SOME RESULTS CONCERNING THE BEHAVIOUR AT INFINITY OF A REAL AND CONTINUOUS SOLUTION OF AN ALGEBRAIC DIFFERENTIAL EQUATION OF THE FIRST ORDER

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[Received August 22nd, 1911.—Read December 8th, 1910.]

I.

1. The results obtained in this paper have reference to the algebraic differential equation

$$(1) \qquad f(x, y, y') \equiv \Sigma A x^m y^n y'^p = 0,$$

where m, n, p are positive integers. I suppose that this equation possesses a solution

$$(2) \qquad y = y(x),$$

which is real and possesses a continuous derivative for $x > x_0$.* The problem is to specify as completely as possible the various ways in which y may behave as $x \to \infty$.

This problem was first attacked by Borel, in his *Mémoire sur les Séries Divergentes*.† Borel proved that the equation (1) cannot have a solution y, such that

$$y > e^{e^x} = e_2(x)$$

for values of x surpassing all limit. He proved further that

$$(3) \qquad f(x, y, y', y'') = 0$$

cannot have a solution y such that

$$y > e_3(x)$$

for values of x surpassing all limit;‡ and there is no doubt of the truth of

* *I.e.*, for all values of x from some value onwards ("Orders of Infinity," *Camb. Math. Tracts*, No. 12, p. 6). We assume the existence of such a solution: it is not part of the problem to consider conditions for its existence.

† *Annales de l'École Normale*, t. 16, pp. 26 *et seq.*

‡ The proof is not complete, but its general lines are clearly indicated.

the corresponding general theorem, though, so far as I am aware, no strict proof has ever been given.

Borel also devoted a section of his memoir to the subject of *oscillating* solutions, but without obtaining any very definite results.

2. In a short memoir published in 1899, Lindelöf* returned to the questions raised by Borel, and proved the following much more precise result:

If the equation (1) *is of degree m in x, then there is a constant C, such that*

$$y < e^{Cx^{m+1}} \tag{4}$$

for $x > x_0$.

Further, he proved that† either

$$|y| < e^{x^{\delta}},\ddagger \tag{5}$$

or

$$e^{x^{\rho-\delta}} < |y| < e^{x^{\rho+\delta}} \quad (\rho > 0), \tag{6}$$

for $x > x_0$.

The solutions of the first class may oscillate, but those of the second are ultimately monotonic, together with all their derivatives.

3. The subject of the increase (*croissance*) of solutions of the equation (1) has also been considered by Boutroux.§

Boutroux confines himself to the equation

$$y' = P(x, y)/Q(x, y), \tag{7}$$

where P and Q are polynomials; but he considers the whole subject from the point of view of the theory of functions of a complex variable. The distinction between the two classes of solutions (5) and (6) of course appears again, in a more precise form—there are solutions whose increase is less than that of some power of $|x|$, and solutions which, in certain angles, behave like exponentials.

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* *Bulletin de la Société Mathématique de France*, t. 17, p. 205.

† Some of these results are contained in an additional note which is in part due to Borel.

‡ The notation is that explained in my tract cited above and my paper "Properties of Logarithmico-Exponential Functions," *Proc. London Math. Soc.*, Ser. 2, Vol. 10, p. 54.

§ *Leçons sur les fonctions définies par les équations différentielles du premier ordre*, Paris, Gauthier-Villars, 1908.

4. In this paper I consider first the equation (7); but, like Borel and Lindelöf, I consider it exclusively from the point of view of the real variable. I am thus able to obtain results very much more precise than those stated in the preceding sections. I show that *all* solutions of (7) are ultimately monotonic, and specify their possible modes of increase by simple asymptotic formulæ. I also show that substantially the same results hold for the equation

$$y'^{\mu} = P(x, y)/Q(x, y), \tag{8}$$

where μ is *odd*.

I then return to the general equation (1). I find asymptotic formulæ, more precise than Lindelöf's, for the solutions which behave at infinity like exponentials, and I prove that any oscillating solution is ultimately less in absolute value than a power of x—in symbols,

$$y = O(x^{\Delta}).$$

In particular I show that, in the case of the equation (8), with μ *even*, every oscillating solution remains finite, *i.e.*,

$$y = O(1).$$

Finally, I discuss certain particular types of oscillating solutions.

Much of the argument is capable of extension, and results still more accurate may be obtained without the intervention of any fresh difficulty of principle. But, after a certain point, the work becomes too tedious to be justified by the interest of the results.

It would, however, be exceedingly interesting to see how far the methods used in the paper will go in proving the analogous results immediately suggested for equations of order higher than the first. Here I do not go beyond the first order, but I hope to return to the subject at a later opportunity.

II.

5. Let us consider the equation

$$y' = P(x, y)/Q(x, y). \tag{7}$$

I shall prove first that it is impossible that y' should vanish for a series of values of x whose limit is infinity, except of course in the trivial case in which (7) has a solution $y = \text{const.}$ In other words, *every solution is ultimately monotonic.*

Suppose the contrary. Then the curves $y = y(x)$, $P = 0$ intersect at points corresponding to an infinity of values of x surpassing all limit.

But $P = 0$ consists of a finite number of branches, and so $y = y(x)$ must intersect at least one of these infinitely often.

Now the branches of $P = 0$, which extend to infinity in the direction of the axis of x consist of (i) a finite number of straight lines

$$y = c_s \quad (\gamma_s),$$

(ii) a finite number of branches

$$y = \phi_t(x) \quad (\delta_t),$$

along which y ultimately increases or decreases steadily.

In the first place, $y = y(x)$ *cannot cut any* δ_t *in an infinity of points.* Suppose, for example, that y ultimately increases along δ_t, and let P, Q be two successive points of intersection. Then $y = y(x)$ crosses δ_t at P and Q, in each case from above to below (Fig. 1), and a glance at the figure is

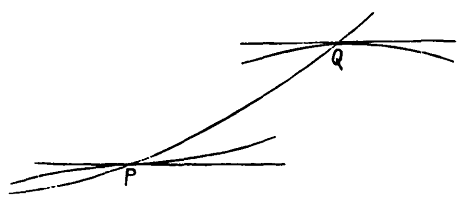

FIG. 1.

enough to show that this is impossible.*

We have now to consider the possible intersections of $y = y(x)$ and γ_s. These fall under the four types represented in Fig. 2.

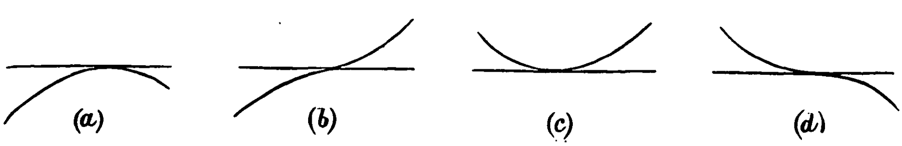

FIG. 2.

Of these we can at once rule out (*a*) and (*c*), since at such points y' would change its sign, and P would not. For a similar reason we can rule out (*b*) and (*d*), *unless the factor* $y - c_s$ *occurs an even number of times in* P. If this is so, and an intersection of (*e.g.*) type (*b*) occurs, it can

* We can suppose x large enough to ensure that P and Q cannot vanish simultaneously. Then it is easy to see that y is regular for a value of x which makes $P = 0$, and hence that there cannot be an infinity of intersections for values of x in the neighbourhood of any finite value. Hence there must be *successive* intersections. We need not elaborate this kind of point in future.

occur *once only*, so far as γ_s is concerned ; for when y has once passed above γ_s it can plainly only come back to γ_s after an intersection of type (a) with some other γ_s.

Hence y is ultimately monotonic.

6. We can go further and say that every derivative $y^{(r)}$ is ultimately monotonic. For, by differentiation and substitution, we find

$$y^{(r)} = P_r(x, y)/Q_r(x, y),$$

where P_r, Q_r are polynomials, and in fact $Q_r = Q^{2r+1}$. Our assertion will therefore follow as a corollary from the following general theorem :—

THEOREM.—*Any rational function*

$$H(x, y) = K(x, y)/L(x, y)$$

is ultimately monotonic along the curve $y = y(x)$—*unless* $L = 0$ *is a solution of the equation* (7).

This theorem I shall now proceed to prove.

7. We have

$$\frac{dH}{dx} = \frac{\partial H}{\partial x} + R\frac{\partial H}{\partial y} = T = \frac{U}{W}, \tag{8}$$

where U and W are polynomials, and d/dx implies differentiation along the curve (2). If dH/dx is not ultimately of constant sign on the curve (2), it must vanish or become infinite infinitely often on (2). In the first case (2) must have an infinity of intersections with at least one of the finite number of branches of

$$U = 0. \tag{9}$$

Now this branch may, for sufficiently large values of x, be represented in the form

$$y = A_0 x^{a_0} + A_1 x^{a_1} + \ldots, \tag{9'}$$

a convergent series of (not generally integral) descending powers of x. If $\delta/\delta x$ refers to differentiation along (9),

$$\frac{\delta y}{\delta x} = A_0 a_0 x^{a_1 - 1} + A_1 a_1 x^{a_1 - 1} + \ldots . \tag{10}$$

Again, along (9), $R(x, y)$ is an algebraic function of x, which may, for sufficiently large values of x, be expressed in the form

$$R = B_0 x^{\beta_0} + B_1 x^{\beta_1} + \ldots, \tag{11}$$

another series of descending powers. And, unless the series (10), (11) are identical, we shall have, at all points of (9) from some definite point onwards,

$$\frac{\delta y}{\delta x} > R \quad \text{or} \quad \frac{\delta y}{\delta x} < R.$$

From this it follows that, at the points of intersection, (2) always crosses (9) from one and the same side to the other and the same side; which is plainly impossible.

On the other hand, if the series (10) and (11) are identical, we have

$$\frac{\delta y}{\delta x} = R,$$

and $U = 0$ is a solution of (7). In other words, H is constant along (2).

There remains only the possibility that

$$\frac{dH}{dx} = \left(L\frac{dK}{dx} - K\frac{dL}{dx}\right) \Big/ L^2$$

should become infinite infinitely often, as we describe (2). This cannot be true owing to K or L or

$$\frac{dK}{dx} = \frac{\partial K}{\partial x} + \frac{\partial K}{\partial y}\frac{dy}{dx}$$

or dL/dx becoming infinite, and so can only occur if L vanishes infinitely often. But then we can show as above that $L = 0$ is a solution of the equation (7).

Thus the proof of the theorem is completed.

COROLLARY.—*Any rational function*

$$H(x, y, y')$$

is ultimately monotonic, unless its denominator vanishes identically in virtue of (7).

The same is true of $H(x, y, y', y'', \ldots)$.

8. We can obtain much more accurate information concerning the increase of the solutions of

$$(7) \qquad Qy' = P.$$

The ratio of any two terms is of one of the forms

$$Ax^m y^n, \quad Ax^m y^n y';$$

and is consequently ultimately monotonic, and so, between any two terms X_i, X_j, there subsists one of the relations

$$X_i \succ X_j, \quad X_i \asymp X_j, \quad X_i \prec X_j.$$

It follows that there must be one pair of terms at any rate such that

$$X_i \asymp X_j.$$

If these two terms come from the same side of (7), we obtain at once

$$y \sim Ax^s, \tag{12}$$

where s is *rational*. If they come from opposite sides, we obtain a relation of the form

$$y^m y' \sim Ax^n. \tag{13}$$

Here four cases present themselves. If $m \neq -1$, $n \neq -1$, we obtain a relation of the type (12). If $m \neq -1$, $n = -1$, we obtain a relation

$$y \sim A(\log x)^{1/p}, \tag{14}$$

where p is an integer. If $m = -1$, $n \neq -1$, we obtain a relation

$$\log y \sim Ax^p,$$

$$y = e^{Ax^p(1+\epsilon)}. \tag{15}$$

Here p may be supposed a positive integer, as if p is negative $y \sim 1$.* Finally, if $m = -1$, $n = -1$, we obtain

$$\log y \sim A \log x,$$

$$y = x^{A+\epsilon}. \tag{16}$$

9. The relations (15), (16) are less precise than (12) and (14). We shall now proceed to examine them more closely.

Let us consider first the exponential solutions (15). We have

$$y' = \frac{P_0 y^r + P_1 y^{r-1} + \dots}{Q_0 y^s + Q_1 y^{s-1} + \dots},$$

where $P_0, \dots, Q_0, \dots$ are polynomials in x. It is clear that $s = r-1$, and that, for sufficiently large values of x, we have

$$y' = R_0 y + R_1 + O(x^{-\Delta}),$$

$$y'/y = R_0 + O(x^{-\Delta}),$$

* p is clearly at most equal to $r+1$, where r is the degree of (1) in x—this, of course, agrees with Lindelöf's result quoted in § 2.

where R_0, R_1 are rational functions of x. Hence

$$y'/y = \Pi(x) + \frac{A}{x} + O\left(\frac{1}{x^2}\right),$$

where Π is a polynomial and A a constant (not necessarily rational). Hence, integrating, we deduce

(17) $$y \sim Ax^a e^{\Pi(x)}.*$$

It is clear that this form includes the form (12).

10. We have now to consider the last case of § 8, which is rather more difficult. There are two terms

(18) $$\lambda x^s y^t y', \quad \mu x^{s-1} y^{t+1}$$

of equal order:† obviously we may suppose that no other term is of greater order. We may go further, and suppose that no other term is of equal order, since the contrary assumption leads at once to a relation of the type (12). We have also

$$y = x^{A+\epsilon}, \quad A = \mu/\lambda.$$

If follows from the theorem of §§ 6, 7 that, if X_i is any third term in the differential equation, the quotient

$$(\lambda x^s y^t y' - \mu x^{s-1} y^{t+1}) / X_i$$

tends to a limit as $x \to \infty$. In other words, the difference of the two principal terms is definitely of order greater than, equal to, or less than that of any third term. We can now distinguish two possibilities.

(a) *There is a third term whose order is equal to that of the difference of the principal terms.*

In this case we have a relation of one of the forms

(19) $$\lambda x^s y^t y' - \mu x^{s-1} y^{t+1} \sim M x^\sigma y^\tau,$$

(20) $$\lambda x^s y^t y' - \mu x^{s-1} y^{t+1} \sim M x^\sigma y^\tau y'.$$

First, suppose (19) holds. Putting

$$y = x^A u = x^{\mu/\lambda} u,$$

* Not, of course, with the *same* A: cf. *Proc. London Math. Soc.*, Ser. 2, Vol. 10, p. 54.

† We say, of course, that X_i is of order greater than, equal to, or less than that of X_j, according as $X \succ X_j$, $X_i \asymp X_j$, or $X_i \prec X$.

and substituting, we obtain

$$u^{t-\tau}u' \sim Nx^{\sigma-s+(\tau-t-1)A}.$$

But, as $u = x^{\epsilon}$, this is only possible if

$$\sigma - s + (\tau - t - 1)A = -1,$$

which shows that A is *rational*, its denominator being $\tau - t - 1$. Also, integrating,

$$u^{t-\tau+1} \sim N \log x.$$

Thus

(21) $$y \sim A\,(x^p \log x)^{1/q},$$

where p and q are integers.

Next, suppose (20) holds. Making the same substitution, we obtain

$$\{\lambda x^{s+(t+1)A}u^t - Mx^{\sigma+(\tau+1)A}u^{\tau}\}\,u' \sim MAx^{\sigma-1+(\tau+1)A}u^{\tau+1}.$$

But $$x^s y^t \succ x^{\sigma} y^{\tau}, \quad x^{s+tA}u^t \succ x^{\sigma+\tau A}u^{\tau},$$

and so $$u^{t-\tau-1}u' \sim Nx^{\sigma-s-1+(\tau-t)A};$$

and the argument may now be completed as before.

(*b*) *There is no third term whose order is equal to that of the difference of the principal terms.*

Let us denote the principal terms by X_1, $\bar{X}_1$. Then there must be at least one term X_2, such that

$$X_2 \succ X_1 - \bar{X}_1;$$

and therefore another term $\bar{X}_2$, such that

$$X_2 \sim \bar{X}_2:$$

and we may suppose, as in the case of X_1, $\bar{X}_1$, that these terms come from opposite sides of the equation. We may also suppose that X_2, $\bar{X}_2$ are of higher order than any other terms other than X_1, $\bar{X}_1$. Further, we may suppose them to be of the form

$$\lambda_2 x^{s_2} y^{t_2} y', \qquad \mu_2 x^{s_2-1} y^{t_2+1},*$$

where $$\mu_2/\lambda_2 = \mu/\lambda = A.$$

Putting $y = x^A u$, we obtain

$$X_1 - \bar{X}_1 = x^{s+(t+1)A}u^t u', \qquad X_2 - \bar{X}_2 = x^{s_2+(t_2+1)A}u^{t_2}u';$$

* If they were not thus related, the increase of y could be determined at once as in §§ 8, 9.

from which it at once follows that

$$X_1 - \bar{X}_1 \succ X_2 - \bar{X}_2.$$

But in this case there must be a fifth term X_3, whose order is greater than that of $X_1 - \bar{X}_1$, and a sixth term $\bar{X}_3$, such that

$$X_3 \sim \bar{X}_3,$$

and we can prove that

$$X_1 \succ X_2 \succ X_3, \qquad X_1 - \bar{X}_1 \succ X_2 - \bar{X}_2 \succ X_3 - \bar{X}_3.$$

And as this argument may be repeated indefinitely, and the number of terms is finite, we must find sooner or later that the supposition (*b*) leads either to the conclusion we desire or to a contradiction.

11. We have thus proved the following theorem :—

Any solution y of the equation*

$$\frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)}$$

is ultimately monotonic, together with all its derivatives, and satisfies one or other of the relations

$$y \sim Ax^a e^{\Pi(x)}, \qquad y \sim A\,(x^p \log x)^{1/q},$$

where $\Pi(x)$ *is a polynomial, and* p, q *are integers.*

These rates of increase are naturally included among the standard asymptotic forms for logarithmico-exponential functions of order 1,† of which they are quite special cases.

12. It is natural to attempt to extend our results to the more general equation

(8) $$y'^{\mu} = P(x, y)/Q(x, y).$$

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* We are, of course, confining ourselves to continuous solutions: see § 1.

Examples.—The solution of $x^2 y' = (x+1)^2 y$ is

$$y = Ax^2 e^{x-(1/x)} \sim Ax^2 e^x;$$

the solution of $2x(x+1)yy' = xy^2 + (x+1)^2$ is

$$y = \sqrt{\{(x+1)(\log x + A)\}} \sim \sqrt{(x \log x)}.$$

† *Proc. London Math. Soc.*, Ser. 2, Vol. 10, p. 76.

If μ is odd this offers no new difficulties: all our arguments apply, with appropriate modification of detail.*

But if μ is even our results are obviously no longer true. Thus

$$y'^2 = 1-y^2$$

possesses the oscillating solution

$$y = \sin x.$$

I shall now proceed to consider the general equation (1), and the particular equation (8), with μ even, with the especial idea of discovering to what limitations the existence of oscillating solutions is subject.

III.

18. I return now to the general equation (1). We can distinguish various possibilities.

(*a*) It may be possible to find a positive ρ such that

$$y > e^{x^\rho}, \tag{22}$$

for an infinity of values of x surpassing all limit. In this case Lindelöf has shown that this inequality holds for *all* sufficiently large values of x, and that y and all its derivatives are ultimately monotonic.†

In this case any rational function

$$H(x, y, y')$$

is ultimately monotonic. For, if we eliminate y' between

$$H = H(x, y, y'), \quad f(x, y, y') = 0,$$

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* The standard forms of increase are

$$y \sim Ax^a e^{Bx^{p/\mu}+Cx^{(p/\mu)-1}+\dots}, \quad y \sim A(x^p lx)^{1/q}, \quad y \sim A(lx)^{\mu/q}.$$

In the first of these x^a can occur only if p/μ is integral. The form

$$y \sim A(x^p lx)^{\mu/q},$$

can only occur if $\mu = 1$ or $p = 0$.

† Lindelöf (*l.c.*) shows that if $y = y(x)$ cuts $y = e^{x^\rho}$ at points whose abscissæ surpass all limit, we can find values of x surpassing all limit for which

$$y' = \rho x^{\rho-1} y, \quad y > e^{x^\rho}.$$

Substituting in (1), we obtain $\quad f_1(x, x^{\rho-1}, y) = 0,$

where f_1 is a polynomial; and it is impossible that f_1 should vanish for an infinity of pairs of values

$$(\xi_i, \eta_i), \quad \eta_i > e^{\xi_i^\rho},$$

unless it vanishes identically.

we obtain an algebraic relation

$$F(x,\, y,\, H) = 0\,;$$

and so the points at which $\dfrac{dH}{dx} = 0$

lie on an algebraic curve, which plainly contradicts (22).

We can now argue as in § 8. The equation (1) must contain two terms of equal order, and so we deduce

$$A_1 x^{m_1} y^{n_1} y'^{p_1} \sim A_2 x^{m_2} y^{n_2} y'^{p_2},$$

$$y^{\mu} y' \sim A x^{\nu}.$$

Here μ and ν are rational, and μ must plainly be -1. Hence

(23) $$y = e^{Ax^s(1+\epsilon)}.$$

It is clear that s can be at most greater by unity than the degree of (1) in x.*

(*b*) It may be possible to find a number K such that

$$y = O(x^K).$$

14. It is obvious that (*a*) and (*b*) do not exhaust the *a priori* possibilities. It is our object now to prove that no other case is really possible.

If we are not in Case (*b*), it is possible, however large be Δ, to find values of x such that

$$y > x^{\Delta}.$$

We can therefore choose an increasing sequence (Δ_ν), whose limit is infinity, and a corresponding sequence (x_ν), such that

$$y(x) > x^{\Delta_\nu} \quad (x = x_\nu).$$

We shall now construct a curve

(24) $$y = x^{\Delta(x)} = e^{\Delta(x)\log x} = e^{\phi(x)},$$

passing through the points $(x_\nu,\, x_\nu^{\Delta_\nu})$ and satisfying certain conditions.

In the first place, we can suppose $\Delta'(x)$, and *a fortiori* $\phi'(x)$, positive and continuous. And we may suppose $\phi \prec x^{\delta}$, and *a fortiori* $\Delta \prec x^{\delta}$, since otherwise we should find ourselves again in Case (*a*).

* We can treat similarly the case in which y is ultimately negative.

Further, since we are at liberty to suppose the increase of the sequence (Δ_ν) as slow as we like, we may suppose that

$$x^{1-\delta}\,\Delta'(x) \to 0$$

for any positive δ.*

Now $$\phi'(x) = \Delta'(x)\log x + \frac{\Delta(x)}{x}.$$

Hence

(25) $$x\phi' \to \infty, \quad x^{1-\delta}\phi' \to 0.$$

15. We have

(26) $$y(x) > e^{\phi(x)},$$

for an infinity of values of x surpassing all limit. We shall now show, by a modification of Lindelöf's argument, that this inequality must hold for all sufficiently large values of x.

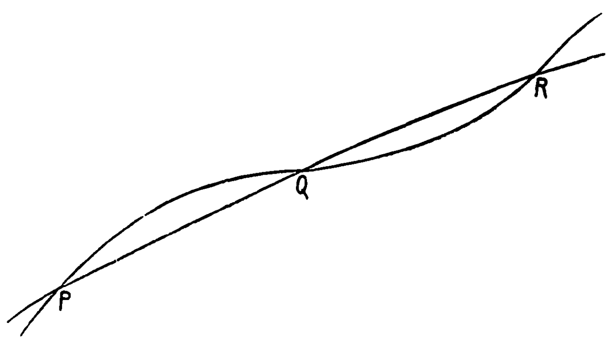

FIG. 3.

If this is not so the curves (2) and (24) must intersect in an infinity of points such as P, Q, R.†

At P (Fig. 3), we have

$$y = e^{\phi}, \quad y' \geqslant \phi' e^{\phi} = \phi' y,$$

and at Q we have $$y = e^{\phi}, \quad y' \leqslant \phi' e^{\phi} = \phi' y.$$

As $y' - \phi' y$ is continuous, there must be a point between P and Q where

$$y' = \phi' y, \quad y \geqslant e^{\phi},$$

* A supposition equivalent, in ordinary cases, to $\Delta(x) \prec x^{\Delta}$.

† The argument is not affected if some of these points are points of *contact*.

and so there must be values of x surpassing all limit for which these relations hold. For these values of x,

$$f(x, y, \phi' y) = \Sigma A x^m y^{n+p} \phi'^p = 0.$$

All the terms in this equation (except those for which $n = 0$, $p = 0$) are large compared with any power of x,* and it is clear that, for any value of x for which the equation holds, there must be two terms such that

$$H x^m y^{n+p} \phi'^p < x^{m'} y^{n'+p'} \phi'^{p'} < K x^m y^{n+p} \phi'^p,$$

where H and K are numbers depending only on the form of the equation. Further, it is clear that

$$n+p = n'+p',$$

and so we have

$$H_1 < x^{m'-m} \phi'^{p'-p} < K_1,$$

say. But this plainly contradicts the relations (25), unless $m = m'$, $p = p'$, which is impossible.

16. The inequality (26) therefore holds for all sufficiently large values of x. But we can now prove, as in § 13, that any rational function $H(x, y, y')$ is ultimately monotonic, and thus arrive at the equation (23). We have thus proved the following theorem :†—

If y is any solution of the equation

$$f(x, y, y') = 0,$$

we have either

$$y = O(x^{\Delta})$$

or

$$y = e^{Ax^s(1+\epsilon)},$$

where s is rational. All solutions of the latter class are monotonic, together with all their derivatives.

IV.

17. I shall now resume the consideration of the special equation

$$(8) \qquad y'^{\mu} = P/Q,$$

where μ is *even*—if μ is odd, we have already seen that there can be no oscillating solutions.

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* Since $y \geqslant x^{\Delta(x)}$ and $\Delta(x) \to \infty$.

† It is hardly necessary to point out again that we are considering only continuous solutions.

We have seen that any oscillating solution of the general equation (1)* must satisfy

$$y = O(x^K).$$

When the equation has the special form (8) we can go much further, and assert that any oscillating solution satisfies

$$y = O(1),$$

i.e., oscillates *finitely*.

In fact if, as in § 5, we denote by γ_s, δ_t the branches of $P = 0$ which stretch to infinity in the direction of the axis of x, we can still show, by the argument used there, that $y = y(x)$ cannot cut any δ_t infinitely often. It follows that y cannot (for sufficiently large values of x) increase beyond the greatest of the numbers c_s. For if it did so it would necessarily continue to increase until $y = y(x)$ met one of the branches δ_t. Hence y can oscillate at most finitely.

We can go further, and assert that, along any branch which does not remain finite, $H(x, y)$, any rational function of x and y—and so also any $H(x, y, y')$—is ultimately monotonic. For

$$\frac{dH}{dx} = \frac{\partial H}{\partial x} + R^{1/\mu}\frac{\partial H}{\partial y}.$$

Let

$$S = \left(\frac{\partial H}{\partial x}\right)^{\mu} - R\left(\frac{\partial H}{\partial y}\right)^{\mu},$$

and suppose, if possible, that $y = y(x)$ meets a branch of $S = 0$ infinitely often.

Along such a branch we have, as in § 7,

$$y = A_0 x^{a_0} + A_1 x^{a_1} + \ldots,$$

$$\frac{\delta y}{\delta x} = A_0 a_0 x^{a_0 - 1} + A_1 a_1 x^{a_1 - 1} + \ldots . \dagger \tag{26}$$

Also, along this branch $R = B_0 x^{\beta_0} + B_1 x^{\beta_1} + \ldots,$

$$\frac{dy}{dx} = R^{1/\mu} = \pm (C_0 x^{\gamma_0} + C_1 x^{\gamma_1} + \ldots), \tag{27}$$

as $R^{1/\mu}$ has two real values, equal and opposite.

We can now prove without difficulty that the assumption of an infinity of intersections leads to a contradiction. Let P, Q, R, ... be successive

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* Such as $y = x \sin x$, which is a solution of

$$(xy' - y)^2 = x^2 (x^2 - y^2).$$

† As in § 7, $\delta y / \delta x$ refers to $S = 0$, and dy/dx to $y = y(x)$.

intersections. These must correspond alternately to the two signs in (27). For, if, *e.g.*, P and Q corresponded to the same sign, $y = y(x)$ would cross $S = 0$ in the same sense at P and Q (Fig. 4*a*), which is manifestly impossible. On the contrary hypothesis (Fig. 4*b*) it is clear that we could find a system of values x_ν, tending to infinity, and such that

$$y(x_\nu) \to \infty, \qquad y'(x_\nu) = 0;$$

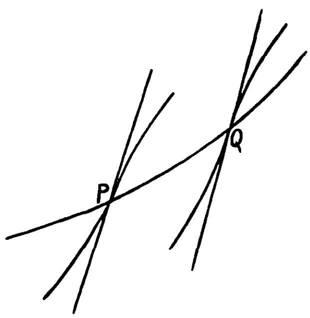

FIG. 4*a*.

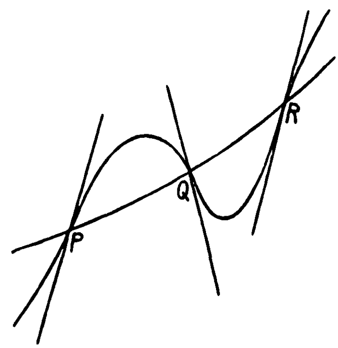

FIG. 4*b*.

and this possibility has already been excluded. Thus it has been shown that, unless y remains finite, $H(x, y)$, $H(x, y, y')$, ... are ultimately monotonic.

We can now show, as in §§ 8, 13, that *any solution of* (8), *which does not remain finite, is determined asymptotically by one or other of the formulæ*

$$y = e^{Ax^s(1+\epsilon)}, \ \ldots$$

obtained in §§ 11, 12.

18. I shall conclude this paper by considering a few cases in which it is possible to obtain more precise information concerning the oscillating solutions.

First, let us suppose, in the equation (8), that P has no factors

$$(y - c_s)^{k_s},$$

in which k_s is even. Then $y = y(x)$ cannot *cross* a line $y = c_s$, since this would involve a change of sign on the part of P. Thus y remains continually between two adjacent lines $y = c_s$, attaining in succession maxima on the upper line and minima on the lower. In Borel's terminology, *the oscillation of y is of a simple and regular sinusoidal type.*

Suppose, in particular, that $\mu = 2$. Then it can be proved that, if y attains the value c_s, $y - c_s$ can occur in P as a simple factor only. For, if

$y = c_s$ for $x = \xi$, we have, near $x = \xi$,

$$y - c_s = A(x-\xi)^p + \ldots,$$
$$y'^2 = B(x-\xi)^{2p-2} + \ldots,$$

and $2p-2 = pk_s$, which is only possible if $p = 2$, $k_s = 1$.*

We may suppose, without loss of generality, that the lines between which y oscillates are $y-1 = 0$ and $y+1 = 0$. We have then

$$y'^2 = (1-y^2)\, S(x, y),$$

where $S > 0$. Further,

$$S = \frac{P_0 x^m + P_1 x^{m-1} + \ldots}{Q_0 x^n + Q_1 x^{n-1} + \ldots},$$

where $P_0, P_1, \ldots, Q_0, \ldots$ are polynomials in y. Suppose, to avoid complications of detail, that $P_0 > 0$, $Q_0 > 0$ for $-1 \leqslant y \leqslant 1$.† Then

$$S = R_0 x^{m-n} + O(x^{m-n-1}),$$

where R_0 is a rational function of y. Putting $y = \sin\theta$, we obtain

$$\theta'^2 = R_0(\sin\theta)\, x^{\frac{1}{2}(m-n)}(1+\epsilon),$$

or

$$\int \frac{d\theta}{\sqrt{R_0}} \sim Ax^s,$$

say. This involves a relation of the type

$$\theta \sim Bx^s.‡$$

Thus y behaves, to put it roughly, like

$$\sin(Bx^s).§$$

19. When $\mu > 2$, we can, of course, obtain more complicated types of oscillating solutions.

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* Consider, for example, the equation

$$y'^2 = (1-y^2)^2 x.$$

We find as the general solution $\quad y = \pm \tanh(\tfrac{2}{3}x^{\frac{3}{2}} + C)$,

and y never attains the values ± 1. This is an example of a finite non-oscillating solution.

† P_0/Q_0 cannot change its sign, but P_0 or Q_0 might vanish, when we should have to take account of the other terms.

‡ If $\theta = 2\nu\pi + \phi$, where $0 < \phi < 2\pi$,

$$\int \frac{d\theta}{\sqrt{R_0}} = \nu \int_0^{2\pi} \frac{d\theta}{\sqrt{R_0}} + O(1).$$

§ A simple example of such a solution is provided by the trochoidal curve

$$x^m = \theta - a\cos\theta, \quad y = \sin\theta \quad (|a| < 1),$$

which satisfies the equation $\quad y'^2 = \dfrac{m^2 x^{2m-2}(1-y^2)}{(1+ay)^2}.$

It is easily verified that $y = \sin^3 x$

satisfies $$y'^6 + 27y^2y'^4 + 243y^4y'^2 = 729y^4(1-y^2).$$

This suggests that the equation

$$y'^6 = Ay^4(1-y^2)$$

has an oscillating solution of the type shown in Fig. 5, and it is easy to

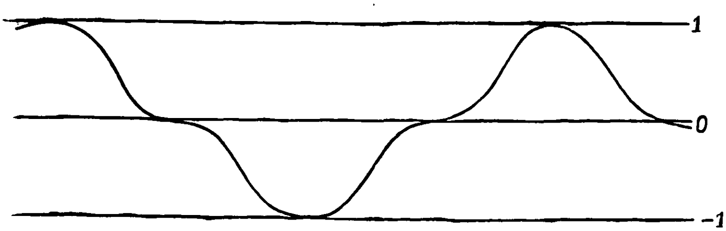

FIG. 5.

verify that this is the case. If we take the more general equation

$$y'^{\mu} = Ay^a(1-y^2)^b, \quad (a, b > 0),$$

we find, as the conditions for the existence of continuous oscillating solutions

$$\frac{a}{\mu} < 1, \quad \frac{b}{\mu} < 1, \quad \frac{a}{\mu} = \frac{2p}{2q+1}, \quad 1 - \frac{2b}{\mu} = \frac{2r}{2s+1},$$

where p, q, r, s are integers. These cannot be satisfied if μ is odd (as is of course to be expected). For even values of μ we find, as possible cases, $\mu = 2$, $a = 0$, $b = 1$; $\mu = 6$, $a = 0$ *or* 4, $b = 1$, 3, *or* 5; and so on. The case mentioned above corresponds to $\mu = 6$, $a = 4$, $b = 1$.

[It is only since writing this paper that I have become acquainted with an important series of memoirs by Kneser and Horn, which deal with a variety of questions concerning the asymptotic behaviour of functions defined by differential equations. These memoirs are for the most part developments of the work of Poincaré on linear differential equations. The point of view adopted is very different from that of Borel and Lindelöf, and so far as I know none of the preceding results are contained in any of them. But the bibliographical indications of §§ 1–4 would be incomplete if I did not refer to them, and I accordingly add the following references, without professing that the list is complete:—

A. Kneser.—*Math. Annalen*, 42, p. 409. *Crelle's Journal*, 116, p. 178; 117, p. 72; 120, p. 267.

J. Horn.—*Math. Annalen*, 49, p. 453; 50, p. 525; 51, p. 346 and p. 360; 52, p. 271 and p. 340. *Crelle's Journal*, 116, p. 265; 117, p. 104 and p. 254; 118, p. 257; 119, p. 196 and p. 267; 120, p. 1.

Added January, 1912.]