(ii.) The envelope of the circumcircle ABC is a Limaçon having S for a double point.

Otherwise, the circumcircle ABC cuts a given circle J orthogonally, and its centre moves on the circumference of another given circle touching J.

4. The envelope of the radical axis of the circumcircles DEF and ABC is a conic of which the centre of the orthogonal circle J is a focus. In other words, this focus of the conic is also a focus of the Limaçon.

The conic meets the Limaçon in eight points which lie on the circle DEF and another given circle.

On the Harmonics of a Ring. By W. D. NIVEN. Read February 9th, 1893. Received, in revised form, December 11th, 1893.

1. This paper may be regarded as an extension of the work contained in the first memoir on "Toroidal Functions" in the *Philosophical Transactions*, by Professor W. M. Hicks, an attempt being here made to show how the problems solved by Mr. Hicks for a single anchor ring may be dealt with when there are two rings having the same rectilineal axes.

It is obvious that with two such rings, in order to satisfy the surface conditions for each ring, it is necessary that the harmonics of either should be capable of being expressed in terms of those of the other. This is accomplished as follows :—

It is first shown that the harmonics of any degree, referred to a circle A in dipolar coordinates, may be derived from their predecessors, one degree lower, by certain differentiations with regard to the radius of the circle and the distance of its plane from a fixed point.

The potential at any point due to a coaxal circle B of uniform line density is next found in terms of A's harmonics.

Finally, the zonal harmonics of B can all be deduced from this potential by the system of differentiations just referred to, and therefore they can all be expressed in terms of the harmonics of A.

The present communication is confined to a discussion of the quantities arising in the process indicated, and deals, for the most part, only with zonal harmonics, although the methods employed will also apply to the general tesseral and sectorial system.

2. I begin with the potential at any point due to a circle loaded with matter of line density $\cos\sigma\phi'$ at the point whose longitude along the circumference is ϕ'. If the circle, of radius a, be in the plane of xy, with its centre at the origin of coordinates, the potential at x, y, z or $\rho\cos\phi$, $\rho\sin\phi$, z is seen at once to be

$$\sqrt{\frac{a}{2\rho}}\int_0^{2\pi}\frac{\cos\sigma\phi'\,d\phi'}{\sqrt{\dfrac{z^2+\rho^2+a^2}{2a\rho}-\cos(\phi'-\phi)}};$$

or, putting $\phi'-\phi=\theta$, adopting dipolar coordinates u, v, and discarding the term with $\sin\sigma\phi$ as a factor, as being zero, we have

$$\cos\sigma\phi\sqrt{\frac{a}{2\rho}}\int_0^{2\pi}\frac{\cos\sigma\theta}{\sqrt{\coth u-\cos\theta}}\,d\theta,$$

which is the same as

$$\cos\sigma\phi\sqrt{2(\cosh u-\cos v)}\int_0^{\pi}\frac{\cos\sigma\theta}{\sqrt{\cosh u-\sinh u\cos\theta}}\,d\theta.$$

3. Let now the origin be shifted to a point on the rectilineal axis at a distance b from its old position; then the relations between the rectangular and dipolar coordinates are given by

$$\frac{\rho\pm i(z-b)+a}{\rho\pm i(z-b)-a}=e^{u\mp iv},$$

or, if

$$a+ib=p,\quad a-ib=q,$$

$$\frac{\rho+iz+q}{\rho+iz-p}=e^{u-iv},\quad \frac{\rho-iz+p}{\rho-iz-q}=e^{u+iv}.$$

If we suppose ρ, z to be fixed, while a, b are varied, it is clear that

u, v must be regarded as functions of p, q, and it is easy to find the following results:—

$$\frac{\partial u}{\partial p} = \frac{1}{2a} e^{-iv} \sinh u,$$

$$\frac{\partial v}{\partial p} = \frac{1}{2a} i\,(e^{-iv} \cosh u - 1),$$

$$\frac{\partial u}{\partial q} = \frac{1}{2a} e^{iv} \sinh u,$$

$$\frac{\partial v}{\partial q} = -\frac{1}{2a} i\,(e^{iv} \cosh u - 1);$$

and thence

$$\frac{\partial}{\partial p} \sqrt{\cosh u - \cos v} = \frac{1}{4a} (e^{-iv} \cosh u + 1) \sqrt{\cosh u - \cos v},$$

$$\frac{\partial}{\partial q} \sqrt{\cosh u - \cos v} = \frac{1}{4a} (e^{iv} \cosh u + 1) \sqrt{\cosh u - \cos v}.$$

Also, if $I(n, \sigma)$ denote the integral,

$$\int_0^\pi \frac{\cos \sigma\theta}{(\cosh u - \sinh u \cos\theta)^{n+\frac{1}{2}}} d\theta,$$

$$\frac{\partial}{\partial p} I(n, \sigma) = -(n+\tfrac{1}{2}) \frac{1}{2a} e^{-iv} \left[\cosh u\, I(n, \sigma) - I(n+1, \sigma)\right].$$

4. Let $X(n, \sigma)$ denote

$$a^{n-\frac{1}{2}} e^{-niv} \sqrt{2(\cosh u - \cos v)}\, I(n, \sigma) \cos \sigma\phi;$$

then, remembering that $2a = p+q$, we shall have

$$\frac{\partial}{\partial p} X(n, \sigma) = AX(n, \sigma) + BX(n+1, \sigma),$$

where $A = \frac{1}{2a}(n-\frac{1}{2}) + \frac{1}{2a} n\,(e^{-iv} \cosh u - 1) + \frac{1}{4a}(e^{-iv} \cosh u + 1)$

$$-\frac{1}{2a}(n+\tfrac{1}{2}) e^{-iv} \cosh u = 0,$$

and $$B = (n+\tfrac{1}{2}) \frac{1}{2a^2}.$$

Similarly it may be shown that if $Y(n, \sigma)$ denote the same thing as $X(n, \sigma)$, except that e^{niv} takes the place of e^{-niv}, we shall find

$$\frac{\partial}{\partial q} Y(n, \sigma) = (n+\tfrac{1}{2}) \frac{1}{2a^2} Y(n+1, \sigma).$$

Next, let $$X(n, \sigma) = a^{n-\frac{1}{2}} \{U(n, \sigma) - i V(n, \sigma)\},$$

$$Y(n, \sigma) = a^{n-\frac{1}{2}} \{U(n, \sigma) + i V(n, \sigma)\},$$

and note that $$\frac{\partial}{\partial a} - i\frac{\partial}{\partial b} = 2\frac{\partial}{\partial p}, \qquad \frac{\partial}{\partial a} + i\frac{\partial}{\partial b} = 2\frac{\partial}{\partial q};$$

then the two following relations may be deduced, viz.,

$$(n+\tfrac{1}{2})\, a^{n+\frac{1}{2}}\, U(n+1, \sigma) = a^2 \left\{ \frac{\partial}{\partial a} a^{n-\frac{1}{2}} U(n, \sigma) - \frac{\partial}{\partial b} a^{n-\frac{1}{2}} V(n, \sigma) \right\},$$

$$(n+\tfrac{1}{2})\, a^{n+\frac{1}{2}}\, V(n+1, \sigma) = a^2 \left\{ \frac{\partial}{\partial b} a^{n-\frac{1}{2}} U(n, \sigma) + \frac{\partial}{\partial a} a^{n-\frac{1}{2}} V(n, \sigma) \right\}.$$

Here $U(n, \sigma)$ stands for the harmonic

$$\cos nv \sqrt{2(\cosh u - \cos v)}\, I(n, \sigma) \cos \sigma\phi,$$

and $V(n, \sigma)$ for the same thing, with $\sin nv$ instead of $\cos nv$.

When it is necessary to express u, v, ϕ explicitly, the notation $U_n^\sigma(uv\phi)$, $V_n^\sigma(uv\phi)$ will also be used.

5. These results show that the successive harmonics are derivable from their predecessors by displacements similar, with some modifications, to those usually employed in the case of the sphere. There is this difference, however, that, whereas the masses of the compound sphere are each zero, those of the compound circles, and therefore also of the distributions on the anchor ring which they represent, may not be zero. They will be zero if the expressions for the corresponding potentials contain the longitudinal factor $\cos \sigma\phi$ or $\sin \sigma\phi$, for in that case the undisplaced circle from which they all take their origin is of zero mass. If, however, $\sigma = 0$, the potential $(n+\frac{1}{2})\, a^{n-\frac{1}{2}}\, U(n+1, 0)$ arises, as we found in §4, from displacements of two circles which produce respectively the potentials $a^{n-\frac{1}{2}} U(n, 0)$ and $a^{n-\frac{1}{2}} V(n, 0)$, the former by an enlargement of its radius, the latter by a movement of its centre along its axis. The second displacement is clearly of the same character as sphere points, and no resulting mass is thereby obtained. But, in the first, since the mass of the ring which

produces potential $a^{n-\frac{1}{2}}\,U(n, 0)$ is clearly increased to $\left(1+\frac{\delta a}{a}\right)^{n+\frac{1}{2}}$ times the former value when its radius is increased from a to $a+\delta a$, it follows that by the superposition of ring negative a and positive $a+\delta a$, there will be a residue of mass $(n+\frac{1}{2})\frac{\delta a}{a}$ times the mass of the circle a, viz., $2\pi a\,.\,a^{n-\frac{1}{2}}$. This residue corresponds, by the previous article, to potential

$$(n+\tfrac{1}{2})\,a^{n-\frac{1}{2}}\,U(n+1, 0)\,\frac{\delta a}{a}.$$

Hence, the masses of the compound circles which produce the potentials $U(n, 0)$ and $U(n+1, 0)$ are equal to one another, and when we pursue the argument down to the value $n=0$, are each equal to $2\pi a$, the line density of the original circle being taken as unity.

6. The result just arrived at throws some light upon the convergence which may be expected from a series expressed in terms of harmonics of the anchor ring, and, on account of its importance, as well as for the purpose of introducing the working quantities of the subject, I shall prove it by finding the charge on the anchor ring which will produce the external harmonic $U(n, 0)$. It is first to be observed that the integral which was denoted in § 3 by $I(n, 0)$ is in point of fact $\pi P_{n-\frac{1}{2}}(\cosh u)$, *i.e.* the zonal spherical harmonic of fractional degree $n-\frac{1}{2}$. The printing of the fractional suffix being inconvenient, I shall at this stage follow Mr. Hicks in representing the integral referred to by πP_n. We may therefore write

$$U(n, 0) \quad \text{or} \quad U_n = \pi \cos nv\,\sqrt{2\,(\cosh u-\cos v)}\,P_n$$

as the type of a zonal harmonic, with a similar specification for V_n, in which $\sin nv$ takes the place of $\cos nv$.

The harmonic U_n is suitable to the exterior of the ring, the corresponding inside form being given by

$$U'_m = \pi \cos nv\,\sqrt{2\,(\cosh u-\cos v)}\,Q_n\,.\left(\frac{P_n}{Q_n}\right)_{u_0},$$

where
$$Q_n = P_n\int_\mu^\infty \frac{d\mu}{(\mu^2-1)\,P_n^2},$$

$$\mu = \cosh u \quad \text{and} \quad \mu_0 = \cosh u_0,$$

u_0 being the parameter of the ring.

We find readily the expression for the density, viz.,

$$\sigma_n = -\tfrac{1}{4}\cos nv\sqrt{2(\cosh u_0 - \cos v)}\,\frac{1}{\sinh u_0}\,\frac{1}{Q_n(\cosh u_0)}\left(\frac{\partial u}{\partial s}\right)_{u_0}.$$

Now it is easy to show that

$$-\frac{\partial u}{\partial s} = \frac{\partial v}{\partial s'} = \frac{\cosh u_0 - \cos v}{a},$$

∂s, $\partial s'$ being elements of lines normal to the surfaces u, v. Also, element of surface

$$\partial S = \rho\,\delta\phi\,\delta s' = \frac{a^2 \sinh u_0}{(\cosh u_0 - \cos v)^2}\,\delta\phi\,\delta v.$$

Hence
$$\iint \sigma_n\,dS = 2\pi a\,\frac{1}{Q_n(u_0)}\int_0^{\pi}\frac{\cos nv}{\sqrt{2(\cosh u_0 - \cos v)}}\,dv$$
$$= 2\pi a \quad (\S\ 14).$$

7. The potential due to a circle of unit line-density with its centre on the rectilineal axis of the ring and its plane normal to that axis, may be found as follows:—Let r be the distance of a point $u_1 v_1 \phi_1$ on the circle referred to from any point $uv\phi$ on the ring; then

$$\frac{1}{r} = \Sigma\Sigma A_n^{\sigma} U_n^{\sigma}(uv\phi) + \Sigma\Sigma B_n^{\sigma} V_n^{\sigma}(uv\phi).$$

The expression for the circle, since u_1, v_1 are the same for every point of it, may be deduced from this by multiplying by $2\pi\rho$, and suppressing all the terms in $\cos\sigma\phi$ and $\sin\sigma\phi$ from $\sigma = 1$ upwards.

The coefficients to be determined in the expansion of $\frac{1}{r}$ are therefore only the following

$$A_0 U_0 + A_1 U_1 + A_1' V_1 + \dots.$$

Multiply both sides by $\sigma_n dS$, and integrate over the surface, observing that

$$\iint \frac{\sigma_n\,dS}{r} = U_n(u_1 v_1),$$

and that, by § 6,

$$\iint \sigma_n U_n\,dS = \pi^2 a\left(\frac{P_n}{Q_n}\right)_u \int_0^{2\pi}\cos^2 nv\,dv.$$

Hence
$$A_n = \frac{1}{\pi^3 a}\,U_n(u_1 v_1)\left(\frac{Q_n}{P_n}\right)_u,$$

except when $n = 0$, and then

$$A_0 = \frac{1}{2\pi^3 a} U_0(u_1 v_1) \left(\frac{Q_0}{P_0}\right)_u.$$

If Q_n be substituted for P_n in the expressions for U_n and V_n, the resulting harmonic is finite in the neighbourhood of the dipolar circle of reference. We shall denote these forms of the harmonic by $\overline{U}_n$ and $\overline{V}_n$ respectively. The expansion for the potential at $uv\phi$ due to the circle of the last article will then be written as follows :—

$$\frac{\rho}{\pi^2 a}\left\{U_0(u_1v_1)\,\overline{U}_0(uv) + 2\sum_1^\infty \left[U_n(u_1v_1)\,\overline{U}(uv) + V_n(u_1v_1)\,\overline{V}_n(uv)\right]\right\}.$$

Problems with Two Rings.

8. The last result, taken in conjunction with the results of § 4, enables us to express the harmonics of one anchor ring in terms of those of another which is coaxal with it. For it was shown in § 4 that all the harmonics of an anchor ring could be deduced from the potential of a circle by certain rules of differentiation. We have therefore only to apply those rules to the expansion just found in order to obtain similar expansions, in terms of the harmonics of the circle a, for the series of compound circles of radius ρ, that is to say, for the complete system of harmonics pertaining to the system of anchor rings the radius of whose dipolar circle is ρ.

In order to produce a symmetrical arrangement, we may suppose the positive directions of the rectilineal axes of the two rings to be towards one another, so that, if A and B be the two rings, the displacements of A are towards B, and those of B towards A. When this arrangement is turned round, so that A and B interchange planes, the relative arrangement of harmonics will then be the same as before.

9. Let W_0, $W_1\mathfrak{W}_1$, $W_2\mathfrak{W}_2$, &c. represent the harmonics of B, referred to its dipolar circle and expressed in terms of its own dipolar system, then, § 9,

$$W_0 = \frac{\rho}{\pi^2 a}\left\{U_0(u_1v_1)\,\overline{U}_0(uv) + 2U_1(u_1v_1)\,\overline{U}_1(uv) + \&c.\right\}$$

$$= \frac{1}{\pi^2}\left({}_0\alpha_0\,\overline{U}_0 + {}_0\alpha_1\,\overline{U}_1 + {}_0\beta_1\overline{V}_1 + \ldots\right), \text{ say,}$$

where $$ {}_0\alpha_0 = \frac{\rho}{a} U_0(u_1v_1), $$

$$ {}_0\alpha_1 = \frac{2\rho}{a} U_1(u_1v_1), $$

$$ {}_0\beta_1 = \frac{2\rho}{a} V_1(u_1v_1), $$

&c.,

and $\overline{U}_n$ is put, for brevity, instead of $\overline{U}_n(uv)$, &c. All the coefficients which have zero for the first suffix can be readily expressed in terms of the radii ρ and a and the distance z between the planes of the circles. It may, in fact, be shown that

$$ U_n(u_1v_1) + iV_n(u_1v_1) = 2a(\rho^2+z^2-a^2+i\,2az)^n J_n, $$

where $$ J_n = \int_0^\pi \frac{d\theta}{(\rho^2+z^2+a^2-2a\rho\cos\theta)^{n+1}}. $$

For, since $$ e^{u\mp iv} = \frac{\rho\pm iz+a}{\rho\pm iz-a}, $$

we have $$ e^u = r_1/r, $$

where $$ r^2 = (\rho-a)^2+z^2, \quad r_1^2 = (\rho+a)^2+z^2, $$

and $$ e^{2iv} = \frac{\rho^2+z^2-a^2+i\,2az}{\rho^2+z^2-a^2-i\,2az}; $$

therefore $$ \sinh u = \frac{2a\rho}{rr_1}, $$

and $$ \cos nv = \tfrac{1}{2}\frac{(\rho^2+z^2-a^2+i\,2az)^n+(\rho^2+z^2-a^2-i\,2az)^n}{(rr_1)^n}. $$

Hence

$$ U_n(u_1v_1) = \frac{\cos nv}{\rho}\frac{(2a\rho)^{n+1}}{\sinh^n u}\int_0^\pi \frac{d\theta}{(\rho^2+z^2+a^2-2a\rho\cos\theta)^{n+1}} $$

$$ = a\left\{(\rho^2+z^2-a^2+i\,2az)^n+(\rho^2+z^2-a^2-i\,2az)^n\right\}J_n; $$

similarly,

$$ V_n(u_1v_1) = ia\left\{(\rho^2+z^2-a^2-i\,2az)^n-(\rho^2+z^2-a^2+i\,2az)^n\right\}J_n. $$

10. The next step is to determine the expansions of $'W_1$ and $'\mathcal{W}_1$. If we put

$$ \pi^2 W_1 = {}_1\alpha_0\overline{U}_0 + {}_1\alpha_1\overline{U}_1 + {}_1\beta_1\overline{V}_1 + \&c., $$

$$ \pi^2 \mathcal{W}_1 = {}_1\gamma_0\overline{U}_0 + {}_1\gamma_1\overline{U}_1 + {}_1\delta_1\overline{V}_1 + \&c., $$

then, by § 4 and § 9, ${}_1a_0$, ${}_1a_1$, ${}_1\beta_1$ will be respectively

$$\frac{1}{a}\left(2\rho\frac{\partial}{\partial\rho}-1\right)\rho\left\{U_0(u_1v_1),\ 2U_1(u_1v_1),\ 2V_1(u_1v_1),\ \ldots\right\};$$

and ${}_1\gamma_0$, ${}_1\gamma_1$, ${}_1\delta_1$, &c. respectively

$$-\frac{2\rho^2}{a}\frac{\partial}{\partial z}\left\{U_0(u_1v_1),\ 2U_1(u_1v_1),\ 2V_1(u_1v_1),\ \ldots\right\}.$$

11. To show, however, how the coefficients are formed in the general case, it should be noticed that the equations of § 4 may be thrown into the form

$$(2n+1)(U_{n+1}+iV_{n+1})=\left(2a\frac{\partial}{\partial a}+2n-1+i\,2a\frac{\partial}{\partial b}\right)(U_n+iV_n).$$

If the harmonics be those at the point ρ, z, ϕ, referred to the circle A, it is clear that we may write $-\frac{\partial}{\partial z}$ for $\frac{\partial}{\partial b}$ in this relation. Next, putting Ω for the operator $2a\frac{\partial}{\partial a}-i\,2a\frac{\partial}{\partial z}$, and Ω' for the same operator with the sign of i changed, we shall have

$$U_n+iV_n=\frac{(\Omega+2n-3)(\Omega+2n-5)\ldots(\Omega-1)}{(2n-1)(2n-3)\ldots 1}U_0$$

$$=f_n(\Omega)\,2aJ_0,\ \text{suppose.}$$

Next, as regards the harmonics W, as the displacements of B are towards A, the sign of $\frac{\partial}{\partial b}$ in § 4 must be changed, and, since b may then be put equal to z, we have in like manner

$$W_n+i\mathfrak{W}_n=f_n(\varpi)\,W_0,$$

where ϖ is put for $2\rho\frac{\partial}{\partial\rho}-i\,2\rho\frac{\partial}{\partial z}$, and ϖ' may be taken to represent the same operator with the sign of i changed.

If now
$$W_n=\Sigma\,{}_na_r\,\overline{U}_r+\Sigma\,{}_n\beta_r\,\overline{V}_r,$$
$$\mathfrak{W}_n=\Sigma\,{}_n\gamma_r\,\overline{U}_r+\Sigma\,{}_n\delta_r\,\overline{V}_r,$$

we shall have

$${}_na_r+i\,{}_n\gamma_r=f_n(\varpi)\frac{2\rho}{a}U_r(u_1v_1),$$

$${}_n\beta_r+i\,{}_n\delta_r=f_n(\varpi)\frac{2\rho}{a}V_r(u_1v_1),$$

except when $r=0$, when the 2 on the right-hand side must be omitted.

12. The equations just written down determine the coefficients completely. As a matter of convenience, however, it may happen that it is easier to find, for instance, ${}_n a_r$ from ${}_n a_0$ than from ${}_0 a_r$.

The coefficient ${}_n a_0$ is itself to be found from

$$\tfrac{1}{2}\left\{f_n(\varpi)+f_n(\varpi')\right\}\frac{\rho}{a}U_0,$$

i.e.
$$\left\{f_n(\varpi)+f_n(\varpi')\right\}\rho J_0.$$

Now J_0 is a symmetrical function of ρ and a, and the operation to be performed is precisely the same as that for finding ${}_0 a_n$, except that ρ and a are everywhere interchanged. Hence, if for a minute we denote the expression for ${}_0 a_n$, found above, by $F(a, \rho, z)$, that for ${}_n a_0$ will be $F(\rho, a, z)$. Now

$$\begin{aligned}{}_n a_r &= \left\{f_n(\varpi)+f_n(\varpi')\right\}\frac{\rho}{a}U_r(u_1v_1)\\ &= \left\{f_n(\varpi)+f_n(\varpi')\right\}\frac{\rho}{a}\left\{f_r(\Omega)+f_r(\Omega')\right\}aJ_0\\ &= \frac{1}{a}\left\{f_r(\Omega)+f_r(\Omega')\right\}a\left\{f_n(\varpi)+f_n(\varpi')\right\}\rho J_0\\ &= \frac{1}{a}\left\{f_r(\Omega)+f_r(\Omega')\right\}a\,{}_n a_0.\end{aligned}$$

13. The integral J_n, which constantly appears in the coefficients, is only a modification of P_n, and both P_n and Q_n, or their first differential coefficients, occur in the equations expressing the boundary conditions in any physical problem, connected with the ring, to which potential functions are applicable. It is therefore important to be able to evaluate these integrals. Mr. Hicks has shown how to express them in terms of elliptic integrals; in what follows it is shown how they may be expressed in series, a form which seems to possess advantages when the section of the ring is small.

Evaluation of the Integrals P_n, Q_n.

The integral $I(n, \sigma)$, which appeared in § 3, is one factor of a tesseral harmonic, but it will be sufficient at present to confine our attention to zonal forms. We had

$$\pi P_n = \int_0^\pi \frac{d\theta}{(\cosh u - \sinh u \cos\theta)^{n+\frac{1}{2}}} \quad\text{..................(1),}$$

which is readily thrown into the form

$$2e^{-(n+\frac{1}{2})u} \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\{1-(1-e^{-2u})\sin^2\theta\}^{n+\frac{1}{2}}} \quad\text{..............(2).}$$

14. The corresponding form for Q_n will now be found. It is to be observed that we have already defined Q_n, § 6, but another form can be found for it from the external zonal harmonic of a prolate ellipsoid, and we shall, in fact, show afterwards that the two are connected by the relation

$$Q_n = P_n \int_\mu^\infty \frac{d\mu}{(\mu^2-1)P_n^2} = \frac{1}{2^{n+\frac{1}{2}}} \int_{-1}^1 \frac{(1-v^2)^{n-\frac{1}{2}}}{(\mu-v)^{n+\frac{1}{2}}}\,dv \quad\text{.........(3),}$$

where $\mu = \cosh u$.

From the second of these we at once pass to

$$\frac{(-1)^n}{1.3.5\ldots(2n-1)} \int_{-1}^1 \frac{\frac{\partial^n}{\partial v}(1-v^2)^{n-\frac{1}{2}}\,dv}{\sqrt{2(\mu-v)}},$$

and then to

$$\int_0^\pi \frac{\cos n\theta}{\sqrt{2(\cosh u - \cos\theta)}}\,d\theta \quad\text{......................(4).}$$

From (3) we also get

$$2^{-n-\frac{1}{2}} \int_0^\pi \frac{\sin^{2n}\theta}{(\cosh u - \cos\theta)^{n+\frac{1}{2}}}\,d\theta \quad\text{....................(5),}$$

and a still more useful expression for Q_n may be obtained by employing a transformation similar to Landen's, viz.,

$$\sin(\psi-\theta) = e^{-u}\sin\psi.$$

We thus find, on reduction,

$$Q_n = e^{-(n+\frac{1}{2})u} \int_0^\pi \frac{\sin^{2n}\psi}{\sqrt{1-e^{-2u}\sin^2\psi}}\,d\psi \quad\text{................(6).}$$

This corresponds to the expression (2), § 13, for P_n, and furnishes a convenient expansion in powers of e^{-u}.

*15. I shall now find the expansion of P_n in powers of e^u and e^{-u}.

* Since this was written, Mr. A. B. Basset, in a paper on "Toroidal Functions," in the *American Journal of Mathematics*, has given the expansion of P_n. The work in the text is, however, retained, as the method is somewhat different.

If y be put for Q_n/π, which may be written

$$e^{-\frac{1}{2}u}\left(a_n e^{-nu}+a_{n+2} e^{-(n+2)u}+\ldots\right),$$

where a_n, a_{n+2}, &c. are known coefficients, then y satisfies the differential equation

$$\frac{d^2y}{du^2}+\coth u\,\frac{dy}{du}-(n^2-\tfrac{1}{4})\,y=0,$$

and, after Cayley (*Elliptic Functions*, § 77), there exists another solution of the form

$$(\log 4+u)\,y+z,$$

where z satisfies the differential equation

$$(e^u-e^{-u})\,\frac{d^2z}{du^2}+(e^u+e^{-u})\,\frac{dz}{du}-(n^2-\tfrac{1}{4})(e^u-e^{-u})\,z$$

$$=-2\,(e^u-e^{-u})\,\frac{dy}{du}-(e^u+e^{-u})\,y \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$=-e^{-\frac{1}{2}u}\left(b_{n-1}e^{-(n-1)u}+b_{n+1}e^{-(n+1)u}+\ldots\right) \text{ suppose,}$$

where b_{n-1}, b_{n+1}, &c. are known coefficients.

Let $z=e^{-\frac{1}{2}u}\left[A_n e^{nu}+A_{n-2}e^{(n-2)u}+\ldots+A_{n-2r}e^{(n-2r)u}+\ldots\right.$

$$\left.+B_{n-2r}e^{-(n-2r)}+\ldots+B_{n+2s}e^{-(n+2s)u}+\ldots\right).$$

If n be even, there will be a term A_0, or B_0, which means the same thing; if n be odd, there will be a term A_1e^u and a term B_1e^{-u}. The values of r may range from $\frac{n}{2}$ (n even) or $\frac{n-1}{2}$ (n odd) to 0, and s from 1 to ∞.

Entering the value of z in the differential equation, we find

$$A_{n-2r+2}=A_{n-2r}\,\frac{4r\,(n-r)}{(2r-1)(2n-2r+1)} \quad \ldots\ldots\ldots\ldots\ldots(2),$$

and

$$B_{n-2r-2}=B_{n-2r}\,\frac{4r\,(n-r)}{(2r+1)(2n-2r-1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

One or other of these equations holds from the top of the series A_n down to B_{n-4} inclusive. After that we begin to take account of the second side of equation (1). It will be found that

$$B_{n-2}=-\frac{b_{n-1}}{2n-1}=\frac{2n}{2n-1}\,a_n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4);$$

but after this point we shall have equations of the form

$$(2s-1)(2n+2s-1)\,B_{n+2s-2}-4s\,(n+s)\,B_{n+2s}=b_{n+2s-1} \quad \ldots\ldots(5),$$

which it will be impossible to solve unless we can assume the value of one B of the system to be known. To get over this difficulty, I suppose B_n temporarily withdrawn, and all the coefficients after it determined by the equations (5). We may then restore B_n in the form of a term $B_n y$ in the expression for z, where y is the solution referred to at the beginning of this article. All the coefficients in z are now found from the equations (5) solved successively, with the exception of B_n, and this may be determined by taking $k^2 = k_1^2 = \frac{1}{2}$, calculating independently the corresponding values of P_n, Q_n for this value of k^2, and entering these values in the equation

$$\pi P_n = (B_n + \log 4 + u)\, Q_n + \pi z,$$

where z does not now include B_n. The term B_n may, however, be easily found in the cases $n=1$, 2, by differentiation of $F(k)$ with regard to k^2.

When n is even, (2) gives

$$A_n = A_0 2^{2n-1} \frac{n!\, n!}{(2n)!},$$

and $$B_0 = B_{n-2} \frac{2^{n-1}(n-1)!}{1.3\ldots(2n-3)} = \frac{2^{2n} n!\, n!}{n\,(2n)!} a_n = \frac{1}{n};$$

therefore $$A_n = 2^{2n-1} \frac{n!\,(n-1)!}{(2n)!}.$$

The same expression holds when n is odd, the transition from the A's to the B's being effected by the equation

$$n^2 A_1 = (n^2-1)\, B_1,$$

except when $n = 1$.

16. When n is large, the first term is the most important in the expansion; the series will therefore conveniently proceed from that term.

If we put $e^{-u} = k_1$ and $k^2 = 1 - k_1^2$, then, § 14,

$$\pi P_n = 2k_1^{n+\frac{1}{2}} F_n(k),$$

$F_n(k)$ being written for

$$\int_0^{\frac{1}{2}\pi} \frac{d\theta}{(1-k^2\sin^2\theta)^{n+\frac{1}{2}}},$$

and $$F_n(k) = 2^{2n-1} \frac{n!\,(n-1)!}{(2n)!} k_1^{-2n} \left\{ 1 + \frac{1\,(2n-1)}{4.1\,(n-1)} k_1^2 + \frac{1.3\,(2n-1)(2n-3)}{4^2.1.2\,(n-1)(n-2)} k_1^4 + \&c. \right\}$$
$$+ \left(B_n + \log\frac{4}{k_1}\right) \frac{(2n)!}{2^{2n}\, n!\, n!} \left\{ 1 + \frac{2n+1}{n+1} k_1^2 + \ldots \right\},$$

where B_n is to be determined by the method of § 15.

17. J_n, being
$$= \frac{1}{r^{2n+1}} 2 \int_0^{\frac{1}{2}\pi} \frac{d\theta}{(1-k^2\sin^2\theta)^{n+\frac{1}{2}}},$$
where
$$k_1^2 = 1-k^2 = r^2/r_1^2,$$
is also found by this expansion. A few terms of the earlier values of J_n, in which B_n appears at an early stage of the series, may be obtained by the method of differentiation previously referred to. The series are arranged in powers of r/r_1, supposed small:—

$$J_0 = \frac{2}{r_1}\log\frac{4r_1}{r} + \frac{1}{2}\,\frac{r^2}{r_1^3}\left(\log\frac{4r_1}{r}-1\right)+\&c.,$$

$$J_1 = \frac{2}{r_1 r^2} + \frac{1}{r_1^3}\left(\log\frac{4r_1}{r}-\frac{1}{2}\right)+\&c.,$$

$$J_2 = \frac{4}{3r_1r^4} + \frac{1}{r_1^3r^2} + \frac{3}{4}\,\frac{1}{r_1^5}\left(\log\frac{4r_1}{r}-\frac{7}{12}\right).$$

18. The expansion for P_n enables us to justify the assumption made in § 14. For let

$$P_n\int_\mu^\infty \frac{d\mu}{(\mu^2-1)P_n^2} = A\,e^{-(n+\frac{1}{2})u}\int_0^\pi \frac{\sin^{2n}\psi}{\sqrt{1-e^{-2u}\sin^2\psi}}\,d\psi,$$

where A is a constant to be determined. If we suppose u to be so large that the first term of P_n need only be considered, and also that $2\sinh u$ may be put equal to e^u, then we shall have

$$A = 1,$$

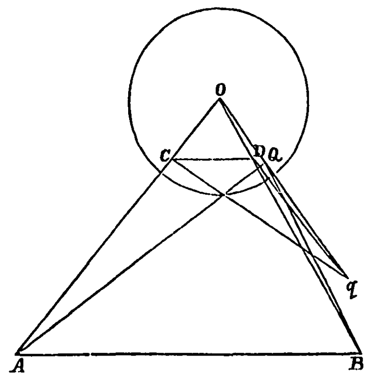
thus proving the equality of the two expressions for Q_n assumed in § 14.

Sphere and Ring.

19. A similar process of solution may be applied when, instead of the ring B, we substitute a sphere with its centre on the rectilineal axis of the ring A.

Referring to the figure, we suppose C, D to be the inverted positions of A, B, the extremities of the diameter of the dipolar axis of the ring. Let the angle subtended by this diameter at the centre O be denoted by 2α, and let Q be any point inside the spheres, q its inverted position. Then, by similar triangles,

$$\frac{Cq}{Oq} = \frac{AQ}{AO}, \text{ and } \frac{Dq}{Oq} = \frac{BQ}{BO};$$



therefore
$$\frac{Cq}{Dq} = \frac{AQ}{BQ}.$$

Hence, if u', v' be the dipolar coordinates of q referred to CD, while those of Q are u, v referred to AB, we have

$$u' = u.$$

Also, from the geometry of the figure, we readily find that, if q is below CD,

$$v' = v - 2a,$$

and, if above,
$$v' = 2a - v.$$

Moreover, $OQ^2 = \rho^2 + (c\cot a - z)^2 = \dfrac{c^2}{\sin^2 a}\,\dfrac{\cosh u - \cos(v-2a)}{\cosh u - \cos v}.$

Hence any harmonic, say

$$\pi \cos mv \sqrt{2(\cosh u - \cos v)}\,P_n(\cosh u),$$

becomes, by the rule of inversion, if q is below CD,

$$\pi \cos m(v_1+2a)\sqrt{2\{\cosh u_1 - \cos(v_1+2a)\}}\,P_n(\cosh u_1)$$
$$\times \frac{c}{R\sin a}\sqrt{\frac{\cosh u_1 - \cos v_1}{\cosh u_1 - \cos(v_1+2a)}},$$

viz.,
$$\frac{c}{R\sin a}\cos m(v'+2a)\sqrt{2(\cosh u_1 - \cos v_1)}\,P_n(\cosh u_1);$$

and, if q is above CD, the same expression with $2a - v_1$ written instead of $v'+2a$.

This result will greatly simplify the electrostatic problem for a sphere and ring.

The following presents to the Library were received during the recess:—

"Vierteljahrschrift der Naturforschenden Gesellschaft in Zurich," 38er Jahrgang, Heft 1 and 2; 1893.

"Proceedings of the Edinburgh Mathematical Society," Vol. XI.; Session 1892-3.

"Journal of the Institute of Actuaries," Vol. XXX., Pt. 6, July, 1893; Vol. XXXI., Pt. 1, October, 1893.

"Proceedings of the Royal Society," Vol. LIII., No. 322-325; Vol. LIV., No. 326.

"Beiblätter zu den Annalen der Physik und Chemie," Band XVII., Stücke 5-8; Leipzig, 1893.

"Nyt Tidsskrift for Mathematik," A. Fjerde Aargang, No. 3; B. Fjerde Aargang, No. 2; Copenhagen, 1893.

"Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig," Mathematisch-Physische Classe, 1893, 2 and 3.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," 4me Serie, Tomes I. and III., 1ère cahier.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. VII., Nos. 2 and 3; 1892-3.

"Nieuw Archief voor Wiskunde," Deel XX., Stuk 2; Amsterdam, 1893.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome XXVII., Livraisons 1 and 2; Harlem, 1893.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. XI., No. 4; Coimbra, 1893.

"Bulletin de la Société Mathématique de France," Tome XXI., Nos. 4, 5, 6.

"Bulletin of the New York Mathematical Society," Vol. II., Nos. 9 and 10; 1893.

"Bulletin des Sciences Mathématiques," Tome XVII., Avril, Mai, Juin, et Juillet; Paris, 1893.

"Mathematical Questions, with their Solutions," edited by W. J. C. Miller, Vol. LIX.; London, 1893.

"Treatise on the Kinetic Theory of Gases," by H. W. Watson, 2nd edition, 8vo; Oxford, 1893.

"Treatise on the Mathematical Theory of Elasticity," by A. E. H. Love, Vol. II., R. 8vo; Cambridge, 1893.

"Ueber einige Eigenschaften der Bessel'schen Function erster Art, insbesondere für ein grosses Argument," von Dr. J. H. Graf. (Offprint from "Zeitschrift für Mathematik und Physik," aus dem 2 Hefte des 38 Jahrgangs.)

"Ueber die Addition und Subtraction der Argumente bei Bessel'schen Functionen, nebst einer Anwendung," von Dr. J. H. Graf. (Offprint from "Math. Annalen," Vol. XLIII., pp. 136-44.)

"Wiskundige Opgaven met de Oplossingen," Zesde Deel, Stuk 1; Amsterdam, 1893.

"Revue Semestrielle des Publications Mathématiques," Tome I., 2ème partie; Amsterdam, 1893.

"Observations Pluviométriques et Thermométriques dans le Département de la Gironde, de Juin, 1891, à Mai, 1892," par Mons. G. Rayet; Bordeaux, 1892.

"Prace Matematyczno-Fizyczne," Tom. IV.; Warsaw, 1893.

"Sphärische Trigonometrie, orthogonale Substitutionen, und elliptische Functionen," von E. Study; Leipzig, 1893.

Mons. M. d'Ocagno.—"Sur la détermination géométrique du point le plus probable donné par un système de droites non convergentes." (Extrait du "Journal de l'Ecole Polytechnique," LXIII.^e cahier, 1893.)

"Jahrbuch über die Fortschritte der Mathematik," Bd. XXII., Heft 3; Berlin, 1893.

"Journal of the Japan College of Science," Vol. v., Part 4; Vol. VI., Part 2; Tokyo, 1893.

"Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," Parts 1–38, 1893.

"Atti della Reale Accademia dei Lincei," 5ª Serie, Rendiconti, Vol. II., Fasc. 1, 2, 4, 5, 6, 8, 9, 10, 11, 12; Roma, 1893.

"Atti della Reale Accademia dei Lincei," Anno CCXC., Rendiconti; Roma, 1893.

"Journal für die reine und angewandte Mathematik," Bd. CXII., Hefte 1–4.

"Annals of Mathematics," Vol. VII., No. 4; May, 1893; University of Virginia.

"Annales de la Faculté des Sciences de Toulouse," Tome VII., Année 1893, 2^{me} Fasc.

"Annali di Matematica," Tomo XXI., Fasc. 2, 3; Milano, 1893.

"Educational Times," July to October, 1893.

"Indian Engineering," Vol. XIII., Nos. 20–25, and Vol. XIV., Nos. 1–12.

"Rendiconti dell' Accademia delle Scienze Fisiche e Matematiche di Napoli," Serie 2, Vol. VII., Fasc. 5–7; Napoli.

"Atti della Reale Accademia delle Scienze Fisiche e Matematiche di Napoli," Serie 2, Vol. v., 1893.

"Electrical Engineer," No. 24, Vol. XI.; June, 1893.

"Annuaire de l'Académie Royale de Belgique," 1892–3, Bruxelles.

"Bulletin de l'Académie Royale de Belgique," 1891–2, Parts 1, 2, 1893, Bruxelles.