to lie on this conic, when P traces out a cubic having a double point at O_1 and passing through $O_2P_1P_2 \ldots P_5$. There cannot be two such cubics; for, if there were, they would require to be considered as intersecting in ten points, whereas they cannot cut in more than nine points.

§ 40. *Prop. II.*

When the locus conic, as in Prop. I, passes through neither O_1 nor O_2, then the same method of proof shows that the curve traced by Q is a curve of the fourth order, having double points at O_1 and O_2, and also at a third point.

For let O_1O_2 cut the conic in A_1 and A_2. To A_1 and A_2 corresponds a common point B on the quartic, which is thus a third double point.

Cor. 4. If O_1Q and O_2Q coincide simultaneously with O_1O_2, the locus is only of the third degree, with ordinary points at O_1 and O_2 and a double point at B.

[Maclaurin might have shown how to use Prop. II to construct a quartic having three double points O_1, O_2, O_3, and through five other points $P_1 \ldots P_5$.

Construct $\Delta O_1O_2O_3$, and let $\hat{O}_1=\alpha$, $\hat{O}_2=\beta$, $\hat{O}_3=\gamma$. Use the quadrilateral PO_1QO_2 in which $\hat{O}_1=\alpha$ and $\hat{O}_2=\beta$, and place P in coincidence with P_1, P_2, ... P_5, when Q takes up the positions Q_1, Q_2, ... Q_5. Let the five points Q determine the conic C. Now restrict Q to lie on C, and P will trace out a quartic having double points at $O_1O_2O_3$ and through $P_1P_2 \ldots P_5$.

There cannot be two such quartics, for, if so, they would require to be considered as intersecting in seventeen points, which is impossible.]

§ 41. *Prop. III.*

If P lies on a curve of degree n, Q traces out a curve of degree $2n$.

For let Q lie on a straight line l; then P will generate a conic which cuts the n-ic in $2n$ points $P_1, P_2, \ldots P_{2n}$, to which correspond on l $2n$ points, $Q_1, Q_2, \ldots Q_{2n}$. ∴ etc.

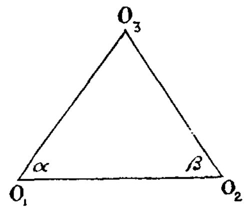

FIG. 37.

Cor. 1. Construct $\Delta O_1O_2O_3$, and let $\hat{O}_1 = \alpha$, $O_2 = \beta$, as in figure.

Then each side of $O_1O_2O_3$ cuts the n-ic in n points, to which corresponds the unique point the opposite vertex. The curve therefore has n-ple points at O_1, O_2, O_3.

Cor. 2. There cannot be four n-ple points on the new curve; for through these and a fifth point on the curve we could describe a conic cutting the curve in $4n+1$ points, which is impossible.

Cor. 4. It has been assumed that O_1 and O_2 do not lie on the given curve of degree n. If O_1 is an ordinary point on the latter the curve obtained is of degree $2n-1$; and if O_1 is an r-ple point the curve is of degree $2n-r$.

[We might, of course, have established this proposition by showing that the co-ordinates of P and Q are connected by an ordinary Cremona quadratic transformation. We therefore have before us established, for the first time, the fundamental features of a Cremona transformation more than a century before it was to become the property of all mathematicians through Cremona's researches.]

REMARK.

"Newton has given Props. I and II and indicated their generalisation.

"This generalisation we have attempted to effect in Prop. III. We have to this end made use of a given curve and two constant angles. In the following we shall attempt to generalise all the propositions of Part I, just as we have generalised the Proposition I of Part I."

SECTION II.

Wherein Curves are Investigated such as may be Obtained from Certain Others by the Use of Given Angles.

§ 42. *Prop. IV.*

With data similar to those in Prop. V of Part I, viz. O_1, O_2 fixed points, angles at P and O_2 in quadrilateral PRO_2Q constant. Let P lie

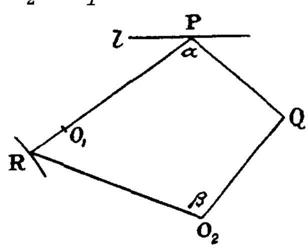

Fig. 38.

on a straight line l. But let R now lie on a curve C_n of degree n. Then Q will generate a curve of degree $3n$.

Dem.

Let Q lie on a straight line l_1, and P on its locus l, then R will generate a cubic * cutting C_n in $3n$ points $R_1, R_2, \ldots R_{3n}$, to which correspond on l_1 the $3n$ points $Q_1, Q_2, \ldots Q_{3n}$. Hence l_1 cuts the locus in these $3n$ points. ∴ etc. *Q.E.D.*

Cor. 1. Construct on O_1O_2 a circle containing an angle equal to O_1PQ and cut by l in two points A and B. To each of the n points in which O_1A cuts C_n corresponds the point O_2, and similarly for O_1B. Hence O_2 is a $2n$-ple point on the curve. But O_1 is not in general a point on the new curve.

Cor. 4. If O_1 is on C_n the degree of the new curve is less by 2, if O_2 is on the curve less by unity. If both points are on C_n the new curve is of degree $3n-3$.

Cor. 5. If of the three vertices P, Q, R of O_2RPQ one is restricted to lie on a straight line, a second on a curve C_n, the remaining vertex generates a curve C_{3n} of degree $3n$.

§ 43. *Prop. V.*

If R in the preceding is restricted to lie on a curve C_n, and P on a curve C_m, then Q generates a curve C_{3mn} of degree $3mn$.

* This cubic has a double point at O_1 and passes through O_2.

Dem.

Let Q lie on a line l, and R on C_n, then, by Cor. 5 of Prop. IV, P generates a curve C_{3n} which cuts C_m in $3mn$ points $P_1, P_2, \ldots P_{3mn}$, to which correspond $Q_1, Q_2, \ldots Q_{3mn}$ on l.

The new curve is $\therefore$ cut by l in $3mn$ points. $\therefore$ etc.

Cor. 2. O_2 is a multiple point on the locus of order $2mn$.

For on O_1O_2 describe a segment of a circle containing an angle equal to O_1PQ. It cuts C_m in $2n$ points $A_1, A_2, \ldots A_{2m}$. The lines O_1A cut C_n in $2mn$ points, to each of which corresponds the point O_2. $\therefore$ etc.

Cor. 3. If of the vertices P, Q, R one lies on a curve C_m, and a second on a curve C_n, the third generates a curve C_{3mn}.

§ 44. *Prop. VI.*

Generalisation of Prop. XIV in Part I.

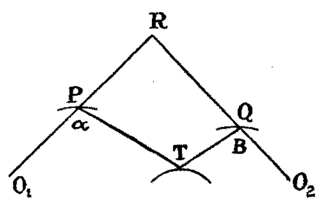

Fig. 39.

In the figure O_1PT and O_2QT are constant angles. If P lies on C_m, Q on C_n, T on C_r, then R generates a curve C_{4mnr}.

Dem.

Let P and Q lie on straight lines, and let R lie on a line l: then T would generate a quartic C_4 cutting C_r in $4r$ points, to which would correspond $4r$ points R on l; *i.e.* R would generate a curve C_{4r}.

Next, let P lie on a straight line, Q on C_n, and T on C_r; then R lies on a curve C_{4nr}. For let R lie on a line l', P on l, and T on C_r: then Q would generate a curve cutting C_{rL} in $4nr$ points, to which correspond $4nr$ points on l'.

Hence the locus of R would be a C'_{4nr}.

Finally, let P lie on C_m. Let R lie on a line l'', Q on C_n, and T on C_r; when P would generate a curve C_{4nr} cutting C_m in $4mnr$ points: and to these correspond $4mnr$ points on l''. $\therefore$ etc.

Cor. 1. Each of the points O_1, O_2 is multiple of order $2mnr$.

§ 45. *Prop. VII.*

Generalisation of Prop. XXI of Part I.

Let $O_1P_1P_2 \ldots P_{n-1}Q$ be a serrate angle, QO_2R a constant angle *rotating round O_2, with R on O_1P_1. If $R, P_1, P_2, \ldots P_{n-1}$ lie on curves $C_r, C_{1p}, \ldots C_{pn-1}$, the locus of Q is a curve of order $rp_1p_2 \ldots p_{n-1}(n+1)$.*

The demonstration is similar to that of Prop. VI.

Cor. 1. The point O_2 is multiple on the curve of order

$$rp_1p_2 \ldots p_{n-1}n.$$

§ 46. *Prop. VIII.*

Generalisation of Prop. XXIV of Part I.

Consider two serrate angles

$$O_1P_1P_2 \ldots P_mP$$

and

$$O_2Q_1Q_2 \ldots Q_nQ$$

in which $P_1P_2 \ldots P_m$ lie on curves of orders $p_1, p_2, \ldots p_m$, and $Q_1Q_2 \ldots Q_n$ on curves of orders $q_1, q_2, \ldots q_n$ respectively. If P_mP and Q_nQ intersect on a curve C_r the intersection of O_1P_1 and O_2Q_1 generates a curve of order

$$r(n+m+2)\Pi p_1\Pi q_1.$$

SECTION III.

Maclaurin's Theory of Pedals.

§ 47. An intelligent perusal of the preceding shows that Maclaurin would inevitably have been led to the pedal transformation of curves,* which he now discusses very thoroughly in general terms, along with its application to conic sections and other familiar curves.

He gives no name to the transformation, and the term pedal (*podaire*) was introduced by the geometers of the nineteenth century.

Almost the only nomenclature he introduces, the "Radial Equation" for the $p-r$ equation to a curve (p. 96), has been quite overlooked and adapted to another purpose by Tucker.

Definition.

The definition is the usual one.

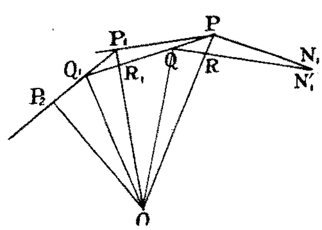

Fig. 40.

* But compare § 2 of Part I.

Let O be a fixed point in the plane of a curve C, on which P and Q are two infinitely near points. PP_1 is the tangent at P to the curve, and OP_1 is drawn perpendicular to P_1P. Then as P moves on its locus P_1 generates a curve, the pedal of the given curve for the pole O.

§ 48. *Prop. IX.*

Draw $PN_1 \perp^r OP$, and $QN_1 \perp^r OQ$; $OQ_1 \perp^r QQ_1$; $OP_2 \perp^r P_1Q_1$.

Then the following pairs of similar triangles arise:—

$$\Delta OPP_1 \approx \Delta Q_1P_1R_1,$$
$$\Delta OR_1Q_1 \approx \Delta PR_1P_1,$$
$$\Delta OR_1P \approx \Delta Q_1R_1P_1.$$

Also $P_1Q_1R_1$, PQR, P_1OP_2, POP_1 are similar; and

$$OP/OP_1 = OP_1/OP_2.$$

Denote OP by r, and OP_1 by p. If the curve C is given by the equation $f(x, y) = 0$ (1) referred to axes through O,

$$OP = \sqrt{(x^2 + y^2)},$$

PP_1 has for equation

$$Y - y = (X - x)y' \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$p = (y - xy')/\sqrt{(1 + y'^2)} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The elimination of x, y, and y' leads to a single relation

$$\phi(p, r) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which is sufficient to characterise the curve, and it is this equation which Maclaurin calls the Radial Equation of the curve.

Cor. 1. From the locus of P_1 may be similarly described its pedal, the locus of P_2. We may thus derive an infinite series of curves (the positive pedals of C).

From the radial equation of C can be easily deduced the radial equation of the locus of P_1.

Let p_1 and r_1 correspond to the locus of P_1.

Then

$$\left.\begin{aligned} p_1/r_1 &= p/r \\ r_1 &= p \end{aligned}\right\}$$
$$\therefore \; p = r_1 \, ; \; r = r_1^2/p_1 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Cor. 2. The series of curves may be continued in the opposite sense, viz. by drawing PN_1 and QN_1 perpendicular to OP and OQ, and finding the locus of N_1 (the negative pedal); or N_1 may be found by drawing ON_1, so that PON_1 is the complement of OPP_1.

Thus the series of pedals may be continued in both directions. They will be all changed if the position of O is altered.

Cor. 3. The tangents at P, P_1, . . . make the same angle with the corresponding radii vectores OP, OP_1, . . .

Cor. 4. If C passes through the pole O so do all the pedals.

Cor. 5. If OP is normal to C at P all the pedals pass through P and have there a common tangent.

Cor. 6. Since $OP_1 \angle OP$, $\therefore$ when C is a finite closed curve so are all its (positive) pedals.

Cor. 7. If C has a parabolic branch so have the pedals. This does not happen for a hyperbolic branch of the curve.

Cor. 8. When the pedal for O is known the pedal for O′ may be found thus:

Draw P_1S $\perp^r$ OP_1, and O′S $\|^1$ OP_1, then S is on the pedal of O′.

§ 49. *Prop. X.*

The Pedal of the Circle.

Properties of the Pedal.

(The Limaçon of Pascal, and the Cardioid.)

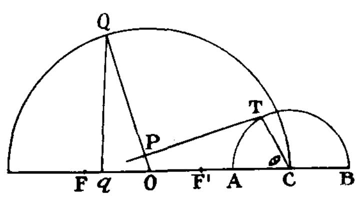

Fig. 41.

Let ATB be the given circle of centre C and radius r. Let $OC = d$, and describe the circle with centre O and radius OC. TP is the tangent at T and OP $\perp^r$ TP cuts the second circle in Q. Qq is $\perp^r$ OC, and $OF = OF' = r$.

Then

$$OP = Fq.$$

For

$$OP = CT - OC\cos\theta = r - d\cos\theta = FO - OQ\cos\theta = Fq.$$

Equation to Pedal.

Let O be the origin, C the point $(d, 0)$.

The equation to PT is of the form

$$(x-d)\cos\phi + y\sin\phi - r = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and OP is given by

$$y\cos\phi - x\sin\phi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence

$$(x^2+y^2-dx)^2=r^2(x^2+y^2) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

So that the pedal is a quartic curve.

[The curve is now called the Limaçon of Pascal. When $d=r$, so that O is on the circumference of the given circle, it is called a cardioid.]

Cor. 1. The origin is a double point on the curve (as are also the circular points at infinity, making in all three double points). The branches through O are real when O is outside the circle, and imaginary when O is inside the circle, while O is a cusp if $d=r$.

Cor. 2. The limaçon touches the circle at A and B, and is a finite closed curve.

Cor. 3. The chords of the quartic that pass through O are of constant length $=2r$.

Cor. 4. Let T′ be a point on the circle infinitely near to T, and find the corresponding points P′ Q′, q'.

Then $\Delta TPP' \simeq \Delta Qqq'$; so that on integrating we find that the area of the limaçon between AB and the semi-circle ATB is equal to a quadrant of the circle of radius d, and the area $ABPA=\pi(d^2/4+r^2/2)$.

Cor. 5. The curve can be rectified only when it is a cardioid. In this latter case the arc BP of the cardioid $=2BT$, *i.e.* double the chord of the corresponding arc of the original circle (proof later), so that the whole length of the curve is $8r$.

Cor. 6. The curve is the epicycloid generated by the rolling of a circle of radius $r/2$ upon an equal circle, the centre of the fixed circle being midway between O and C.

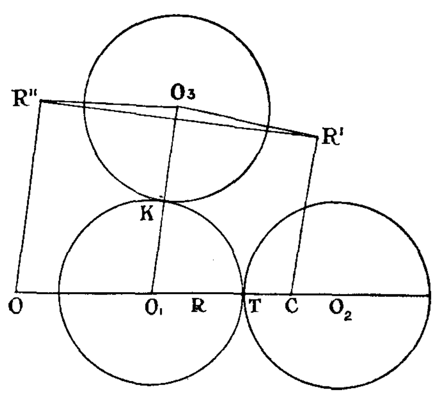

FIG. 42.

Let O_1 bisect OC : $O_1K=KO_3=TO_2=r/2$: $TR=TC$.

Let R be carried into position R′, and let R″ be the image of R′ in O_1O_3.

Then

$$CO_1O_3 = R'O_3O_1 = R''O_3O_1,$$

and

$$O_1C = O_3R' = O_3R''.$$

Hence

$$O_3R'' \parallel^l O_1C \text{ and} = O_1C;$$
$$\therefore \parallel^l OO_1 \text{ and} = OO_1:$$

so that

$$OR'' \parallel^l \text{ and} = O_1O_3 = r.$$

Also

$$CR' \parallel^l O_1O_3.$$

Hence

$$OR''R' = CR'R'' = \pi/2,$$

and the locus traced by R′ is the pedal of the circle of centre O and radius r for a pole at C.

Cor. 7. The limaçon cannot be rectified (*vide* Nicole in *Actis Academiæ Parisiensis*, 1707).

Cor. 8. It is a conchoid of the circle whose diameter is OC. For let OP cut this circle in G. Hence $PG = PO + OG = r - d \cos \theta + d \cos \theta = r$, and the curve is a conchoid for the pole at O and the constant r.

"From this it is obvious that the curve is the conchoid of circular base described by De la Hire in 1708, and which Roberval and Pascal have also discussed. None of these, however, as far as I know, observed that it is an Epicycloid ('Eorum tamen nemo quantum novi eam Epicycloidem esse observavit')."

Loria (*Ebene Kurven*) ascribes the discovery of this property to Cramer, but clearly Maclaurin has a prior claim. It is one of the few occasions on which he *does* lay claim to a discovery, and he should certainly be credited with it.

Cor. 9. The mid-points of the chords through O lie on a circle.

Generalisation.

Cor. 10. Take two fixed points O and O′, P any point on the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let OPT be a constant angle, which, without loss of generalisation, Maclaurin takes to be a right angle. Draw O′T ⊥^{r} PT.

If the origin is chosen at O and O′ is the point (d, o), the equation to the locus of T is

$$(x^2 + y^2)^2 + (2gx + 2fy)(x^2 + y^2)$$
$$+ cx^2 + y^2(d^2 + c + 2gd) - 2fdxy = 0 \quad . \quad . \quad . \quad . \quad (2)^*$$

* The pedals of the central conics give rise to the rational bicircular quartics (Khon Loria, *Encykl. d. Math. Wiss.*).

It is obtained from the circle in the same way as the rational circular cubics are obtained from the straight line (and is likewise a rational or unicursal curve). It appears to be the simplest of the curves of the fourth order, the conchoid excepted, just as the circular cubics with a double point are the simplest of the third order.

[Teixeira has shown that just as the rational circular cubic is the cissoidal of a circle and a straight line, so these bicircular quartics are the cissoidals of a circle and a circle.]

§ 50. *Prop. XI.*

Pedals of the Conic Sections.

(I.) For the Parabola.

The pedal of the focus is the tangent at the vertex. Hence by Cor. 8 of Prop. IX the pedal of any other point O' is the rational circular cubic already discussed in Lemma II of Part I. The curve has a double point at the pole with real, coincident, or imaginary tangents according as the pole is outside, on, or inside the parabola. It has a line of symmetry when the pole is on the axis of the parabola, and is the cissoid of Diocles when the pole is the vertex of the parabola.

(II.) For the Ellipse.

The pedal of the focus is the major auxiliary circle (Maclaurin's theorem). Hence the pedal of any other point is the bicircular quartic of Cor. 10 of Prop. X.

(III.) For the Hyperbola.

The pedal of the focus is again a circle, and we have a conclusion similar to that of (II).

Cor. If O is a fixed point, P any point on a circle, and OPT a constant angle, then PT envelops a conic section.

§ 51. *Prop. XII.*

When a curve rolls on a congruent curve, corresponding points being points of contact, the roulette of any carried point can be obtained easily as a pedal.

The usual proof is given.

Cor. 1. The curves described by this method coincide with the epicycloids of Nicole generated by a curve rolling on a congruent curve.

Cor. 2. Thus the epicycloids generated by a parabola rolling upon a congruent parabola are (1) a straight line for the focus, (2) a cissoid of Diocles for the vertex, (3) a rational circular cubic for any other point.

Cor. 3. If the generating curves are ellipses, the focus of the moving ellipse generates a circle, and any other point a bicircular quartic.

Similar conclusions hold for the hyperbola.

§ 52. *The curves whose radial equation can be represented in the form*

$$p = \mathrm{A}r^{n+1}$$

or

$$p/r = (r/a)^n.$$

These curves have the property that their pedals for the same pole have a similar radial equation.

Let p_1 and r_1 be the elements of the first positive pedal corresponding to p and r of the given curve.

Then

$$p_1/r_1 = p/r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$r_1 = p\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\therefore\ p = r_1,\ r = r_1{}^2/p_1.$$

But

$$p/r = (r/a)^n,$$

$$\therefore\ p_1/r_1 = (r_1{}^2/ap_1)^n\,;$$

and finally

$$p_1/r_1 = (r_1/a)^{\frac{n}{n+1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Similarly the second positive pedal is given by

$$p_2/r_2 = (r_2/a)^{n/(2n+1)}$$

and the mth pedal by

$$p_m/r_m = (r_m/a)^{\frac{n}{mn+1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Similarly, the mth negative pedal is given by

$$\pi_m/\rho_m = (\rho_m/a)^{\frac{n}{-mn+1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Particular examples of $p-r$ equations are:—

(I.) Circle of radius a, the pole being on the circumference,

$$p/r = r/2a \qquad (n = 1).$$

(II.) The straight line at a distance a from the pole,

$$p/r = a/r \qquad (n = -1).$$

(III.) The Parabola (first negative pedal of the straight line),

$$p/r = (r/a)^{-\frac{1}{2}} \qquad (n = -\tfrac{1}{2}).$$

(IV.) The Rectangular Hyperbola,

$$x^2 - y^2 - a^2$$
$$\therefore\ p/r = a^2/r^2 \qquad (n = -2),$$

the pole being at the centre.

(V.) The Cardioid (first positive pedal of the circle),

$$p/r = (r/2a)^{\frac{1}{2}}$$
$$(i.e.\ p^2 = r^3/2a).$$

(VI.) The Lemniscate (first positive pedal of the rectangular hyperbola of IV),

$$p/r = r^2/a^2$$

or

$$(x^2 + y^2)^2 = a^2(x^2 - y^2)$$

in Cartesian co-ordinates.

(VII.) Maclaurin also gives later the logarithmic spiral* whose $p-r$ equation is

$$p/r = C\ ;\ n = 0,$$

(*vide* Section IV), but this is not an algebraic curve.

§ 53. *Prop. XIV.*

Property of the curve $p/r = (r/a)^n$.

Let B be the point $p = r = a$; this point is a vertex on the curve and its pedals.

The following relation holds (*vide* fig. 40):—

$$\angle P_1OQ_1 = (n + 1)\angle POQ.$$

Dem.

Let the polar co-ordinates of P be $(r,\ \theta)$ and of P_1 $(p,\ \phi)$. Since $p/r = (r/a)^n$, therefore

$$\frac{dp}{p} = (n + 1)\frac{dr}{r}.$$

But by the pedal transformation

$$r\frac{d\theta}{dr} = p\frac{d\phi}{dp},$$

and therefore

$$d\phi = (n + 1)d\theta\ ;\ \therefore \text{ etc.}$$

Cor. 1. In particular, if θ and ϕ are measured from the initial position OB, then

$$\phi = (n + 1)\theta.$$

§ 54. *Prop. XV.*

Maclaurin's theorem regarding the rectification of such curves.

If P traces out the curve $p/r = (r/a)^n$, starting from the vertex B, while

* Since $p/r = p_1/r_1 =$ etc., a logarithmic spiral can be described to pass through the points P, P_1, P_2, etc.

P_1 and N_1 are the points corresponding to P on the first positive and negative pedals, then

$$\text{arc } BP_1 = (n+1)(\text{arc } BN_1 + \text{straight line } N_1P).$$

Dem.

By Prop. XIV (fig. 40)

$$\frac{Q_1R_1}{p} = (n+1)\frac{QR}{r}, \text{ or } \frac{r}{p}Q_1R_1 = (n+1)QR,$$

that is

$$P_1Q_1 = (n+1)QR.$$

Now

$$P_1Q_1 = ds_1 \text{ if } s_1 = \text{arc } BP_1,$$

and

$$QR = QN_1' - RN_1' = QN_1' - PN_1' = QN_1' - PN_1 + N_1'N_1.$$

Hence if

$$\sigma_1 = \text{arc } BN_1,$$
$$QR = d\,.\,PN_1 + d\sigma_1.$$

Thus

$$ds_1 = (n+1)(d\,.\,PN_1 + d\sigma_1)$$

and

$$\text{arc } BP_1 = (n+1)(\text{arc } BN_1 + PN_1).$$

[The following analytical proof* may be given.

Let (r, θ) be the polar co-ordinates of any point P on the curve, and let P_1 and N_1 correspond to P on the first positive and first negative pedals respectively. Let the positive direction of the arc s be such that s increases as θ increases, and let the positive direction of the tangent PP_1 be that of the curve at P. Also let ψ be the angle from the positive direction of OP to that of PP_1. Then

$$\overline{OP} = r\,;\ ds = dr \sec\psi.$$

Let

$$p = \overline{OP}_1 = r \sin\psi$$

and

$$t = \overline{P_1P} = -\overline{PP}_1 = r\cos\psi,$$

so that

$$t^2 + p^2 = r^2.$$

Then

$$dt = \frac{r}{t}dr - \frac{p}{t}dp$$

$$= dr \sec\psi - \frac{p}{r}dp \sec\psi$$

$$= ds - \frac{p}{r}ds_1,$$

where ds_1 is the element of arc at P_1.

Similarly, if $\varpi, \rho, \sigma, \tau$ correspond on the first negative pedal to p, r, s, t,

* Suggested by Professor G. A. Gibson.

$$d\tau = d\sigma - \frac{\varpi}{\rho}ds = d\sigma - \frac{p}{r}ds \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But

$$ds_1 = dp \sec\psi = (n+1)\frac{p}{r}ds \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where

$$p/r = (r/a)^n.$$

Hence

$$ds_1 = (n+1)(d\sigma - d\tau),$$

so that

$$\text{arc BP}_1 = (n+1)(\text{arc BN}_1 - \overline{\text{PN}_1})$$
$$= (n+1)(\text{arc BN}_1 + \overline{\text{N}_1\text{P}}).$$

The signs of BP_1, BN_1, and N_1P must be attended to. Thus when $n>1$ the angle BON_1, which is equal to $(1-n)\theta$, is of opposite sign to that of BOP and of BOP_1; the arcs BP_1 and BN_1 are therefore of opposite sign, and N_1P has the same sign as arc BP_1. Numerically, the arc BP_1 is less than $(n+1)$ times the length of the line PN_1 by $(n+1)$ times the arc BN_1.]

This interesting theorem Maclaurin proceeds to apply to deduce various conclusions regarding the rectification of the series of curves formed by a given curve, $p/r=(r/a)^n$, along with its positive and negative pedals.

Cor. 1. If two consecutive curves of the series admit of rectification, so do all.

Cor. 2. If one of the curves admits of rectification, but not the next in the series, then half only of the curves of the series admit of rectification.

Cor. 5. When the first negative pedal passes through the pole, the theorem for the total lengths from B may be written

$$s_1 = (n+1)\sigma_1.$$

For in such a case PN_1 vanishes.

§ 55. *Prop. XVII.*

When the given curve is a circle of radius $a/2$, and the radial equation is $p/r=r/a$, the first negative pedal reduces to a point B.

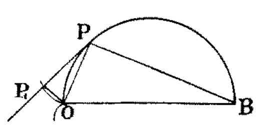

FIG. 43.

The first positive pedal is the cardioid BP_1, and for it

$$\textit{arc } BP_1 = 2 \textit{ chord } BP \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Thus half the complete cardioid $=2a$, and the whole length $=4a$.

The pedal of the cardioid is given by

$$p/r=(r/a)^{\frac{1}{3}}$$

and

$$\text{arc BP}_2=\frac{3}{2}(\text{arc BP}+\text{P}_1\text{P}).$$

Hence it cannot be measured by a straight line alone, nor by an arc of a circle alone save for the complete curve up to O, when

$$\text{BP}_2\text{O}=\frac{3}{2}\text{BPO}=3\pi a/4.$$

The whole curve cannot be cut in any given ratio, for otherwise the quadrature of the circle would follow.

The third positive pedal

$$\frac{p}{r}=\left(\frac{r}{a}\right)^{\frac{1}{4}}$$

has an arc

$$\begin{aligned}\text{BP}_3&=\frac{4}{3}(\text{BP}_1+\text{P}_1\text{P}_2)\\&=\frac{4}{3}(2\text{ chord BP}+\text{line P}_1\text{P}_2)\end{aligned}$$

and

$$\text{BP}_3\text{O}=\frac{8}{3}\text{BO}=\frac{8a}{3}.$$

In general the nth pedal is given by

$$p/r=(r/a)^{1/(n+1)}.$$

If S_n denote the arc of the pedal from B to O,

$$\begin{aligned}\text{S}_n&=\frac{n+1}{n}\text{S}_{n-2}\\&=\frac{n+1}{n}\;\frac{n-1}{n-2}\text{S}_{n-4}\\&=\frac{(n+1)(n-1)\;.\;.\;.\;2}{n(n-2)\;.\;.\;.\;1}\text{OB}\end{aligned}$$

when n is odd,

$$=\frac{(n+1)(n-1)\;.\;.\;.\;3}{n(n-2)\;.\;.\;.\;2}\;\frac{\pi a}{2}$$

when n is even.

The areas of the pedals are also discussed.

§ 56. *Prop. XVIII.*

The pedals of the straight line

$$\frac{p}{r}=\left(\frac{r}{a}\right)^{-1}.$$

The first positive pedal reduces to a point.

The negative pedals are given by

$$\frac{p}{r}=\left(\frac{r}{a}\right)^{\frac{-1}{1+m}}=\left(\frac{a}{r}\right)^{\frac{1}{m+1}}.$$

The first negative pedal is the parabola

$$p/r=\left(\frac{r}{a}\right)^{-\frac{1}{2}}.$$

The second negative pedal is given by

$$p/r=\left(\frac{r}{a}\right)^{-\frac{1}{3}},$$

whose arcs can be expressed by straight lines. Only these may increase beyond all limit, as the curve goes to infinity with the parabola.

We thus form two sets of curves: in one set the arcs can be expressed by parabolic arcs and straight lines, and in the other set by straight lines only.

§ 57. *Prop. XIX.*

The pedals of the equilateral hyperbola

$$x^2-y^2=a^2,$$

or

$$p/r=a^2/r^2.$$

The first positive pedal is the lemniscate

$$(x^2+y^2)^2=a^2(x^2-y^2),$$

or

$$p/r=(r/a)^2.$$

Two series of curves are obtained, in one of which arcs are expressible by hyperbolic arcs and straight lines, and in the other by arcs of lemniscates and straight lines.

§ 58. *Prop. XXI.*

The radius of curvature of the curve

$$p/r=(r/a)^n \text{ is } \frac{a^n}{n+1}\frac{1}{r^{n-1}}.$$

Maclaurin proves the more general formula $\rho=r\,dr/dp$, from which the formula is easily deduced.

[§ 59. *Remarks on Pedals.*

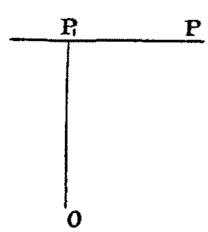

FIG. 44.

If the line PP_1 is given by

$$lx + my - 1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and OP_1 is ⊥r PP_1 through the origin O, the co-ordinates of P_1 are

$$\left.\begin{array}{l}\xi = l/(l^2 + m^2) \\ \eta = m/(l^2 + m^2)\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Also

$$\left.\begin{array}{l}l = \xi/(\xi^2 + \eta^2) \\ m = \eta/(\xi^2 + \eta^2)\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence if the equation in line co-ordinates to a curve is

$$\phi(l, m) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

its first positive pedal has the equation in point co-ordinates

$$\phi\left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence we may, in general, say that *the degree of the first pedal is twice the class of the original curve.*

If

$$f(x, y) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

is the equation in point co-ordinates to the first positive pedal of a curve, the curve itself has the equation in line co-ordinates

$$f\left(\frac{l}{l^2 + m^2}, \frac{m}{l^2 + m^2}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and we should therefore say that *the class of the latter is twice the degree of its first pedal.*

The paradox arising from the application of these theorems is explained in the same way as for the degrees of a curve and its inverse, and both theorems are subject to important modifications. But the analysis given indicates the importance of line co-ordinates in the theory of pedals.

Ex. A conic, being of class 2, has a pedal in general of degree 4, having a special relation to the circular points at infinity.

But if the pole O is at the focus the line equation to the conic is

$$l^2 + m^2 + al + bm + c = 0$$

and its pedal is

$$c(x^2 + y^2) + ax + by + 1 = 0,$$

i.e. a circle.

If the curve is a parabola

$$al^2 + 2hlm + bm^2 + 2gl + 2fm = 0$$

the pedal is a circular cubic; and if the focus is at O the pedal is the straight line

$$2gx + 2fy + a = 0.$$

§ 60. *Differential Geometry of Pedals and Maclaurin's Theorem for the Curves* $p/r=(r/a)^n$.

Use of the $p-r$ *Equation.*

Let, as usual,

$$p,\ r\ ;\ p_1,\ r_1\ ;\ p_2,\ r_2\ ;\ \text{etc.},$$

denote corresponding elements of the curve and its pedals.

Then

$$p/r=p_1/r_1, =p_2/r_2=\text{etc.} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$r_m=p^m/r^{m-1}\ ;\ p_m=p^{m+1}/r^m \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let S, S_1, . . . S_m be corresponding arcs of the curve and its pedals, and let ϕ be the angle between a radius vector and the corresponding tangent.

Then, up to sign

$$\left.\begin{aligned} ds&=dr\ \sec\phi=rdr/\sqrt{r^2-p^2}\\ ds_1&=dr_1\ \sec\phi=rdp/\sqrt{r^2-p^2}\\ ds_2&=dr_2\ \sec\phi=2\frac{p}{r}ds_1-\frac{p^2}{r^2}ds\\ ds_m&=2\frac{p}{r}ds_{m-1}-\frac{p^2}{r^2}ds_{m-2}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3)$$

Cor. 1.

$$ds_1/ds=dp/dr.$$

Cor. 2. The elimination of p/r from (3) gives rise to a variety of results, *e.g.* from the equivalents of ds_2 and ds_3 in (3) we deduce

$$4\begin{vmatrix} ds_1 & ds_2\\ ds_2 & ds_3\end{vmatrix}\times\begin{vmatrix} ds & ds_1\\ ds_1 & ds_2\end{vmatrix}=\begin{vmatrix} ds & ds_2\\ ds_1 & ds_3\end{vmatrix}^2 \quad . \quad . \quad . \quad (4)$$

Also

$$\begin{vmatrix} ds_k & ds_{k+1} & ds_{k+2}\\ ds_l & ds_{l+1} & ds_{l+2}\\ ds_m & ds_{m+1} & ds_{m+2}\end{vmatrix}=0 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Let the tangent

$$\begin{aligned} PP_1&=t\\ \ldots\ P_1P_2&=t_1\\ \ldots\ P_2P_3&=t_2,\ \text{etc.}\end{aligned}$$

Then

$$\left.\begin{aligned} t&=\sqrt{r^2-p^2}\\ t_1&=\frac{p}{r}\sqrt{r^2-p^2}\\ t_2&=\frac{p^2}{r^2}\sqrt{r^2-p^2}\\ &\text{etc.}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence

$$\left.\begin{aligned} dt &= \frac{rdr - pdp}{\sqrt{(r^2 - p^2)}} = ds - \frac{p}{r}ds_1 \\ dt_1 &= ds_1 - \frac{p}{r}ds_2 \\ dt_2 &= ds_2 - \frac{p}{r}ds_3 \\ &\text{etc., etc.} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (7)$$

Also

$$\begin{vmatrix} ds_k - dt_k & ds_{k+1} \\ ds_l - dt_l & ds_{l+1} \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Cor. Owing to the homogeneity, of degree zero in p and r, of the expressions for $ds, ds_1, \ldots ds_m$; $dt, dt_1, \ldots$, it follows that any linear homogeneous equation in these is immediately integrable in terms of p and r.

§ 61. *Maclaurin's Theorem.*

Let us seek to determine the curves for which

$$ds_2 = \text{A}ds + \text{B}dt \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

A and B being constants.

Here

$$2\frac{p}{r}ds_1 - \frac{p^2}{r^2}ds = \text{A}ds + \text{B}dt.$$

Hence

$$2\frac{p}{r}dp - \frac{p^2}{r^2}dr = \text{A}dr + \text{B}\left(dr - \frac{p}{r}dp\right) \quad . \quad . \quad . \quad . \quad (2)$$

In (2) put

$$p = ry,$$

$$\therefore \frac{dr}{r} = dy(2 + \text{B})y/\{\text{A} + \text{B} - (1 + \text{B})y^2\} \quad . \quad . \quad . \quad . \quad (3)$$

the integral of which is

$$r\{\text{A} + \text{B} - y^2(1 + \text{B})\}\frac{2 + \text{B}}{2 + 2\text{B}} = \text{C} \quad . \quad . \quad . \quad . \quad (4)$$

In particular, when $\text{B} = -\text{A}$

$$r(1 - \text{A}y^2)^{\frac{2-\text{A}}{2-2\text{A}}} = \text{C}',$$

or

$$r\left(\frac{p}{r}\right)^{\frac{2-\text{A}}{1-\text{A}}} = \text{K} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The corresponding equation to s_1 is then

$$\frac{r_1^2}{p_1}\left(\frac{p_1}{r_1}\right)^{\frac{2-\text{A}}{1-\text{A}}} = \text{K},$$

or

$$r_1^{-A}p_1 = K' \qquad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\therefore p_1 = K'r_1^A \qquad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Thus Maclaurin's theorem is established by the important converse that only such curves (7) obey this law.

§ 62. The curves of Maclaurin are the so-called sine spirals, an account of which will be found in chap. xviii of Loria's *Ebene Kurven.* From Maclaurin's thorough discussion of them it might have been better to have called them the Curves of Maclaurin.

The sine spirals are defined in polar co-ordinates by an equation of the form

$$r^n = A \sin n\theta.$$

It is easy to see that for any curve

$$r\frac{d\theta}{dr} = \frac{p}{\sqrt{(r^2 - p^2)}} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$d\theta = \frac{dr}{r}\frac{p/r}{\sqrt{(1 - p^2/r^2)}} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence, when

$$\frac{p}{r} = Cr^n \qquad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$d\theta = \frac{Cdr\, r^{n-1}}{\sqrt{(1 - C^2r^{2n})}} \qquad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\therefore n(\theta + a) = \sin^{-1}(Cr^n)$$

or

$$Cr^n = \sin n(\theta + a) \qquad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\therefore \text{etc.}$$

Note.—Maclaurin's Theory of Pedals (including the Theorem for the Sine Spirals) was originally published in 1718 in the *Philosophical Transactions.* In substance it is the same as in the *Geometria Organica,* but the method of fluxions is used more freely in the earlier work.

His "New Universal Method of describing curves of any order by the sole use of given angles and straight lines" appeared in 1719, likewise in the *Philosophical Transactions.* The account given is very brief, and there is inaccuracy in the theory of double points.

SECTION IV.

63. This section is concerned with applications to mechanics.

SECTION V.

On the Description of Geometrical Curves through Given Points

§ 64. *Lemma III.*

A curve C_n meets a conic in $2n$ points and a cubic in $3n$ points.

The proof is analytical.

From it is suggested:

Cor. 1. Two curves C_m and C_n seem to cut in mn points.

This is easily proved when one of the curves is $y=x^m$; but the general demonstration is beyond Maclaurin's powers. The truth of the statement is assumed in what follows.

Cor. 2. Two curves of degree n cut in n^2 points. Thus we may find two curves of degree n through the same n^2 points. Now the equation to C_n involves $\frac{1}{2}(n^2+3n)$ conditions, and $\therefore$ $\frac{1}{2}(n^2+3n)$ points may not be sufficient to determine a curve uniquely when $\frac{1}{2}(n^2+3n)$ is not greater than n^2.

Thus nine points may not uniquely determine a cubic, and yet ten points are *too* many.

[This is the source of the so-called Cramer's Paradox. Cramer, who simply repeats what Maclaurin gives with the additional application to quartics, quotes Maclaurin as his authority (*vide* Cramer, *Courbes algebriques*).

The paradox is therefore Maclaurin's and not Cramer's.]

Cor. 3. If, of the points given to determine a C_n, $nr+1$ lie on C_r, where $n>r$, then either the problem is impossible or the C_n degenerates into C along with C_{n-r}.

Cor. 4. A curve C cannot have more than $\frac{1}{2}(n-1)(n-2)$ double points.

Cor. 5. If, on a curve C_m, three points are multiple of order $m/2$ and one of order $\frac{m}{2}-1$, all the other points will be simple.

§ 65. *Prop. XXV*

shows how to draw a curve C_n through $2n+1$ given points one of which is an $(n-1)$-ple point.

Prop. XXVI

shows how to draw a C_{2n} through as many points as suffice to determine a C_n, and other three points each of which is an n-ple point.

Prop. XXVII

shows how to draw a C_{2n} through $2n+4$ given points, of which three are n-ple points, while a fourth is an $(n-1)$-ple point.

APPENDIX.

In the light of the account just given, the student will find it interesting to examine the following references to Loria's *Ebene Kurven.* The pages refer to the first edition of Loria's treatise.

Page 39.

The locus of the image of the vertex of a parabola in the tangent is a cissoid of Diocles.

Loria refers to Mirman: "Sur la Cissoide de Diokles," *Nouvelles Annales,* 1885.

When a parabola rolls externally on a congruent parabola its vertex describes a cissoid.

Reference to Hendrick's "Demonstration of a Proposition" (*Analyst,* 1877).

Page 48.

The Ophiuride.

Given a right angle OBC on whose sides O and C are fixed points.

Through C is drawn CD cutting OB in D; DM is $\perp^r$ CD, and OM $\perp^r$ DM The locus of M as CD varies is the ophiuride.

Reference to Uhlhorn: *Entwickelungen in der höheren Geometrie,* 1809.

Page 49.

The pedal of a parabola for a pole on the tangent at the vertex is an ophiuride, and a cissoid for the vertex.

Page 60.

The Strophoid.

This name was given by Montucci (*Nouvelles Annales,* 1846).

It is the logocyclic curve of Booth (1877).

Page 69.

The generalised strophoidal curve, given by Maclaurin, is ascribed to Lagrange (*Nouv. Ann.,* 1900).

Page 86.

The trisectrix of Catalan is the first negative pedal of the parabola when the pole is at the focus.

($\therefore$ a sine spiral admitting of rectification.)

Page 89.

The Cubic Duplicatrix of G. de Longchamps.

A is a fixed point, P any point on the y-axis. PQ $\perp^r$ AP meets the x-axis in Q. If QR is drawn parallel to the y-axis and cuts AP in R, R traces the curve in question.

Page 90.

The Parabolic Leaf of De Longchamps, 1890.

A is a fixed point, P any point on the y-axis. PQRS is a rectangle, Q being on OX, R on OY, and S on AP between A and P. The locus of S is the parabolic leaf.

Page 223.

Given a circle of centre O and radius R, the locus of the vertices of the parabolas which touch the circle and have a fixed point on the circumference as focus is a curve whose equation is given by Barisien (*Intermediaire des Math.*, 1896), which Retali (*J. de Math. Spec.*, 1897) observed to be the pedal of the cardioid, when the pole is at the cusp.

Page 498.

The cardioid as a special epicycloid is ascribed to Cramer, and not to Maclaurin.

These extracts may serve to show the importance of Maclaurin's methods in the invention of curves.

The *Geometria Organica* is, in fact, remarkable for the great number and variety of the curves invented by the young Maclaurin, and had he never written another page of mathematics, Maclaurin's name would have been entitled to a conspicuous place in the annals of mathematicians.

If I have succeeded in pointing this out in the foregoing summary of his work, my object in writing it has been attained.

(*Issued separately October* 20, 1916.)