# ON THE THEORY OF THE APPLICATION OF EXPANSIONS TO DEFINITE INTEGRALS

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1. In the application of expansions to definite integrals, either the whole integrand or one of its factors is replaced by an infinite series. The new infinite series, now constituted by the integrand, is then integrated term-by-term. The special case in which the integrand consists of a single function will not be discussed here; the problem of determining whether the substitution is in that case allowable is merely the general problem of the term-by-term integration of infinite series. In the more general case, the series which we substitute for one of the factors may converge everywhere to that factor as sum, or it may do so except at a set of content zero, or it may not have that function as sum, or even converge other than at exceptional points. We have an example of the last-named case when we substitute for one of the factors its series of Fourier; if the factor in question is a function of a general character, its Fourier series will not converge at all. On the other hand, the series got by integrating the Fourier series term-by-term always converges; it converges, in fact, uniformly to one of the integrals of the function associated with the original series. The example given by the application of the Fourier expansion suggests the general problem. Let the integrand of the integral to be considered be $f(x)\,g(x)$. If $f(x)$ can be expanded in a converging series, may we integrate this series when multiplied term-by-term by $g(x)$? If we cannot conveniently replace $f(x)$ by such a series, expand its integral $F(x)$ and differentiate the series so obtained term-by-term. Can we then integrate term-by-term the series got by multiplying term-by-term this last series by $g(x)$, and, if so, shall we in this way obtain $\int f(x)\,g(x)\,dx$? If $F(x)$ cannot be conveniently expanded, and its integral can, take the series representing this latter integral and differentiate it twice. Can the new series, when multiplied term-by-term by $g(x)$, be integrated term-by-term, and is the sum in this case $\int f(x)\,g(x)\,dx$?

This problem is of great practical importance, and its solution enables us to deal in a systematic manner with integrals whose treatment by ordinary methods involves special artifices.

In the case when $f(x)$ itself is capable of expansion the problem has already received partial solution, but, for completeness, I have given here, with indications of proof, the enunciations of the corresponding theorems, as well as the enunciations and detailed proofs of those that are new. I denote by $s_n(x)$ the $n$-th partial summation of what I call *the original series;* this is the series representing $f(x)$, when such a series exists, and otherwise is the series got by differentiating term-by-term a series giving the value of $\int f(x)\,dx$ or $F(x)$; or, more generally, is the series got by differentiating $p$ times a series giving the $p$-th integral of $f(x)$. With this understanding, the process in question is always allowable as a means of evaluating the integral under the following circumstances:—

(i) If $s_n(x)$ is a monotone function of $x$, and the succession $s_n(x)$ is a sequence,* while $g(x)$ is any function possessing a Harnack-Lebesgue integral.

(ii) If $s_n(x)$ converges boundedly† as $n$ increases, while $g(x)$ may be any function possessing a Lebesgue integral, proper or improper.

(iii) If $s_n(x)$ converges, not necessarily boundedly, and $\int [s_n(x)]^2\,dx$ is a bounded function of $(x, n)$, while $g(x)$ is any function whose square has a Lebesgue integral, proper or improper.

(iv) If the functions $|s_n(x)|$ form an integrable sequence, while $g(x)$ is any bounded function.

(v) If $\int s_n(x)\,dx$ converges to $\int f(x)\,dx$, and $\int |s_n(x)|\,dx$ is bounded, while $g(x)$ is any bounded function possessing only discontinuities of the first kind.

(vi) If $\int s_n(x)\,dx$ converges boundedly to $\int f(x)\,dx$, while $g(x)$ is any function of bounded variation.

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* For brevity we have not usually distinguished between a sequence and a succession which converges except at a set of content zero. For the same reason I have not specially called attention to the fact that the functions which occur in the theorems or in the processes may, under certain circumstances, only exist at a set complementary to a set of content zero.

† We say that a sequence converges boundedly if $s_n(x)$ is a bounded function of the ensemble $(x, n)$, so that there are no points of non-uniform convergence with infinite measure; in other words, that the peak and chasm functions are finite. The *boundedness* must, of course, be without exception, even when non-convergence is allowed at a set of points of content zero, in accordance with the previous footnote.

(vii) If at the limits of integration $g(x) = 0$ or $\int s_n(x)dx$ converges to the integral $F(x)$, and the repeated integral $\int dx \int s_n(x)dx$ converges everywhere to the integral of $F(x)$, while $g(x)$ is any function which possesses a differential coefficient of bounded variation ; and so on.

In all these cases we notice that the convergence of the new series when integrated is a bounded convergence, and it is under circumstances which are specified—uniform convergence. The theorems are also true, with, in certain cases, limitations which are given, when the range of integration is infinite.

I have purposely, in the present account, kept myself aloof from questions connected with the various methods in existence of attaching a conventional sum to a non-convergent series. To have done otherwise would have taken me too far afield. I hope none the less that the results obtained will be found interesting and useful, as well as systematic.

I have not thought it necessary to give more than a couple of examples ; those selected will, perhaps, sufficiently serve the purpose of justifying the discussion here carried out. The first of these examples has been recently given* as illustrating the existence of one of "a number of comparatively simple cases that present themselves in practice, and do not come under any really general theorem." The second, almost equally simple, example has been specially constructed with the view of illustrating the final theorem of the present paper.

2. In the first four theorems which follow, it is immaterial whether we suppose the given function $f(x)$ to have itself been expanded in a series which converges everywhere except at most at a set of content zero, whose partial summation is $s_n(x)$, or whether we suppose $s_n(x)$ obtained by differentiating the corresponding partial summation $S_n(x)$ of the series which represents an indefinite integral of $f(x)$. In the latter case we must, of course, suppose that $S_n(x)$ is an integral, so that it is the integral of its differential coefficient $s_n(x)$. In fact, in all four theorems, $s_n(x)$ is seen to trace out an integrable sequence.

Calling its limiting function $s(x)$, we have therefore

$$\int s(x)dx = \underset{n=\infty}{\text{Lt}} \int s_n(x)dx = \underset{n=\infty}{\text{Lt}}\, S_n(x) = \int f(x)dx.$$

Hence $s(x)$ and $f(x)$ differ at most at a set of content zero, so that $f(x)$ may, under the sign of integration, replace the limit of $s_n(x)$ when $n$ increases indefinitely.

* T. J. I'A. Bromwich, *Introduction to the Theory of Infinite Series* (1908), p. 451.

3. THEOREM 1.—*If the partial summations $s_n(x)$ are monotone ascending (or descending) functions of $x$ and form a convergent sequence,* and $g(x)$ possesses a Harnack-Lebesgue integral, the sequence of Harnack-Lebesgue*† *integrals $\int_c^z s_n(x)\, g(x)\, dx$ converges to $\int_c^z f(x)\, g(x)\, dx$, and the convergence is uniform when the interval is finite. The convergence still holds when the interval is infinite, provided only the sequence is bounded at infinity.*‡

We may evidently assume $s_n(x)$ to be monotone decreasing, since in the alternative case we only have to change the variable from $x$ to $-x$. The upper bound of the sequence $s_n(x)$ is then the upper bound of $s_n(c)$, and is therefore either the value of $s_n(c)$ for some integer $n$ or the limiting value $s(c)$; in either case it is finite. Similarly the lower bound of the sequence is finite.

Again, adding a finite constant $A$, we get a bounded sequence of monotone decreasing positive functions. It follows then, by a theorem proved recently by myself in the *Messenger of Mathematics*,§ that the functions $g(x)\{A+s_n(x)\}$ form a sequence which is integrable term-by-term in the Harnack-Lebesgue manner. That is

$$\operatorname*{Lt}_{n=\infty} \int_c^z g(x)\{A+s_n(x)\}\, dx = \int_c^z g(x)\{A+f(x)\}\, dx,$$

whence also

$$\operatorname*{Lt}_{n=\infty} \int_c^z g(x)\, s_n(x)\, dx = \int_c^z g(x)\, f(x)\, dx,$$

where $f(x)$ has been written for the limiting function of $s_n(x)$ under the integral sign in accordance with § 2.

Thus the convergence of $\int_c^z s_n(x)\, g(x)\, dx$ to its limit $\int_c^z f(x)\, g(x)\, dx$ has been proved. In order to discuss the uniformity of the convergence, it will be convenient to re-prove the theorem quoted from the *Messenger*. Instead of employing Moore's conditions, I shall use the theorem‖ relating to the uniformity of approach of the defining sequence of integrals to the Harnack integral, defined as the limit of that sequence.

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* See first footnote above. There may be oscillation, but it must be finite.

† That these are Harnack-Lebesgue integrals follows from the conditions of the problem. See E. W. Hobson, "The Second Mean-Value Theorem in the Integral Calculus," *Proc. London Math. Soc.*, Ser. 2, Vol. 7 (1908), p. 21.

‡ It is, of course, assumed in this case that $g(x)$ has a Harnack-Lebesgue integral in the whole infinite interval, that is, that the integral over a finite interval has a unique finite limit as the upper limit of integration moves off to infinity.

§ "On a Theorem in the Harnack Integration of Series" (1910), *Messenger of Mathematics*, pp. 101-106.

‖ See Hobson's *Theory of Functions of a Real Variable*, p. 383.

Assuming any positive quantity $e$, the theorem asserts that we can find a set of intervals $d$, surrounding the Harnack points, such that, if $g_d$ denotes the function which is equal to $g$, except in the intervals $d$, where it is zero, then

$$\left| \int_c^z (g-g_d)\,dx \right| \leqslant e,$$

for all values of $z$ in the range of integration. Hence, by the Second Theorem of the Mean,

$$\left| \int_c^z s_n(g-g_d)\,dx \right| \leqslant e s_n(c) \tag{1}$$

for all values of $z$. Here it has been supposed, for convenience, that the functions $s_n$ are positive and monotone decreasing.

Similarly,

$$\left| \int_c^z f(x)(g-g_d)\,dx \right| \leqslant e f(c). \tag{2}$$

But, since the function $g_d$ has no Harnack points, it has a Lebesgue integral; therefore, by Theorem 2 of the present paper, we can find $m$ so that, for all values of $n$ greater than $m$ and all values of $z$,

$$\left| \int_c^z g_d(f-s_n)\,dx \right| \leqslant e. \tag{3}$$

From (1), (2), and (3) it follows that

$$\left| \int_c^z g(x)(f-s_n)\,dx \right| \leqslant eA,$$

where $A$ is a finite quantity, independent of $n$ or $z$. This proves that the left-hand side converges uniformly to zero, whence the required result at once follows for the special kind of sequence considered, and therefore, by the argument used above in the first instance, the theorem holds in all its generality for a finite interval of integration. To prove the convergence for an infinite interval, we have recourse to the general theory of change of order of limits, and make use of the Second Theorem of the Mean. In accordance with this theory, $\int_c^z s_n(x)\,g(x)\,dx$ will have its repeated limits, when $z$ and $n$ become infinite, equal, provided only that

$$\underset{z=\infty}{\mathrm{Lt}}\ \underset{n=\infty}{\mathrm{Lt}} \left| \int_z^\infty s_n(x)\,g(x)\,dx \right| = 0.$$

But, if $B$ is the upper bound of the absolute value of $s_n(x)$ for all values of the ensemble $(n, x)$, the modulus of the integral $\leqslant B \left| \int_z^{z'} g(x)\,dx \right|$, and therefore, since $g(x)$ possesses a Harnack-Lebesgue integral in the whole infinite interval, the repeated limit in question is clearly zero.

Hence we may write

$$\int_c^\infty f(x)\,g(x)\,dx = \underset{z=\infty}{\mathrm{Lt}} \int_c^z f(x)\,g(x)\,dx = \underset{z=\infty}{\mathrm{Lt}}\ \underset{n=\infty}{\mathrm{Lt}} \int_c^z s_n(x)\,g(x)\,dx$$

$$= \underset{n=\infty}{\mathrm{Lt}}\ \underset{z=\infty}{\mathrm{Lt}} \int_c^z s_n(x)\,g(x)\,dx = \underset{n=\infty}{\mathrm{Lt}} \int_c^\infty s_n(x)\,g(x)\,dx,$$

which proves the theorem.

4. The following theorem is now well known, but it is rarely stated in its most general form.

THEOREM 2.—*If $s_n(x)$ converges* boundedly as $n$ increases and $g(x)$ is any function possessing a Lebesgue integral, proper or improper, the sequence of Lebesgue integrals $\int s_n(x)\,g(x)\,dx$ converges to $\int f(x)\,g(x)\,dx$, the limits of integration being finite or infinite. Moreover, the convergence is uniform throughout any finite interval.*

For, if $U$ be any quantity greater than the upper bound of $|s_n(x)|$, for all values of $n$ and $x$, $s_n(x)\,g(x)$ lies between $U|g(x)|$ and $-U|g(x)|$, which are both functions possessing Lebesgue integrals in the finite or infinite interval considered. Hence, by a known theorem,† since the functions $s_n(x)\,g(x)$ form a sequence, the unique limiting function of which may, for the purposes of integration (§ 2), be taken to be $f(x)\,g(x)$, this sequence is integrable so that

$$\int f(x)\,g(x)\,dx = \underset{n=\infty}{\mathrm{Lt}} \int s_n(x)\,g(x)\,dx,$$

which proves the theorem as far as the convergence itself is concerned. The uniformity follows, for example, from the fact that,‡ if a sequence of functions, bounded below, is integrable, the sequence of integrals converges uniformly, for it is plain we shall obtain such a sequence if we add to each term of our sequence $U|g(x)|$.

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* See first footnote above, § 1.

† W. H. Young, "On Semi-integrals and Oscillating Successions of Functions" (1910), §§ 31 and 34, *supra*, pp. 286–325.

‡ Vitali, "Sulle Funzioni Integrali" (1905), *Atti di Torino*, Vol. LX, pp. 1021–1034. Cp. "On Semi-integrals and Oscillating Successions of Functions," § 25.

5. The next theorem is easily seen to follow from one given by myself elsewhere, and modified from Lebesgue.*

THEOREM 3.—*If* $s_n(x)$ *converges*† *and* $\int \{s_n(x)\}^2 dx$ *is bounded, then*

$$\operatorname*{Lt}_{n=\infty} \int s_n(x)\, g(x)\, dx = \int f(x)\, g(x)\, dx,$$

*provided the square of* $g(x)$ *possesses a Lebesgue integral, proper or improper; the limits of integration may here be finite or infinite. Moreover, the convergence of* $\int s_n(x)\, g(x)\, dx$ *to its limit is uniform throughout any finite interval.*

For, since $\int \{s_n(x)\}^2 dx$ is bounded, and the integrands, being positive and converging to $[s(x)]^2$, form a sequence which is semi-integrable below, we have

$$0 \leqslant \int [s(x)]^2 dx \leqslant \int [s_n(x)]^2 dx,$$

which proves that $[s(x)]^2$ possesses a Lebesgue integral, proper or improper. But, by hypothesis, $[g(x)]^2$ also possesses a Lebesgue integral, therefore

$$2\{[s(x)]^2 + [g(x)]^2\}$$

also possesses a Lebesgue integral. Hence, since this latter expression is equal to

$$\{s(x)+g(x)\}^2 + \{s(x)-g(x)\}^2,$$

both the squares in this expression have Lebesgue integrals, and therefore the same is true of half their difference, that is, $s(x)\, g(x)$.

Thus the conditions required in the theorem referred to in the footnote‡ are satisfied, so that the sequence of functions $s_n(x)\, g(x)$ is integrable. That is,

$$\int s(x)\, g(x)\, dx = \operatorname*{Lt}_{n=\infty} \int s_n(x)\, g(x)\, dx.$$

Putting $g(x) = 1$, it appears that the sequence of functions $s_n(x)$ is integrable, so that (§ 2) for purposes of integration we may replace $s(x)$ by $f(x)$, which proves the theorem as far as the convergence is concerned.

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* Lebesgue's enunciation is as follows:—If $f(x)\, g(x)$ and $[g(x)]^2$ are summable, and if, further, $[f(x) - s_n(x)]^2$ is summable, and such that its integral is in the whole interval considered bounded, then the sequence $s_n(x)\, g(x)$, supposed to converge to $f(x)\, g(x)$, is integrable term-by-term (*Sur les Intégrales Singulières*, 1910, Fac. de Toulouse, 3e Série I, p. 50).

† See first footnote above, § 1.

‡ The theorem given in my paper on Semi-integrals, § 30, refers to successions of functions which do not necessarily converge anywhere. The special case of this theorem which we here require is the following:—If $f(x)\, g(x)$ and $[g(x)]^2$ are summable, and if, further, $[s_n(x)]^2$ is summable and such that its integral is bounded in the whole interval considered, then the succession $s_n(x)\, g(x)$, supposed to converge, except at a set of content zero, to $f(x)\, g(x)$, is absolutely integrable term-by-term.

That the convergence is uniform follows also from the theorem quoted, since the succession $s_n(x)\,g(x)$ is absolutely integrable. This theorem holds still when the limits of integration are not both finite, for in this case, by Schwarz's lemma,*

$$\left(\int_z^{z'} s_n g\,dx\right)^2 \leqslant \int_z^{z'} s_n^2\,dx \int_z^{z'} g^2\,dx \leqslant B\int_z^{z'} g^2\,dx, \tag{1}$$

where $B$ is any positive quantity greater than the upper bound of $\int s_n^2\,dx$.

Now $g^2$ possesses an integral from $c$ to $\infty$, so that when $z$ moves off to infinity, $z'$ remaining greater than $z$, the right-hand side has zero as unique limit, and therefore so has the left-hand side; therefore $s_n g$ has an integral from 0 to $\infty$.

Again
$$\left(\int_z^{z'} (s-s_n)g\,dx\right)^2 \leqslant \int_z^{z'} (s-s_n)^2\,dx \int_z^{z'} g^2\,dx$$
$$\leqslant \int_z^{z'} (s^2+s_n^2+2\,|ss_n|)\,dx \int_z^{z'} g^2\,dx. \tag{2}$$

But
$$\int_z^{z'} |2ss_n|\,dx \leqslant \int_z^{z'} s^2\,dx+\int_z^{z'} s_n^2\,dx;$$

therefore the left-hand side of (2) is
$$\leqslant \int_z^{z'} 2\,(s^2+s_n^2)\,dx \int_z^{z'} g^2\,dx.$$

Again, $s_n^2 \geqslant 0$, so that the sequence traced out by $s_n^2$ is semi-integrable below. Hence
$$\int_z^{z'} s^2\,dx \leqslant \operatorname*{Lt}_{n=\infty} \int_z^{z'} s_n^2\,dx \leqslant B.$$

Thus we get
$$\left(\int_z^{z'} (s-s_n)g\,dx\right)^2 \leqslant 4B\int_z^{z'} g^2\,dx.$$

As before, this shews that the left-hand side has zero as unique limit when $z$ moves off to infinity, and proves that $(s-s_n)g$ is integrable from $c$ to infinity, and therefore, since $s_n g$ has the same property, that $sg$ is integrable from $c$ to infinity.

Now
$$\int_c^{\infty} (s-s_n)\,g\,dx = \int_c^{z} (s-s_n)\,g\,dx+\int_z^{\infty} (s-s_n)g\,dx.$$

Here the first integral has, when $z$ is fixed and $n$ increases indefinitely, the unique limit zero, and the second integral is, by (3), such that, if $n$ is fixed or not, provided only $z$ moves off to infinity, this integral has the

* See, for instance, my paper on "A New Method in the Theory of Integration" (1910), *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 37.

unique limit zero. Thus, given any positive quantity $e$, we can find $z$ and $n$ so that

$$\int_c^\infty (s-s_n) g\, dx < e,$$

and this will be true for all greater values of $n$. Thus

$$\underset{n=\infty}{\mathrm{Lt}} \int_c^\infty (s-s_n) g\, dx \leqslant e.$$

Since $e$ is at our disposal, this shews that the limits on the left all coincide and have the value zero. Thus

$$\int_c^\infty s(x)\, g(x)\, dx = \underset{n=\infty}{\mathrm{Lt}} \int_c^\infty s_n(x)\, g(x)\, dx.$$

6. The next theorem has, as far as I know, not been stated, but it is an immediate consequence of the theorem, given in my paper on Semi-integrals, that a sequence is integrable if two integrable sequences can be found, between which the given sequence everywhere lies.

THEOREM 4.—*If* $|s_n(x)|$ *traces out an integrable sequence and* $g(x)$ *is any bounded function, then*

$$\underset{n=\infty}{\mathrm{Lt}} \int_c^z s_n(x)\, g(x)\, dx = \int_c^z f(x)\, g(x)\, dx,$$

*the limits of integration being finite or infinite. Moreover, the convergence of* $\int s_n(x)\, g(x)\, dx$ *to its limit is uniform throughout any finite interval.*

For, if $U$ is any quantity greater than the upper bound of $|g(x)|$, $s_n(x)\, g(x)$ lies between $U|s_n(x)|$ and $-U|s_n(x)|$. Hence, by the theorem referred to above, the sequence traced out by $s_n(x)\, g(x)$ is integrable. In particular this is the case if $g(x) = 1$, so that, for purposes of integration, we may replace $s(x)$ by $f(x)$. Thus

$$\underset{n=\infty}{\mathrm{Lt}} \int_c^z s_n(x)\, g(x)\, dx = \int_c^z f(x)\, g(x)\, dx,$$

which proves the theorem as far as the convergence is concerned.

That the convergence is uniform follows in precisely the same way as in Theorem 2, except that we now add the sequence $U|s_n(x)|$ term-by-term to the given sequence. Since the sequence so obtained, as well as the sequence added, is bounded below, each when integrated gives rise to a uniformly convergent sequence: hence the same is true of their difference, which proves the theorem.

7. The last theorem suggests a remark which it may be advisable to make. The theorems obtained state that the process in question can be carried out if, under the circumstances alleged, $g(x)$ is any member of a certain class of functions. It may, of course, also be allowable if $g(x)$ be a suitably chosen member of a different class. Thus in the last theorem it may be possible in a particular case to find an unbounded function $g(x)$ for which the result is true. An obvious example is that in which $|s_n(x)|$ is a monotone function of $n$, and $\underset{n=\infty}{\mathrm{Lt}} \int |g(x)|\,|s_n(x)|\,dx$ exists or $\int |g(x)|\,|f(x)|\,dx$ exists.

This follows, for instance, from the theory of monotone sequences.

8. In the remaining theorems of the paper we may suppose that $f(x)$ cannot be conveniently expanded, and that instead we have recourse to the expansion of its integral or one of its repeated integrals. This may arise from the fact that $f(x)$ cannot be expanded in a series which satisfies any of the requirements of the Theorems 1–4 just given. In Theorems 5 and 6 we suppose $S_n(x)$ to be the partial summation of the expansion of $F(x)$, that is, $\int f(x)\,dx$, and write

$$s_n(x) = \frac{d}{dx} S_n(x).$$

In the most general case $S_n(x)$ must be an integral, so that $s_n(x)$ exists everywhere except at most at a set of content zero, where we may, if we please, attribute to it the value zero. With this understanding, we have the following theorem.

THEOREM 5.—*If* $\int |s_n(x)|\,dx$ *is bounded for all values of the ensemble* $(x, n)$ *considered, and* $\int s_n(x)\,dx$ *or* $S_n(x)$ *converges to* $\int f(x)\,dx$ *or* $F(x)$, *while* $g(x)$ *is any bounded function possessing only discontinuities of the first kind, then*

$$\underset{n=\infty}{\mathrm{Lt}} \int_c^z s_n(x)\,g(x)\,dx = \int_c^z f(x)\,g(x)\,dx.$$

*This theorem remains true when one or both the limits of integration are infinite, provided only the original series when integrated represents a continuous function in the whole closed infinite interval.*

*Further, the convergence of* $\int_c^z s_n(x)\,g(x)\,dx$ *to its limit is necessarily bounded, and it is uniform if the convergence of* $\int_c^z s_n(x)\,dx$ *to its limit is uniform.*

Since $g(x)$ has only discontinuities of the first kind, it has only a finite number of discontinuities at which the jump on one side at least is $\geqslant 1/p$, for at a limiting point of points of this kind the upper and lower limits on one side at least would differ by at least $1/p$, so that such a limiting point would be a discontinuity of the second kind. If we take the interval between two adjacent discontinuities $\geqslant 1/p$ and replace the value of $g(x)$ at the end-points by its limiting values on the sides towards the interval considered, so that these discontinuities are to be replaced by continuities, all the jumps in the closed interval will be less than $1/p$, so that, corresponding to each point we have an interval containing the point in which the oscillation of $g(x)$ is less than $1/p$. Hence, applying the Heine-Borel theorem, we can divide the interval into a finite number of parts in each of which the oscillation of the function is less than $1/p$.

Thus, when we have to do with a function $g(x)$ having only discontinuities of the first kind, we can always divide the continuum into a finite number of segments, such that the oscillation of $g(x)$ *inside* each segment is less than $1/p$, the oscillation inside the segment being supposed to take no account of the values of $g(x)$ at the end-points of the segment, but only of the limiting values there.

If we define a function $g_p(x)$ as equal to $g(x)$ at all the points of division, and equal to the upper bound* of $g(x)$ in each open segment at each point of that open segment, we shall have

$$| g(x)-g_p(x) | < 1/p,$$

so that the functions $g_p(x)$, as $p$ increases, converge uniformly to $g(x)$.

Also, by the Second Theorem of the Mean or otherwise,

$$\left| \int_c^z \{g(x)-g_p(x)\} \, s_n(x)\,dx \right| < U/p, \qquad (1)$$

provided $\int_c^z | s_n(x) | \, dx$ is, for all values of $z$ and $n$, numerically less than $U$. Hence any limit of the left-hand side when $n$ increases indefinitely is $\leqslant U/p$, so that, proceeding after taking the limit to let $p$ increase indefinitely in such a way as to give a unique limit, we see that any repeated limit $\underset{p=\infty}{\mathrm{Lt}} \; \underset{n=\infty}{\mathrm{Lt}}$ of the left-hand side is zero. Thus

$$\underset{n=\infty}{\mathrm{Lt}} \int_c^z g(x)\,s_n(x)\,dx = \underset{p=\infty}{\mathrm{Lt}} \; \underset{n=\infty}{\mathrm{Lt}} \int_c^z g_p(x)\,s_n(x)\,dx = \underset{p=\infty}{\mathrm{Lt}} \int_c^z g_p(x)\,f(x)\,dx. \qquad (2)$$

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* Or any other value between the upper and lower bounds inclusive.

But the functions $g_p(x)$ form a bounded sequence, and $f(x)$ is a summable function; therefore, by a known theorem, already frequently used, the functions $g_p(x)f(x)$ form an integrable sequence, so that the last member of (2) is equal to

$$\int_c^z g(x)f(x)\,dx.$$

Hence also
$$\operatorname*{Lt}_{n=\infty}\int_c^z g(x)\,s_n(x)\,dx = \int_c^z g(x)f(x)\,dx,$$

whatever sequence of values of $n$ be taken so as to give a unique limit to the left-hand side of the inequality last given. Thus the limit in the preceding equation is unique, which proves the theorem.

That the convergence is always bounded and that it is uniform when the convergence of $\int s_n(x)dx$ to its limit is uniform, may be seen as follows. From (1) we deduce that this is the case for $g(x)$ if it is true for $g_p(x)$ for every value of $p$. But for fixed $p$, the result obviously holds, since $g_p$ is constant in stretches.

If $z$ is infinite the above reasoning still holds down to the inequality (1), in which we may put $z=\infty$, if we alter the sign $<$ to $\leqslant$, or which may be written

$$\operatorname*{Lt}_{z=\infty}\left|\int_c^z [g(x)-g_p(x)]\,s_n(x)\right| dx \leqslant U/p.$$

Hence, by the same reasoning as before, we get the equation corresponding to (2), which may be written

$$\operatorname*{Lt}_{p=\infty}\operatorname*{Lt}_{n=\infty}\left|\int_c^\infty [g(x)-g_p(x)]\,s_n(x)\right| dx = 0,$$

or
$$\operatorname*{Lt}_{p=\infty}\operatorname*{Lt}_{n=\infty}\operatorname*{Lt}_{z=\infty}\left|\int_c^z [g(x)-g(p)]\,s_n(x)\right| dx = 0,$$

or
$$\operatorname*{Lt}_{n=\infty}\int_c^\infty g(x)\,s_n(x)dx = \operatorname*{Lt}_{p=\infty}\operatorname*{Lt}_{n=\infty}\operatorname*{Lt}_{z=\infty}\int_c^z g_p(x)\,s_n(x)\,dx;$$

this we may write
$$= \operatorname*{Lt}_{p=\infty}\operatorname*{Lt}_{n=\infty}\operatorname*{Lt}_{z=\infty}\left\{\int_c^{z'_p} g_p(x)\,s_n(x)\,dx+\int_{z'_p}^z g_p(x)s_n(x)\,dx\right\},$$

where $z'_p$ is the nearest of the points of division to the point infinity (with proper sign) considered. The first integral gives us no trouble, since it falls under the former case. In the latter integral $g_p$ is constant, so that we may write the term

$$\operatorname*{Lt}_{p=\infty} g_p(x)\operatorname*{Lt}_{n=\infty}\operatorname*{Lt}_{z=\infty}\int_{z'_p}^z s_n(x)\,dx,$$

which is equal to

$$\underset{p=\infty}{\mathrm{Lt}}\, g_p(x) \underset{z=\infty}{\mathrm{Lt}}\, \underset{n=\infty}{\mathrm{Lt}} \int_{z'_p}^{z} s_n(x)\,dx = \underset{p=\infty}{\mathrm{Lt}}\, g_p(x) \underset{z=\infty}{\mathrm{Lt}} \int_{z'_p}^{z} f(x)\,dx$$

$$= \underset{p=\infty}{\mathrm{Lt}}\, \underset{z=\infty}{\mathrm{Lt}} \int_{z'_p}^{z} g_p(x) f(x)\,dx,$$

if, and only if, the condition given in the enunciation holds, that is, if $\underset{n=\infty}{\mathrm{Lt}} \int_{z'_p}^{z} s_n(x)\,dx$ represents a continuous function of $z$ up to and including the point infinity. In this case, therefore, we have

$$\underset{n=\infty}{\mathrm{Lt}} \int_{c}^{\infty} g(x)\, s_n(x)\,dx = \underset{p=\infty}{\mathrm{Lt}} \int_{c}^{\infty} g_p(x) f(x)\,dx,$$

corresponding to equation (2) in the first case. The rest of the argument is unaltered.

9. In the preceding proof I have followed Lebesgue's argument in a kindred matter closely.* It would have been a little simplified if we had been content to state the theorem on the hypothesis that $g(x)$ is continuous, but the gain would not have been great. In the theorem about to be given the whole difficulty consists in the fact that we only require from $g(x)$ that it should be the most general kind of function of bounded variation. The proof of the theorem when $g(x)$ belongs to that particular sub-class of the class of functions of bounded variation which are obtained by Lebesgue integration of other functions is immediate. The whole difficulty consists in the gradual extension of the theorem, first to continuous functions of bounded variation, and then to discontinuous functions of the same class. In the case which ordinarily arises $g(x)$ will have not only its first differential coefficient, but this differential coefficient will be finite, in which case $g(x)$ itself is an integral, viz., the integral of that differential coefficient; the importance of the theorem is then only matched by the simplicity of its proof. For theoretical purposes, however, the theorem is required in all its generality; I again follow Lebesgue in his treatment of a similar problem.

10. THEOREM 6.—*If* $\int s_n(x)\,dx$ *converges boundedly to* $\int f(x)\,dx$, *and* $g(x)$ *is any function of bounded variation in the interval considered, then*

$$\underset{n=\infty}{\mathrm{Lt}} \int s_n(x)\, g(x)\,dx = \int f(x)\, g(x)\,dx.$$

* *Sur les Intégrales Singulières*, p. 59 *seq.*

*Further the theorem remains true when one or both of the limits of integration is infinite, provid d only that either* (i) $\underset{n=\infty}{\text{Lt}} \int s_n(x)\,dx$ *is continuous at the point, or points, infinity, i.e., provided* $\int f(x)\,dx$ *exists, and is equal to the sum of the limits of the separate terms in the expansion of* $\int f(x)\,dx$, *or* (ii) $g(x)$ *converges to zero as* $x$ *increases. Moreover, the convergence of* $\int s_n(x)\,g(x)\,dx$ *is necessarily bounded, and it is uniform if that of* $\int s_n(x)\,dx$ *is uniform.*

*Note.*—In the above theorem, when the interval of integration is infinite, we must make the obvious convention that a function which is of bounded variation in every finite interval is only of bounded variation in an infinite interval, if the limits (necessarily unique) of the positive and negative variations, are, as $x$ increases or decreases indefinitely, finite.

If $g(x)$ be an integral we are able to use at once the theorem of integration by parts, and the required result follows by the reasoning below in an abbreviated form. Next, let $g(x)$ be a continuous function of bounded variation; we must then take a new variable, viz., $t$ the arc of the curve $y = g(x)$.* Then $t$ is a monotone continuous function of $x$, and therefore $x$ is a monotone continuous function of $t$, and we may suppose it to be monotone increasing; $y$ is also a function of $t$, so that we may write

$$x = x(t), \quad y = y(t).$$

Now the derivates of $y(t)$ lie between $-1$ and $+1$; therefore, by Lebesgue's theorem, $dy/dt$ necessarily exists, except possibly for a set of values of $t$ of content zero, and $y(t)$ is the integral of any one of its derivates. Thus

$$g\,[x(t)] = y(t) = \int \frac{dy}{dt}\,dt, \tag{1}$$

which shews that $g[(x, t)]$ is an integral with respect to $t$.

Also $$\int s_n(x)\,dx = S_n(x);$$

and therefore, since $x$ is a monotone increasing function of $t$,

$$\int s_n[x(t)]\,\frac{dx}{dt}\,dt = S_n[x(t)], \tag{2}$$

so that the integral on the left has $F(x)$ as unique limit, when $n$ increases

* Young's *Theory of Sets of Points*, p. 266.

indefinitely. Hence

$$\underset{n=\infty}{\mathrm{Lt}} \int_c^z s_n(x)\, g(x)\, dx = \underset{n=\infty}{\mathrm{Lt}} \int \{s_n[x(t)]\} \frac{dx}{dt}\, g[x(t)]\, dt$$

$$= \underset{n=\infty}{\mathrm{Lt}} \left\{ g(z)\, S_n(z) - g(c)\, S_n(c) - \int S_n[x(t)] \frac{d}{dt} g[x(t)]\, dt \right\}, \tag{3}$$

since, by (1) and (2), both $g[x(t)]$ and $S_n[x(t)]$ are integrals with respect to $t$.

Since,* by hypothesis, the convergence of the integral series to $S(x)$ is bounded, $S_n(x)$ is a bounded function of the ensemble $(x, n)$; and therefore $S_n[x(t)]$ is a bounded function of the ensemble $(t, n)$. Therefore $S_n[x(t)] \frac{d}{dt} g(x, t)$ is numerically less than $A \left| \frac{d}{dt} g[x(t)] \right|$, where $A$ is any quantity greater than the numerical values of the upper and lower bounds of $S_n[x(t)]$, for all values of the ensemble $(t, n)$; that is, the integrand of the integral on the right of the preceding equation is numerically less than a summable function. Hence, by the theorem so frequently used, the sequence of functions of $t$,

$$S_n[x(t)] \frac{d}{dt} g[x(t)]$$

is integrable, so that the limits in the preceding equation all coincide, and we have

$$\underset{n=\infty}{\mathrm{Lt}} \int_c^z s_n(x)\, g(x)\, dx = g(z) F(z) - g(c)\, F(c) - \int F[x(t)] \frac{d}{dt} g[x(t)]\, dt. \tag{4}$$

Now $$F(x) = \int f(x)\, dx,$$

so that, by substitution, $$F[x(t)] = \int f[x(t)] \frac{dx}{dt}\, dt. \tag{5}$$

Thus $F[x(t)]$, as well as $g[x(t)]$, is an integral with respect to $t$, so that, integrating by parts, we get from (3) the equation

$$\underset{n=\infty}{\mathrm{Lt}} \int_c^z s_n(x) g(x) dx = \int g[x(t)] \frac{d}{dt} F[x(t)] dt = \int g[x(t)] f[x(t)] \frac{dx}{dt} dt,$$

since the differential coefficient of $F$, regarded as a function of $t$, is the integrand of the expression for $F$ as an integral on the right-hand side of (4), except for a set of values of $t$ of content zero.

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* We here repeat the argument used in proving Theorem 2, instead of merely quoting that theorem, with a view to the subsequent extension to an infinite interval given below.

Hence, by the formula for integration by substitution the required result follows,

$$\underset{n=\infty}{\mathrm{Lt}} \int_c^x s_n(x)\, g(x)\, dx = \int_c^x g(x) f(x)\, dx. \tag{6}$$

Finally, let $g(x)$ be discontinuous; it has then only a countable set of discontinuities, and these are of the first kind. Let these, arranged in some order, be $p_1, p_2, \ldots$.

Since a function of bounded variation is the difference of two monotone increasing functions, we may assume that $g(x)$ is a monotone increasing function; if we prove the required formula (6) in this case, it will then follow, by subtraction, for the case when $g(x)$ is any function of bounded variation.

Let $J_r(x)$ denote the sum of those of the first $r$ jumps which lie in the closed interval $(c, x)$, where the jump at $c$ is

$$g(c+0)-g(c),$$

and at $x$ is

$$g(x)-g(x-0),$$

while at any intermediate point $x'$ it is

$$g(x'+0)-g(x'-0),$$

always supposing these points belong to the first $r$ discontinuities. Then $J_r(x)$ is a monotone function of $x$, constant in each of the half-open intervals, open on the right, into which the whole interval is divided by the first $r$ discontinuities, and having at each of those discontinuities the same jumps on the left and right as $g(x)$ itself.

As $r$ increases, $J_r$ increases at each point $x$; therefore the monotone increasing sequence of functions $J_r(x)$ defines a unique limiting function $J(x)$, which, like the constituent functions of the sequence, is a monotone increasing function. $J(x)$ is evidently the sum of all the jumps of $g(x)$ in the closed interval $(c, x)$, and is therefore not greater than $g(x)-g(c)$.

Hence $J(x+h)-J(x)$ is the sum of all the jumps in the closed interval $(x, x+h)$; therefore

$$g(x+0)-g(x) \leqslant J(x+h)-J(x) \leqslant g(x+h)-g(x),$$

whence, moving $h$ up to zero,

$$J(x+0)-J(x) = g(x+0)-g(x), \tag{7}$$

which shews that $J(x)-g(x)$ is continuous on the right, and similarly on the left, and is therefore a continuous function, which, since both $g$ and $J$ are monotone ascending functions, is a function of bounded variation. By what has already been proved, therefore, the theorem is true when $g$ is replaced by $J-g$. It is therefore true, provided it holds when $g(x)$ is replaced by $J(x)$.

The formula certainly holds when $g(x)$ is replaced by $J_r(x)$, since $J_r(x)$ is constant in each of a finite number of segments making up the range of integration. Thus we have

$$\underset{n=\infty}{\text{Lt}} \int_c^z s_n(x)\, J_r(x)\, dx = \int_c^z f(x)\, J_r(x)\, dx. \tag{8}$$

Now $J(x)-J_r(x)$ is the sum of all the jumps after the $r$-th in the completely open interval $(c, x)$, together with the left-hand jump at $x$, if $x$ is neither a point of continuity, nor one of the first $r$ discontinuities of $g(x)$. Hence $J(x)-J_r(x)$ is a monotone function of $x$, so that, by the Second Theorem of the Mean,

$$\left| \int_c^z s_n(x)\, [J(x)-J_r(x)]\, dx \right| \leqslant U\, |J(c)-J_r(c)|, \tag{9}$$

where $U$ is the finite upper bound of the absolute value of $S_n(x)$. Hence the repeated limit, or limits $\underset{r=\infty}{\text{Llt}}\ \underset{n=\infty}{\text{Llt}}$ of the left-hand side are not greater than the limit of the right-hand side when $r$ increases without limit. We have therefore

$$\underset{r=\infty}{\text{Lt}}\ \underset{n=\infty}{\text{Llt}} \int_c^z s_n(x)\, [J(x)-J_r(x)]\, dx = 0,$$

whence, by (8),
$$\underset{n=\infty}{\text{Lt}} \int_c^z s_n(x)\, J(x) dx = \underset{r=\infty}{\text{Lt}} \int_c^z g(x)\, J_r(x)\, dx. \tag{10}$$

Now $J_r(x)$ is bounded, say numerically less than $A$, and $g(x)$ is summable, therefore $g(x)\, J_r(x)$, lying between $Ag(x)$ and $-Ag(x)$, is, by the theorem already quoted, the general function of an integrable sequence. Hence, by (10),

$$\underset{n=\infty}{\text{Lt}} \int_c^z s_n(x)\, J(x)\, dx = \int_c^z g(x)\, J(x)\, dx,$$

which proves the required formula when $g(x)$ is replaced by $J(x)$, and therefore, as pointed out above, proves the theorem.

We have next to shew that the result continues to hold under the additional conditions stated, when the interval of integration is no longer finite. Let us first prove that condition (i) is sufficient. Evidently the proof holds as it stands when $z$ is infinite, provided only we secure that equation (3) continues to hold. This requires

$$\int s_n[x(t)] \frac{d}{dt} g\, [x(t)]\, dt$$

to exist when the upper limit of integration is infinite. But this is the case, since the integrand is less than $A\frac{d}{dt}g[x(t)]$, whose integral in any finite interval is the total variation of $g(x)$, and therefore has a unique finite limit as the interval stretches out to infinity. Since therefore the hypothesis (i) enables us to pass from (3) to (4), it is a sufficient condition for the truth of the theorem.

Next to shew that condition (ii) is sufficient. We may now suppose $g(x)$ expressed in the form $g_1(x)-g_2(x)$, where $g_1(x)$ and $g_2(x)$ are both monotone decreasing with zero as limit; it will therefore be sufficient to prove the theorem on the hypothesis that $g(x)$ itself is monotone decreasing and has the limit zero. Now the necessary and sufficient condition that the repeated limits

$$\operatorname*{Lt}_{z=\infty}\operatorname*{Lt}_{n=\infty}\int_c^z s_n(x)\,g(x)\,dx \quad\text{and}\quad \operatorname*{Lt}_{n=\infty}\operatorname*{Lt}_{z=\infty}\int_c^z s_n(x)\,g(x)\,dx,$$

should exist and be equal, is that

$$\operatorname*{Lt}_{z=\infty}\operatorname*{Lt}_{n=\infty}\int_z^{\infty} s_n(x)g(x)\,dx$$

should exist and be equal to zero.

By the Second Theorem of the Mean, this integral is, on the supposition stated, equal to

$$g(z)\int_z^{z'} s_n(x)\,dx.$$

But $\int_z^{z'} s_n(x)\,dx$ is a bounded function of the ensemble $(n, z, z')$ by the conditions of the theorem in the whole infinite interval. Also $g(z)$ converges to zero as $z$ increases. Hence the required result follows.

This proves the convergence of

$$\int_c^z s_n(x)\,g(x)\,dx \quad\text{to}\quad \int_c^z f(x)\,g(x)\,dx.$$

That the convergence is always bounded, and that it is uniform when that of $\int_c^z s_n(x)\,dx$ to its limit is uniform, follows, almost in the same manner as in the proof of the preceding theorem. From (9) it follows that the convergence will be uniform for $J(x)$, if it is so for $J_r(x)$ for all values of $r$. But for fixed $r$ the convergence of $\int_c^z J_r(x)\,s_n(x)\,dx$ to its limit is of course uniform, if $\int_c^z s_n(x)\,dx$ converges uniformly, since $J_r$ is constant in stretches. Hence $\int_c^z s_n(x)\,J(x)\,dx$ converges uniformly to $\int_c^z f(x)\,J(x)\,dx$.

Since $g(x)-J(x)$ is a continuous function of bounded variation, it only remains to prove the result in the case first treated (p. 476), and it will be true generally.

Now, by the same reasoning as before,

$$\int_c^z [s_n(x)-f(x)]\, g(x)\, dx = g(z)[S_n(z)-F(z)]-g(c)[S_n(c)-F(c)]$$
$$-\int \{S_n[x(t)]-F[x(t)]\} \frac{dg}{dt}\, dt,$$

and when the convergence of $S_n(x)$ to $F(x)$ is uniform, we can confine our attention to such values of $n$ that for all values of $x$

$$| S_n(x)-F(x) | < e,$$

so that the preceding equation gives

$$\int_c^z [s_n(x)-f(x)]\, g(x)\, dx < e\left[ | g(z) | + | g(c) | + \int \left| \frac{dg}{dt} \right| dt \right] < eU,$$

where $U$ is a finite constant. This shews that the left-hand side converges uniformly to zero, and therefore that the convergence of $\int_c^z s_n(x)\, g(x)\, dx$ in this case to its limit is also uniform. This completes the proof that, when $\int_c^z s_n(x)\, dx$ converges uniformly, so does $\int_c^z s_n(x) g(x)\, dx$. The same argument proves, without any condition, the boundedness of the convergence, $e$ being now any quantity greater than a certain finite quantity. Thus the whole theorem is proved.

11. The preceding theorem is the first of a chain of similar theorems. The integral series, that is, the expansion of

$$F(x) = \int f(x)\, dx,$$

may be distinguished as the first integral series. Integrating a second time and expanding, say,

$$\int dx \int f(x)\, dx = \operatorname*{Lt}_{n=\infty} T_n(x),$$

the new series, whose partial summations are denoted by $T_n(x)$, may be called *the second integral series*; what we shall call *the original series* is then the series got by differentiating the second integral series term-by-term twice, so that

$$s_n(x) = T_n''(x).$$

Here, as already remarked, the equality may fail at a set of points of content zero without affecting our results, since the quantities involved only occur under the sign of integration. The functions $T'_n(x)$ are therefore supposed to be differentiable, except possibly at a set of points of content zero.

The series got by differentiating the second integral series term-by-term will not in general be what we have called the first integral series, nor does it even necessarily converge. If, however, the individual term or the individual partial summation $T'_n(x)$ is an integral, this series may be got by term-by-term integration of the original series; in particular this is the case if $s_n(x)$ is finite and summable. With this understanding we have the following theorem:—

THEOREM 7.—*If $T_n(x)$, the partial summation of the second integral series, is a bounded function of the ensemble $(x, n)$, and $T'_n(x)$ is an integral, the series got by multiplying the original series term-by-term by a function $g(x)$ possessing a differential coefficient $g'(x)$ which is a function of bounded variation, when integrated term-by-term, converges to*

$$\int_c^z g(x) f(x)\, dx,$$

*provided at the limits of integration $g(x)$ is zero, or $T'_n(x)$ converges to*

$$\int f(x)\, dx.$$

For, by Theorem 6, denoting as before $\int_c^x f(x)dx$ by $F(x)$,

$$\underset{n=\infty}{\mathrm{Lt}} \int_c^z T'_n(x)\, g'(x) dx = \int F(x) g'(x)\, dx. \qquad (1)$$

But, since $g'(x)$ is a bounded function, $g(x)$ is an integral; also $F(x)$ and $T_n(x)$ are integrals; therefore, integrating by parts,

$$\int_c^z T'_n(x) g'(x) dx = \Big[ T'_n(x)\, g(x) \Big]_c^z - \int_c^z T''_n(x)\, g(x)\, dx, \qquad (2)$$

and

$$\int^z F(x)\, g'(x)\, dx = \Big[ F(x)\, g(x) \Big]_c^z - \int_c^z f(x)\, g(x)\, dx. \qquad (3)$$

From (1), (2) and (3) the theorem follows, under the conditions given in the enunciation, remembering that, under the sign of integration $T''_n(x)$ may be replaced by $s_n(x)$.

12. In precisely the same manner we can prove the following theorem :—

THEOREM 7′.—*If* $T_n$, *the partial summation of the third integral series, is a bounded function of the ensemble* $(x, n)$, *and* $T_n''(x)$ *exists and is an integral, the series got by integrating term-by-term the original series multiplied term-by-term by a function* $g(x)$ *possessing a second differential coefficient* $g''(x)$ *which is a function of bounded variation, converges to* $\int_c^z f(x)\,g(x)\,dx$, *provided at the limits of integration either* $g'(x) = 0$, *or* $T_n'(x)$ *converges to* $\int dx \int f(x)\,dx$, *and either* $g(x) = 0$, *or* $T_n''(x)$ *converges to* $\int f(x)\,dx$.

The general theorem is at once evident :—

THEOREM $7^{(n-2)}$.—*If* $T_n$, *the partial summation of the r-th integral series, is a bounded function of the ensemble* $(x, n)$, *and* $T_n^{(r-1)}(x)$ *exists and is an integral, the series got by integrating the original series term-by-term after multiplying term-by-term by a function* $g(x)$, *possessing an* $(r-1)$*-th differential coefficient which is a function of bounded variation, converges to* $\int_c^z g(x)\,f(x)\,dx$, *provided at the limits of integration either* $g^{(k-1)} = 0$, *or* $T_n^{(r-k)}(x)$ *converges to* $T^{(r-k)}(x)$, *for all values of k between* 1 *and* $(r-1)$ *inclusive.*

Here $T^{(r-k)}$ denotes the $k$-ple integral of $f(x)$.

It is unnecessary to write out the proofs of these theorems at length, as no new principle is involved.

13. We conclude the paper with two illustrative examples.

Ex. 1.*—Consider the integral $\int_0^1 x^p \log x\,(1+x)^{-2}\,dx$, where $p+1 > 0$.

Here let
$$f(x) = (1+x)^{-2} = 1-2x+3x^2-\ldots, \tag{1}$$
$$g(x) = x^p \log x, \tag{2}$$

and divide the integral into two parts, writing

$$\int_0^1 f(x)\,g(x)\,dx = \int_0^c f(x)\,g(x)\,dx + \int_c^1 f(x)\,g(x)\,dx.$$

* Bromwich, *loc. cit.*, p. 451.

If $x$ lies between 0 and $c$, where $c$ is less than unity, the series (1) converges boundedly, so that, by § 2, we may take it to be the original series. Also, by Theorem 2, if $g(x)$ is a summable function we may integrate term-by-term. This is the case in the first of the two integrals, which may accordingly be evaluated by term-by-term integration.

In the second integral the original series does not converge for $x = 1$, but the second integral series, viz., the expansion of $-\log(1+x)$ in a power series, converges boundedly (in fact uniformly). Hence, by Theorem 7, we may integrate term-by-term provided $g(x)$ has an integral for differential coefficient, which is certainly the case, since $x^p \log x$ possesses finite differential coefficients of every order, and provided secondly $g(x)$ is zero, or the first integral series converges, at the limits of integration. Now at the point $x = 1$, $g(x)$ is zero; and at the point $x = c$ the first integral series converges to $-(1+c)^{-1}$. Therefore the conditions of Theorem 7 are fulfilled, and we may evaluate the second integral, as well as the first by term-by-term integration, and therefore we may evaluate the given integral by term-by-term integration.

Ex. 2.—Let $$f(x) = (1+x)^{-1-q},$$
and $$g(x) = x^p (\log x)^q.$$

Here, as before, we write

$$\int_0^1 f(x)\, g(x)\, dx = \int_0^c f(x)\, g(x)\, dx + \int_c^1 f(x)\, g(x)\, dx. \tag{1}$$

In treating the first of the two integrals on the right, we remark that $g(x)$ is summable; it has, in fact, for integral

$$\frac{x^{p+1}}{p+1}(\log x)^q - \frac{qx^{p+1}}{(p+1)^2}(\log x)^{q-1} + \frac{q(q-1)x^{p+1}}{(p+1)^3}(\log x)^{q-2} - \dots$$
$$+ (-)^q \frac{x^{p+1}}{(p+1)^{q+1}}.$$

Also the series $$1-(1+q)\,x + \frac{(1+q)(2+q)x^2}{2} - \dots$$

converges boundedly to $(1+x)^{-1-q}$. Hence, by Theorem 2, term-by-term integration is allowable in dealing with the first integral.

In dealing with the second integral on the right of (1), we remark that the $(1+q)$-th integral series of $f(x)$, *i.e.*, the expansion of $\log(1+x)$ multiplied by a constant, converges boundedly, while $g(x)$ possesses differential coefficients of each order, so that the $q$-th is certainly an integral. Moreover all the differential coefficients up to the $(q-1)$-th inclusive are zero at $x=1$, while at $x=c$ the $k$-th integral series converges to $(1+x)^{-1-q+k}$, for all values of $k$ from 1 to $q$ inclusive. Therefore, by Theorem $7^{(q-1)}$, term-by-term integration is again allowable.