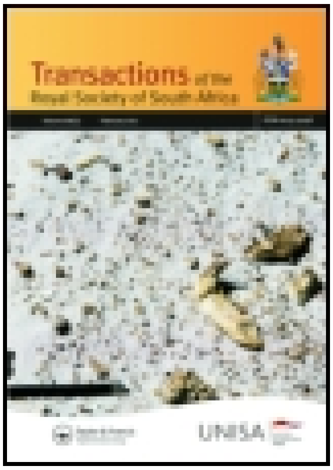

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SECOND NOTE ON THE DETERMINANT OF THE SUM OF TWO CIRCULANT MATRICES

Sir Thomas Muir LL.D. [a]
[a] Rondebosch, S.A.February 22, 1920.
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SECOND NOTE ON THE DETERMINANT OF THE SUM OF TWO CIRCULANT MATRICES.

By Sir Thomas Muir, LL.D.

(1) Almost forty years ago the curious identity

$$\begin{vmatrix} 1 & a_2+b_2 & a_3+b_3 & a_4+b_4 & a_5+b_5 \\ 1 & a_1+b_3 & a_2+b_4 & a_3+b_5 & a_4+b_1 \\ 1 & a_5+b_4 & a_1+b_5 & a_2+b_1 & a_3+b_2 \\ 1 & a_4+b_5 & a_5+b_1 & a_1+b_2 & a_2+b_3 \\ 1 & a_3+b_1 & a_4+b_2 & a_5+b_3 & a_1+b_4 \end{vmatrix} = \begin{vmatrix} 1 & a_2-b_2 & a_3-b_3 & a_4-b_4 & a_5-b_5 \\ 1 & a_1-b_3 & a_2-b_4 & a_3-b_5 & a_4-b_1 \\ 1 & a_5-b_4 & a_1-b_5 & a_2-b_1 & a_3-b_2 \\ 1 & a_4-b_5 & a_5-b_1 & a_1-b_2 & a_2-b_3 \\ 1 & a_3-b_1 & a_4-b_2 & a_5-b_3 & a_1-b_4 \end{vmatrix}$$

was pointedly drawn attention to* in the hope that a purely determinant proof might be forthcoming. In the long-continued absence of such a proof I propose to supply one, not so much, however, on account of the importance of the identity itself as of the incidental and subsequent theorems to which the attempt has led up.

(2) At the outset it is clear that each of the two determinants involved is expressible as a sum of sixteen (2^4) determinants with monomial elements: and, further, that as they differ only in the signs of the b's, a number of the one set of sixteen must cancel the same number of the other set. For shortness' sake and definiteness of statement let us denote the sixteen on the left, namely,

$$\begin{vmatrix} 1 & a_2 & a_3 & a_4 & a_5 \\ 1 & a_1 & a_2 & a_3 & a_4 \\ 1 & a_5 & a_1 & a_2 & a_3 \\ 1 & a_4 & a_5 & a_1 & a_2 \\ 1 & a_3 & a_4 & a_5 & a_1 \end{vmatrix}, \quad \begin{vmatrix} 1 & a_2 & a_3 & a_4 & b_5 \\ 1 & a_1 & a_2 & a_3 & b_1 \\ 1 & a_5 & a_1 & a_2 & b_2 \\ 1 & a_4 & a_5 & a_1 & b_3 \\ 1 & a_3 & a_4 & a_5 & b_4 \end{vmatrix}, \quad \ldots\ldots\ldots$$

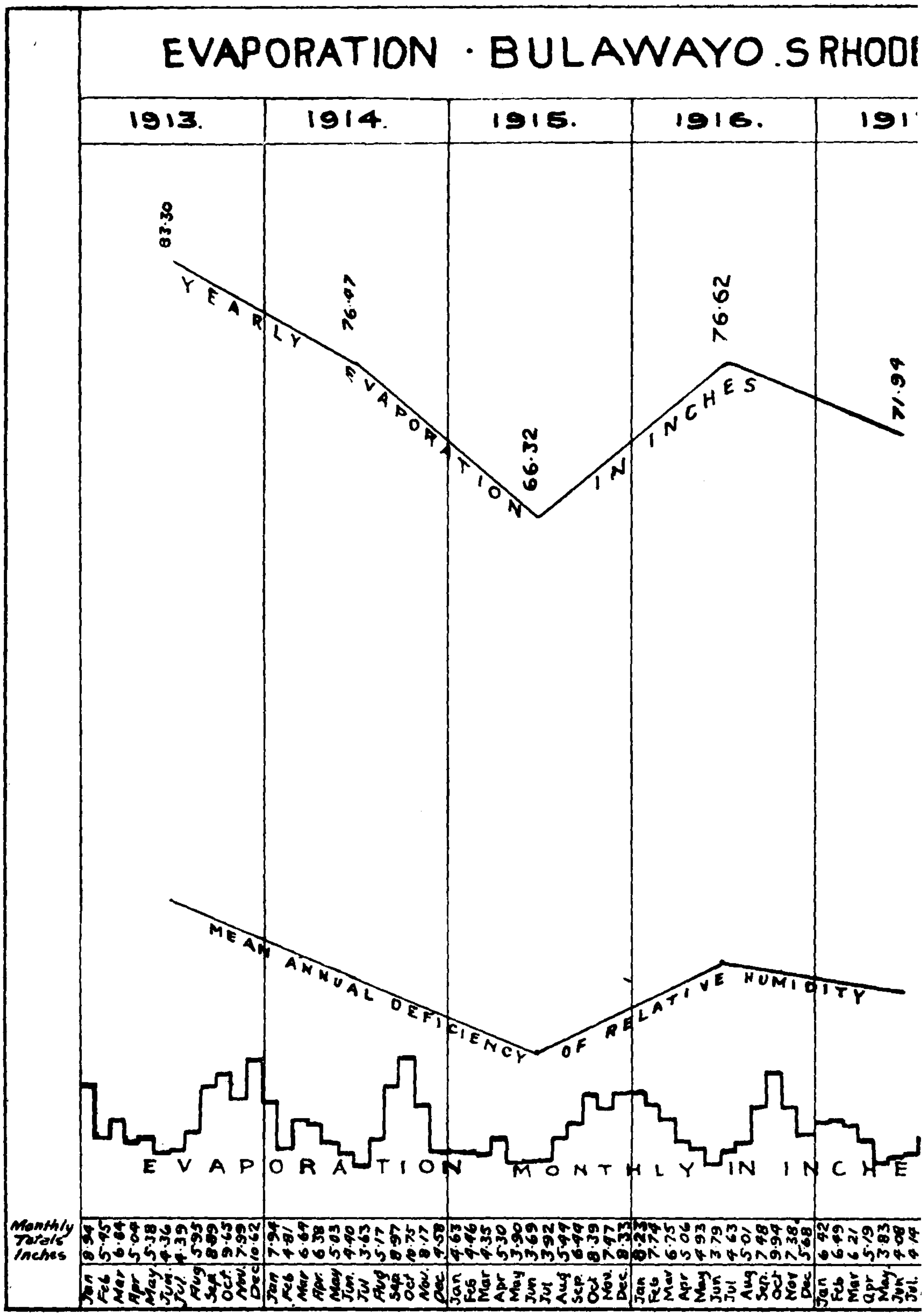
by

$$|\,0\ 1\ 1\ 1\ 1\,|, \quad |\,0\ 1\ 1\ 1\ 2\,|, \quad \ldots\ldots\ldots$$

the constant column being indicated by 0, a column of a's by 1, and a column of b's by 2. We then see that, if in the symbol for a determinant on the right there be an even number of 2's, the said determinant cannot differ from the corresponding determinant on the left; and if there be an odd number of 2's the difference existing is merely a difference of sign. It thus follows by subtraction that what we are reduced to showing is that the double of the sum of the determinants on the right that have an odd number of 2's is equal to 0: that is, that

$$2\{\,|\,0\,1\,1\,1\,2\,| + |\,0\,1\,1\,2\,1\,| + |\,0\,1\,2\,1\,1\,| + |\,0\,1\,2\,2\,2\,|$$
$$+ |\,0\,2\,1\,1\,1\,| + |\,0\,2\,1\,2\,2\,| + |\,0\,2\,2\,1\,2\,| + |\,0\,2\,2\,2\,1\,|\,\} = 0.$$

* 'Analyst,' x (1882), pp. 8–9.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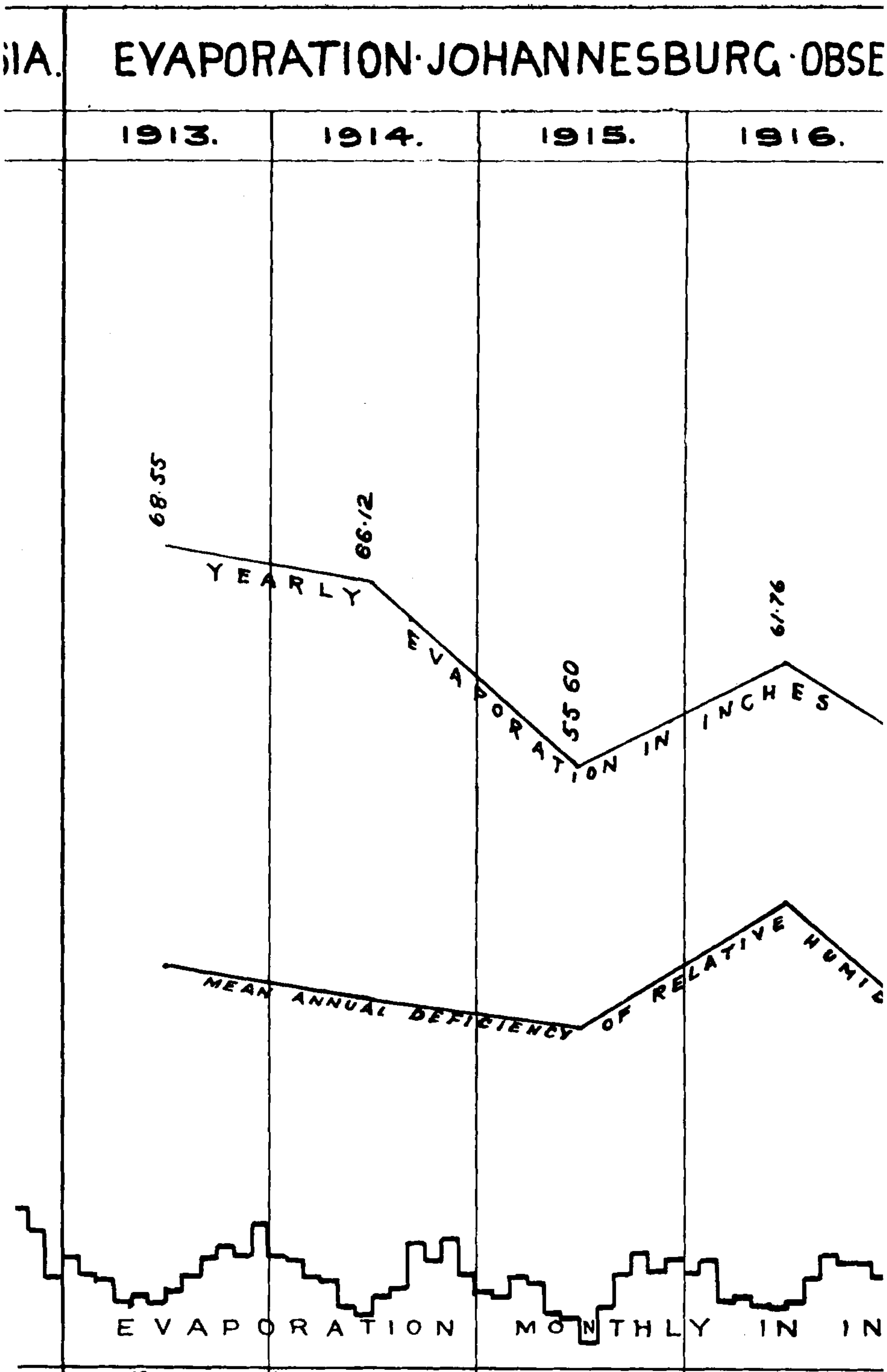

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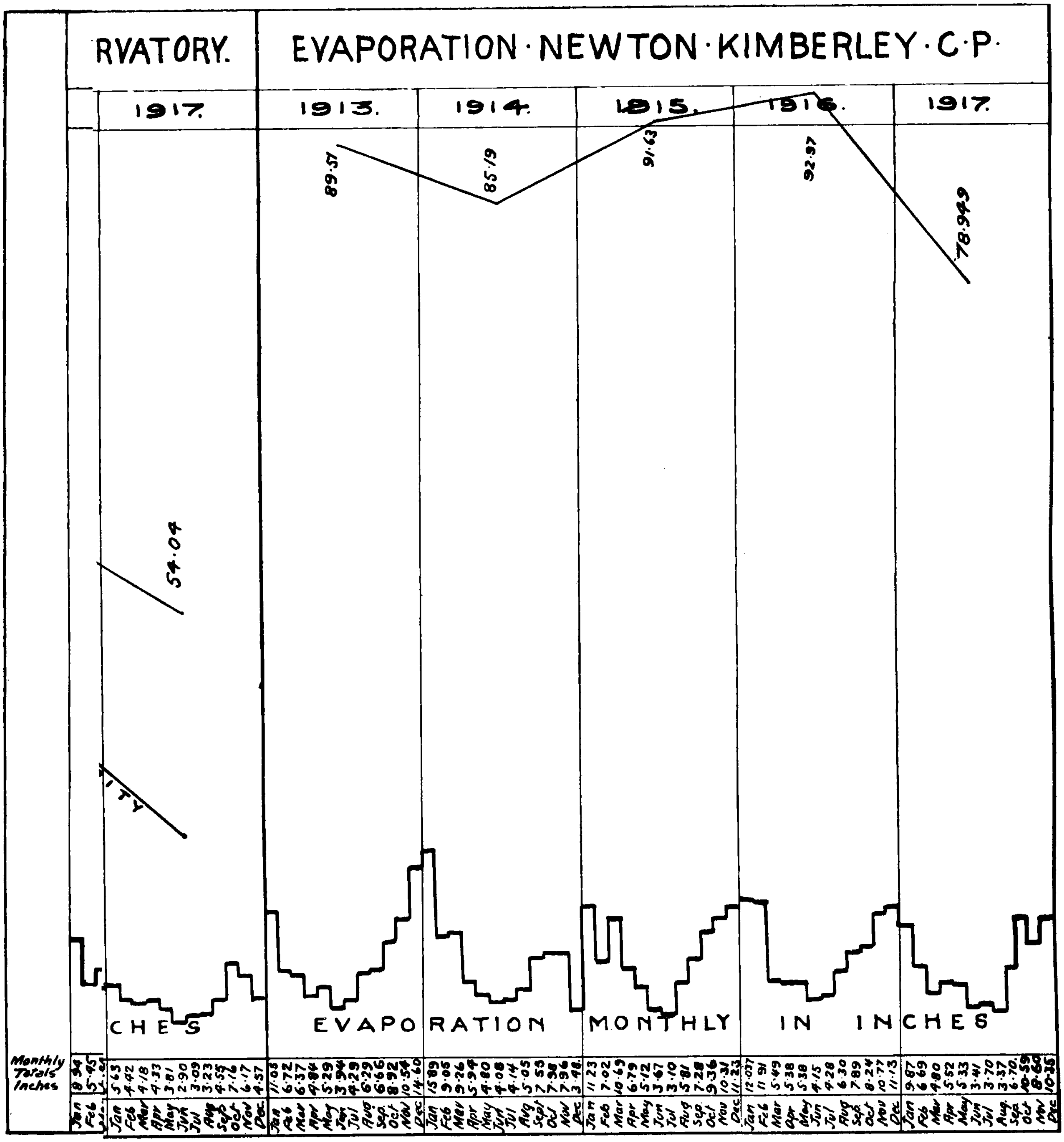

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In the next place, it being impossible that a determinant with one 2 in its symbol—that is, with one column of b's—can have any of its terms cancelled by terms of a determinant with three columns of b's, we infer that we are face to face with *two* vanishing aggregates, namely,

$$|\,0\,1\,1\,1\,2\,| + |\,0\,1\,1\,2\,1\,| + |\,0\,1\,2\,1\,1\,| + |\,0\,2\,1\,1\,1\,| = 0$$

and

$$|\,0\,1\,2\,2\,2\,| + |\,0\,2\,1\,2\,2\,| + |\,0\,2\,2\,1\,2\,| + |\,0\,2\,2\,2\,1\,| = 0.$$

It is seen, however, on a second glance that either of these is got from the other by merely interchanging a's and b's, so that in reality there is only one to be proved, the full-length form of which is

$$0 = \begin{vmatrix} 1 & a_2 & a_3 & a_4 & b_5 \\ 1 & a_1 & a_2 & a_3 & b_1 \\ 1 & a_5 & a_1 & a_2 & b_2 \\ 1 & a_4 & a_5 & a_1 & b_3 \\ 1 & a_3 & a_4 & a_5 & b_4 \end{vmatrix} + \begin{vmatrix} 1 & a_2 & a_3 & b_4 & a_5 \\ 1 & a_1 & a_2 & b_5 & a_4 \\ 1 & a_5 & a_1 & b_1 & a_3 \\ 1 & a_4 & a_5 & b_2 & a_2 \\ 1 & a_3 & a_4 & b_3 & a_1 \end{vmatrix} + \begin{vmatrix} 1 & a_2 & b_3 & a_4 & a_5 \\ 1 & a_1 & b_4 & a_3 & a_4 \\ 1 & a_5 & b_5 & a_2 & a_3 \\ 1 & a_4 & b_1 & a_1 & a_2 \\ 1 & a_3 & b_2 & a_5 & a_1 \end{vmatrix} + \begin{vmatrix} 1 & b_2 & a_3 & a_4 & a_5 \\ 1 & b_3 & a_2 & a_3 & a_4 \\ 1 & b_4 & a_1 & a_2 & a_3 \\ 1 & b_5 & a_5 & a_1 & a_2 \\ 1 & b_1 & a_4 & a_5 & a_1 \end{vmatrix}.$$

To effect the proof it would seem as if we had to show that the cofactors of b_1, b_2, b_3, b_4, b_5 vanish separately: but, as before, although the said five cofactors do vanish separately, it is sufficient only to show that one of them vanishes, say the cofactor of b_1. Our problem is thus further reduced to proving the equality

$$0 = -\begin{vmatrix} 1 & a_2 & a_3 & a_4 \\ 1 & a_5 & a_1 & a_2 \\ 1 & a_4 & a_5 & a_1 \\ 1 & a_3 & a_4 & a_5 \end{vmatrix} - \begin{vmatrix} 1 & a_2 & a_3 & a_5 \\ 1 & a_1 & a_2 & a_4 \\ 1 & a_4 & a_5 & a_2 \\ 1 & a_3 & a_4 & a_1 \end{vmatrix} - \begin{vmatrix} 1 & a_2 & a_4 & a_5 \\ 1 & a_1 & a_3 & a_4 \\ 1 & a_5 & a_2 & a_3 \\ 1 & a_3 & a_5 & a_1 \end{vmatrix} - \begin{vmatrix} 1 & a_3 & a_4 & a_5 \\ 1 & a_2 & a_3 & a_4 \\ 1 & a_1 & a_2 & a_3 \\ 1 & a_5 & a_1 & a_2 \end{vmatrix};$$

and this is final, for the cofactors of the elements in the first columns, it will be found, cancel each other in pairs or in triads. As a matter of fact, if we call the four determinants P, Q, R, S, and append the suffixes 1, 2, 3, 4 to each to obtain a notation for the said cofactors, we have

$$\begin{array}{rrrrl} -P_1 & -Q_3 & +R_2 & & = 0 \\ P_2 & -Q_1 & & +S_2 & = 0 \\ P_4 & & +R_1 & +S_4 & = 0 \\ & Q_4 & -R_3 & -S_1 & = 0 \\ -P_3 & & +R_4 & & = 0 \\ & Q_2 & & -S_3 & = 0, \end{array}$$

where the vanishing trinomials are instances of a known theorem * regarding vanishing aggregates of secondary minors in a persymmetric determinant, the persymmetric determinant here being the circulant $C(a_1, a_2, a_3, a_4, a_5)$.

* Cazzaniga, T., 'Rendic. . . . Istituto Lombardo' (2), xxxi, pp. 610–614; Muir, T., 'Trans. R. Soc. Edin.,' xxxix, p. 226; xl, pp. 511–533.

(3) In the foregoing analytic search for a proof the primary minors of the determinant

$$\begin{vmatrix} 1 & a_2 & a_3 & a_4 & a_5 \\ 1 & a_1 & a_2 & a_3 & a_4 \\ 1 & a_5 & a_1 & a_2 & a_3 \\ 1 & a_4 & a_5 & a_1 & a_2 \\ 1 & a_3 & a_4 & a_5 & a_1 \end{vmatrix}$$

will be found on closer investigation to play an important part. It is naturally viewable as the determinant got from the circulant $C(a_1, a_2, a_3, a_4, a_5)$ by removing the factor $a_1 + a_2 + a_3 + a_4 + a_5$, and it is what our determinants of §1 reduce to on making all the b's vanish.

Two properties of the said minors have been made use of for our main purpose, but in their quite general forms they are worth enunciating on their own account. The first is that *in the determinant* $C(a_1, a_2, a_3, a_4, a_5)/(a_1 + a_2 + a_3 + a_4 + a_5)$ *the cofactor of the* $(r,s)^{th}$ *element differs only in sign from the cofactor of the* $(r + s - 1, 7 - s)^{th}$ *element,* (r,s) *being any place in the secondary diagonal or on the upper side of it but not in the first column.* The second is that *in the determinant* $C(a_1, a_2, a_3, a_4, a_5)/(a_1 + a_2 + a_3 + a_4 + a_5)$ *we have*

$$cof(r, 2) + cof(r + 1, 3) + cof(r + 2, 4) + cof(r + 3, 5) = 0,$$

where r *may have any one of the values* 1, 2, 3, 4, 5, *and cof* (r,s) *stands for the cofactor of the element in the* $(r,s)^{th}$ *place.*

(4) Now let M denote the determinant of the sum of the matrices of

$$C(a_1, a_2, a_3, a_4, a_5), \quad C(b_1, b_2, b_3, b_4, b_5),$$

the former being taken symmetrical with respect to the secondary diagonal, and the latter with respect to the primary diagonal; and let N denote the determinant of the differences of the same two matrices. Then multiplying M by N in the ordinary way we find that only three of the twenty-five (25) products of pairs of rows are distinct, one occurring 5 times, and each of the two others 10 times. Further, we find that the product-determinant comes out in the form of a circulant, namely, that

$$\text{MN } i.e. \begin{vmatrix} a_1+b_1 & a_2+b_2 & \dots & a_5+b_5 \\ a_5+b_2 & a_1+b_3 & \dots & a_4+b_1 \\ a_4+b_3 & a_3+b_4 & \dots & a_3+b_2 \\ a_3+b_4 & a_4+b_5 & \dots & a_2+b_3 \\ a_2+b_5 & a_5+b_1 & \dots & a_1+b_4 \end{vmatrix} \cdot \begin{vmatrix} a_1-b_1 & a_2-b_2 & \dots & a_5-b_5 \\ a_5-b_2 & a_1-b_3 & \dots & a_4-b_1 \\ a_4-b_3 & a_3-b_4 & \dots & a_3-b_2 \\ a_3-b_4 & a_4-b_5 & \dots & a_2-b_3 \\ a_2-b_5 & a_5-b_1 & \dots & a_1-b_4 \end{vmatrix}$$

$$= \begin{vmatrix} U & V & W & W & V \\ V & U & V & W & W \\ W & V & U & V & W \\ W & W & V & U & V \\ V & W & W & V & U \end{vmatrix} = C\,(U, V, W, W, V),$$

where U, V, W stand for

$$\Sigma a_1^2 - \Sigma b_1^2, \quad \mathring{\Sigma} a_1 a_2 - \mathring{\Sigma} b_1 b_2, \quad \mathring{\Sigma} a_1 a_3 - \mathring{\Sigma} b_1 b_3$$

respectively.* In the next place, since it is readily verifiable that

$$(\Sigma a_1 + \Sigma b_1)(\Sigma a_1 - \Sigma b_1) = \mathrm{U} + 2\mathrm{V} + 2\mathrm{W},$$

we may strike out from the three determinants of the equality the factors $\Sigma a_1 + \Sigma b_1$, $\Sigma a_1 - \Sigma b_1$, $\mathrm{U} + 2\mathrm{V} + 2\mathrm{W}$ respectively, thus arriving at

$$\frac{\mathrm{M}}{\Sigma a_1 + \Sigma b_1} \cdot \frac{\mathrm{N}}{\Sigma a_1 - \Sigma b_1} = \frac{\mathrm{C}(\mathrm{U}, \mathrm{V}, \mathrm{W}, \mathrm{W}, \mathrm{V})}{\mathrm{U} + 2\mathrm{V} + 2\mathrm{W}}$$

and therefore by using the result specified in §1, and by factorising C (U, V, W, W, V) with the help of ϵ, an imaginary fifth root of 1, we obtain

$$\begin{aligned}\left(\frac{\mathrm{M}}{\Sigma a_1 + \Sigma b_1}\right)^2 = \; & (\mathrm{U} + \mathrm{V}\epsilon + \mathrm{W}\epsilon^2 + \mathrm{W}\epsilon^3 + \mathrm{V}\epsilon^4) \\ & .\,(\mathrm{U} + \mathrm{V}\epsilon^2 + \mathrm{W}\epsilon^4 + \mathrm{W}\epsilon + \mathrm{V}\epsilon^3) \\ & .\,(\mathrm{U} + \mathrm{V}\epsilon^3 + \mathrm{W}\epsilon + \mathrm{W}\epsilon^4 + \mathrm{V}\epsilon^2) \\ & .\,(\mathrm{U} + \mathrm{V}\epsilon^4 + \mathrm{W}\epsilon^3 + \mathrm{W}\epsilon^2 + \mathrm{V}\epsilon) \\ = \; & \{\mathrm{U} + \mathrm{V}(\epsilon + \epsilon^4) + \mathrm{W}(\epsilon^2 + \epsilon^3)\}^2 \\ & .\,\{\mathrm{U} + \mathrm{V}(\epsilon^2 + \epsilon^3) + \mathrm{W}(\epsilon + \epsilon^4)\}^2\end{aligned}$$

whence

$$\begin{aligned}\frac{\mathrm{M}}{\Sigma a_1 + \Sigma b_1} = \; & \{\mathrm{U} + \mathrm{V}(\epsilon + \epsilon^{-1}) + \mathrm{W}(\epsilon^2 + \epsilon^{-2})\} \\ & .\,\{\mathrm{U} + \mathrm{V}(\epsilon^2 + \epsilon^{-2}) + \mathrm{W}(\epsilon + \epsilon^{-1})\},\end{aligned}$$

and so, finally, the interesting result regarding the resolution of M into factors, namely

$$\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 & a_4+b_4 & a_5+b_5 \\ a_5+b_2 & a_1+b_3 & a_2+b_4 & a_3+b_5 & a_4+b_1 \\ a_4+b_3 & a_5+b_4 & a_1+b_5 & a_2+b_1 & a_3+b_2 \\ a_3+b_4 & a_4+b_5 & a_5+b_1 & a_1+b_2 & a_2+b_3 \\ a_2+b_5 & a_3+b_1 & a_4+b_2 & a_5+b_3 & a_1+b_4 \end{vmatrix}$$

$$\begin{aligned}= \; & (\Sigma a_1 + \Sigma b_1) \\ & .\left\{(\Sigma a_1^2 - \Sigma b_1^2) + 2(\mathring{\Sigma} a_1 a_2 - \mathring{\Sigma} b_1 b_2)\cos\frac{2\pi}{5} + 2(\mathring{\Sigma} a_1 a_3 - \mathring{\Sigma} b_1 b_3)\cos\frac{4\pi}{5}\right\} \\ & .\left\{(\Sigma a_1^2 - \Sigma b_1^2) + 2(\mathring{\Sigma} a_1 a_2 - \mathring{\Sigma} b_1 b_2)\cos\frac{4\pi}{5} + 2(\mathring{\Sigma} a_1 a_3 - \mathring{\Sigma} b_1 b_3)\cos\frac{2\pi}{5}\right\}\end{aligned}$$

the generalisation of which for any order is enunciated in the 'Messenger of Math.,' xi, pp. 105–108.

* $\mathring{\Sigma}$ is used as the symbol of a *cyclic sum*, for example, $\mathring{\Sigma} a_1 a_3 = a_1 a_3 + a_2 a_4 + a_3 a_5 + a_4 a_1 + a_5 a_2$.

RONDEBOSCH, S.A.;
February 22, 1920.