On the General Linear Differential Equation of the Second Order.

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The Sections and Articles of this paper are numbered consecutively with those of my paper "On the Equation of Riccati" (Vol. xviii., pp. 180—202), to which it is a complement.

§ VIII. *On Decomposable Forms and their Notation.*

84. I call a form which can be expressed in two ways, viz., either by means of synthemes alone, or by means of synthemes and of u, u', u'', $\dot{u}$, $\dot{u}'$ and $\dot{u}''$ (to the exclusion of v, $\dot{v}$ and their derivatives), a pure decomposable. Of such forms (1, 2, 3) of Art. 9 and (5) of Art. 73 afford examples.

85. The simplest of pure decomposable forms, which I therefore call primary, are herein represented by the following expressions:—

$$\begin{aligned}
(0, 0)(1, 1)-(1, 0)(0, 1) &= AD-BC = \quad i = a_1,\\
(0, 0)(2, 2)-(2, 0)(0, 2) &= AI-EF = \quad d = a_2,\\
(1, 1)(2, 2)-(2, 1)(1, 2) &= DI-GH = \quad a = a_3,\\
(0, 0)(2, 1)-(2, 0)(0, 1) &= AG-CE = -h = a_4,\\
(0, 0)(1, 2)-(1, 0)(0, 2) &= AH-BF = -g = a_5,\\
(1, 0)(2, 1)-(2, 0)(1, 1) &= BG-DE = \quad f = a_6,\\
(0, 1)(1, 2)-(1, 1)(0, 2) &= CH-DF = \quad e = a_7,\\
(1, 0)(2, 2)-(2, 0)(1, 2) &= BI-EH = -c = a_8,\\
(0, 1)(2, 2)-(2, 1)(0, 2) &= CI-FG = -b = a_9.
\end{aligned}$$

86. These expressions, which exhaust the primaries of the second degree in the synthemes and which are in fact the minors of the determinant

$$\begin{vmatrix} A, & C, & F \\ B, & D, & H \\ E, & G, & I \end{vmatrix},$$

may be easily deduced. Thus

$$\dot v''v'.\dot v v = (G-\dot u''u')(A-\dot u u) = AG - A\dot u''u' - G\dot u u + \dot u''u'\dot u u$$

$$= \dot v''v.\dot v v' = (E-\dot u''u)(C-\dot u u') = CE - C\dot u''u - E\dot u u' + \dot u''u\dot u u',$$

whence $$AG - CE = A\dot u''u' - C\dot u''u - E\dot u u' + G\dot u u,$$

and h or a_4 is decomposable. This result I shall express by

$$-h = -\delta h, \quad \text{or} \quad a_4 = \delta a_4;$$

and so in other cases.

87. More generally, put

$$\dot v^{(r)}v^{(s)} = (r, s) - \dot u^{(r)}u^{(s)}, \quad \dot v^{(p)}v^{(q)} = (p, q) - \dot u^{(p)}u^{(q)},$$

then

$$\dot v^{(r)}v^{(s)}.\dot v^{(p)}v^{(q)} = (p, q)(r, s) - (p, q)\,\dot u^{(r)}u^{(s)} - (r, s)\,\dot u^{(p)}u^{(q)} + \dot u^{(r)}u^{(s)}\dot u^{(p)}u^{(q)};$$

again, put $\dot v^{(p)}v^{(s)} = (p, s) - \dot u^{(p)}u^{(s)}, \quad \dot v^{(r)}v^{(q)} = (r, q) - \dot u^{(r)}u^{(q)},$

then

$$\dot v^{(p)}v^{(s)}.\dot v^{(r)}v^{(q)} = (p, s)(r, q) - (p, s)\,\dot u^{(r)}u^{(q)} - (r, q)\,\dot u^{(p)}u^{(s)} + \dot u^{(p)}u^{(s)}\dot u^{(r)}u^{(q)},$$

and, subtracting and transposing, we get

$$(p, q)(r, s) - (p, s)(r, q)$$

$$= (p, q)\,\dot u^{(r)}u^{(s)} - (p, s)\,\dot u^{(r)}u^{(q)} - (r, q)\,\dot u^{(p)}u^{(s)} + (r, s)\,\dot u^{(p)}u^{(q)},$$

whence the nine primaries of Art. 85, and no others, can be deduced.

88. If we frame the two schemes*

	u, v	u', v'	u'', v''
$\dot u, \dot v$	A	C	F
$\dot u', \dot v'$	B	D	H
$\dot u'', \dot v''$	E	G	I

and

	u	u'	u''
$\dot u$	a	c	f
$\dot u'$	b	d	h
$\dot u''$	e	g	i

the first scheme may be regarded as giving the equations of syn-

* It has been suggested that a change of my original notation would considerably facilitate the reading of this paper and the detection of identities; and that when the synthemes of the former paper are denoted as in the first of the above schemes (viz., $A = \dot u u + \dot v v$, $B = \dot u' u + \dot v' v$, &c.), then the decomposables of the present paper are the first minors of this array. This criticism has enabled me to simplify, and other criticisms to amend, the paper. In giving an additional example, I follow a suggestion that an example or two would be useful.

themes (Art. 8), and the other as representing the six equations

$$\dot{i}u''+h\dot{u}'+f\dot{u}=0, \qquad iu''+gu'+eu=0,$$
$$g\dot{u}''+d\dot{u}'+c\dot{u}=0, \qquad hu''+du'+bu=0,$$
$$e\dot{u}''+b\dot{u}'+a\dot{u}=0, \qquad fu''+cu'+au=0.$$

89. Now, taking the two determinants

$$\begin{vmatrix} A, & C, & F \\ B, & D, & H \\ E, & G, & I \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a, & c, & f \\ b, & d, & h \\ e, & g, & i \end{vmatrix},$$

or J and j, wherein a, b, and so on, are the several minors corresponding to A, B and so on, respectively, we get (six, or, changing rows into columns) twelve expressions of the type $Ab+Cd+Fh$, or, say, ΣAb, each of which vanishes. We also get (three, or, changing as before) six of the type $Aa+Cc+Ff$, or, say ΣAa, each of which represents J. And J vanishes (not identically but) in virtue of the nine equations of synthemes given in Art. 8. This will be seen on actually substituting for A, B, and so on, their respective values $\dot{u}u+\dot{v}v$, $\dot{u}'u+\dot{v}'v$, and so on, or may be shown as in Arts. 98, 99.

90. If δ be a distributive operator affecting only the letters A, B, ... I (and not $\dot{u}$ or u), and such that its effect on a single syntheme (say A) is to destroy the term containing $\dot{v}$ and v (so that, for instance, $\delta A = \dot{u}u$); and, if moreover the effect of δ on a product be made to resemble that of the d of differentiation (so that, for example, $\delta . AD = A\delta D + D\delta A$), then we may represent any one of the decompositions of Art. 85 by $\alpha = \delta\alpha$.

§ IX. *On certain Identities.*

91. The above formulæ involve identities, which I write here. Thus

$$Aa-Gg = Ii-Bb = Dd-Ff\,(=ADI-BFG),$$
$$Aa-Hh = Ii-Cc = Dd-Ee\,(=ADI-CEH),$$
$$Gg-Hh = Bb-Cc = Ff-Ee\,(=BFG-CEH),$$
$$Gg-Ff = Aa-Dd = Hh-Ee\,(=DEF-AGH),$$
$$Aa-Ii = Gg-Bb = Hh-Cc\,(=IBC-AGH),$$
$$Dd-Ii = Ee-Cc = Ff-Bb\,(=IBC-DEF),$$

whence we can form systems of four equations homogeneous and linear in A, B ... and also in a, b ..., and independent: for example,

$$Ee-Ff+Gg-Hh = 0 = Ee-Ff+Bb-Cc,$$

$$Dd-Ee-Ii+Cc = 0 = Dd-Ee-Aa+Hh.$$

92. There is another class of identities, which also I write down here, premising a fully worked out example. We have

$$Ab+Cd+Fh = 0 = A\delta b+C\delta d+F\delta h$$
$$= A\delta\,(FG-CI)+C\delta\,(AI-EF)+F\,(CE-AG).$$

But
$$A\delta\,(FG-CI) = A\,(F\dot{u}''u'+G\dot{u}u''-C\dot{u}''u''-I\dot{u}u'),$$
$$C\delta\,(AI-EF) = C\,(A\dot{u}''u''+I\dot{u}u-E\dot{u}u''-F\dot{u}''u),$$
$$F\delta\,(CE-AG) = F\,(C\dot{u}''u+E\dot{u}u'-A\dot{u}''u'-G\dot{u}u);$$

therefore

$$Ab+Cd+Fh = (AG-CE)\,\dot{u}u''+(EF-AI)\,\dot{u}u'+(CI-FG)\,\dot{u}u$$
$$= -\dot{u}\,(hu''+du'+bu).$$

Proceeding in this manner, I get the following twelve equations, whereof the sinisters, being all of the type ΣAb, vanish; and if we reject the monomial solutions $\dot{u}, \dot{u}' \ldots u', u'' = 0$, we are led to the system of Art. 88.

Now, unless j (and therefore J) vanishes the system of Art. 88 will lead to the rejected monomial solutions. But J (and therefore j) does in fact vanish when we postulate the nine equations of synthemes (Art. 8). This will be seen on substituting for A, B, ... I their values, or may be shown as in Arts. 98, 99.

$$Ba+Dc+Hf = -\dot{u}'\,(fu''+cu'+au),$$
$$Ea+Gc+If = -\dot{u}''\,(fu''+cu'+au),$$
$$Ab+Cd+Fh = -\dot{u}\,(hu''+du'+bu),$$
$$Eb+Gd+Ih = -\dot{u}''\,(hu''+du'+bu),$$
$$Ae+Cg+Fi = -\dot{u}\,(iu''+gu'+eu),$$
$$Be+Dg+Hi = -\dot{u}'\,(iu''+gu'+eu),$$

$$Ca+Db+Ge=-u'\,(e\dot{u}''+b\dot{u}'+a\dot{u}),$$

$$Fa+Hb+Ie=-u''(e\dot{u}''+b\dot{u}'+a\dot{u}),$$

$$Ac+Bd+Eg=-u\,(g\dot{u}''+d\dot{u}'+c\dot{u}),$$

$$Fc+Hd+Ig=-u''(g\dot{u}''+d\dot{u}'+c\dot{u}),$$

$$Af+Bh+Ei=-u\,(i\dot{u}''+h\dot{u}'+f\dot{u}),$$

$$Cf+Dh+Gi=-u'\,(i\dot{u}''+h\dot{u}'+f\dot{u}).$$

93. The simultaneous interchanges (B, C), (E, F), and (G, H) have no effect upon α_1, α_2, α_3. In other cases they change α_{2m} into α_{2m+1} or α_{2m+1} into α_{2m}. These interchanges would be made by a shifting of accents in Art. 8 (say, for instance, by changing $\dot{u}'u$ into $u\dot{u}'$). But such shifting would have no effect upon A, D, or I.

94. I remark that

$$ADI-BFG=(2,2)(1,1)(0,0)-(2,1)(1,0)(0,2),$$

$$ADI-CEH=(2,2)(1,1)(0,0)-(2,0)(1,2)(0,1),$$

and that a similar form for $BFG-CEH$ may be found by subtraction.

§ X. *Differentiations and Verifications; Simplifications.*

95. The foregoing results are general and not confined to the cases in which the accents denote differentiations. They are true when the accents are regarded merely as marks to distinguish different quantities. But, treating the accents as differentiations and recurring to § II., we get

$$i'=-(g+h),$$

$$d'=-(\dot{p}+p)\,d+\dot{q}g+qh-(b+c),$$

$$a'=-(\dot{p}+p)\,a+\dot{r}e+rf,$$

$$-h'=f+d+\dot{p}h-\dot{q}i,$$

$$-g'=e+d+pg-qi,$$

$$f'=-c-\dot{p}f+\dot{r}i,$$

$$-e'=-b-pe+ri,$$

$$-c'=(\dot{p}+p)\,c-qf-\dot{r}g+a,$$

$$-b'=(\dot{p}+p)\,b-\dot{q}e-rh+a.$$

96. If we differentiate the identity

$$Ff - Ee - Gg + Hh = 0,$$

which (with signs changed) is one of the identities of Art. 91, we get, after substitutions and reductions,

$$\begin{aligned} &(Ff - Ee - Gg + Hh)' \\ &= -(\dot{p}+p)(Ff - Ee - Gg + Hh) - Fc + Eb + (G-H)\,d - I\,(g-h) \\ &\quad + \dot{q}\,(Hi + Dg + Be) - q\,(Gi + Cf + Dh) \\ &\quad + \dot{r}\,(Fi + Cg + Ae) - r\,(Ei + Af + Bh)\,; \end{aligned}$$

but $-Fc + Eb + (G-H)\,d - I\,(g-h)$ vanishes identically, as will be found on substitution, and the rest vanishes in virtue of the identities of Arts. 91, 92. All this is right.

97. Again, take the identity

$$Ei + Bh + Af = 0\,;$$

we get

$$(Ei + Bh + Af)' = Gi + Dh + Cf - (Ei + Bh + Af)\,\dot{p} = 0 + 0\dot{p} = 0,$$

as is seen on turning to Art. 92. This too is right.

98. And I here add that

$$\begin{aligned} Aa + Bb + Ee &= u''\,(i\dot{u}'' + h\dot{u}' + f\dot{u}) + u'\,(g\dot{u}'' + d\dot{u}' + c\dot{u}), \\ Aa + Cc + Ff &= \dot{u}''\,(iu'' + gu' + eu) + \dot{u}'\,(hu'' + du' + bu), \\ Bb + Dd + Hh &= \dot{u}\,(fu'' + cu' + au) + \dot{u}''\,(iu'' + gu' + eu), \\ Cc + Dd + Gg &= u\,(e\dot{u}'' + b\dot{u}' + a\dot{u}) + u''\,(i\dot{u}'' + h\dot{u}' + f\dot{u}), \\ Ee + Gg + Ii &= \dot{u}'\,(hu'' + du' + bu) + \dot{u}\,(fu'' + cu' + au), \\ Ff + Hh + Ii &= u'\,(g\dot{u}'' + d\dot{u}' + c\dot{u}) + u\,(e\dot{u}'' + b\dot{u}' + a\dot{u}). \end{aligned}$$

The first of these formulæ is obtained from

$$Aa + Bb + Ee = A\delta a + B\delta b + C\delta c,$$

by processes corresponding with those of Art. 92; the rest in a similar manner.

99. Rejecting the results $\dot{u}, \dot{u}', \dot{u}'', \ldots u'' = 0$, there remain in Arts. 88 and 92, six equations which show that the six forms of J given in Art. 98 vanish. Again, subtract the second form from the first. We get

$$Ee - Ff + Bb - Cc = \dot{u}\,(fu'' + cu') - u\,(e\dot{u}'' + b\dot{u}'),$$

whereof the sinister vanishes (see Art. 91), and the dexter reduces to $\dot{u}(-au)-u(-a\dot{u})$ or zero.

Formulæ such as those given in Arts. 88 and 92 may aid in keeping down elevation of degree arising from elimination.

100. In this paper, however, my object is not so much to consider the actual calculations incident to the eliminations as to conform, as nearly as may be, with the prior memoir and to ingraft upon it the proof that the general linear differential equation of the second order is soluble by means of an algebraical equation, the coefficients of which, however, will not in general be algebraical.

101. It is not meant to be asserted that such equation will itself be algebraically soluble. For, although one of its roots will be a rational function of two other of them and of the coefficients, still, unless its degree be prime, its solubility cannot be affirmed.

102. The sinisters of $a = \delta a$ will consist of terms of the form $\beta\gamma$ (using β, γ, ϵ, &c., to represent letters of the set $A, B, \ldots I$), and the dexters will consist of terms of the form $\epsilon\dot{u}^{(r)}u^{(s)}$.

103. The expressions $a_{m}\delta a_{n}-a_{n}\delta a_{m}$ will be linear and homogeneous in $\dot{u}$ and u; so that on dividing by $\dot{u}u$ we shall get equations which the elimination of $\dot{u}_1$, $\dot{u}_2$, u_1 and u_2 will enable us to express in terms of $A, B, \ldots I$. But (anticipating Art. 133) the expressions

$$(\Theta_0\Theta_3-\Theta_1\Theta_2)\,a_m-c_1\dot{c}_1 e^{-\int(\dot{p}+p)\,dx}\,\delta a_m,$$

similar in the linearity and homogeneity to the former, will be more advantageous, being of lower dimensions in $A, B, \ldots I$. By means of Art. 88, the $\dot{u}_1$, $\dot{u}_2$, u_1 and u_2 of Art. 79 can be expressed rationally in terms of the minors. But the forms $(0:0)$ are useless.

104. As the δ of Arts. 86, 90 does not affect $\dot{u}$ or u, so neither does it affect $p, q \ldots \dot{q}, \dot{r}$, nor indeed any function X of x; so that, for example, $\delta . Xa = X\delta a$.

§ XI. *Introduction of the General Biordinal.*

105. Let p and r be independent and arbitrary functions of x, and let m be any constant (other than 0 or ∞). Then the general linear and homogeneous biordinal may be represented by

$$z''+\left(p+\frac{m}{x}\right)z'+\left\{\frac{1}{x}\left(p+\frac{m}{x}\right)-\frac{xr}{m}-\frac{2}{x^2}\right\}z=0.$$

106. Effecting the transformation indicated by

$$\left\{\frac{d^2}{dx^2}+\left(p+\frac{m}{x}\right)\frac{d}{dx}+\frac{1}{x}\left(p+\frac{m}{x}\right)-\frac{xr}{m}-\frac{2}{x^2}\right\}\left(\frac{d}{dx}-\frac{m}{x}\right)y=0,$$

we get
$$y'''+py''+qy'+ry=0,$$
wherein
$$q=-\left\{\frac{m-1}{x}\left(p+\frac{m-2}{x}\right)+\frac{xr}{m}\right\}.$$

107. The complete solution of this terordinal involves that of the biordinal. Let $y=u, v, w$ denote three independent solutions of the terordinal. Then $(y=)\,w=x^m$ may be taken as one of these solutions.

108. It follows (Arts. 4 and 15; also Art. 55) that $\dfrac{x^{m-1}}{r}$ is an integrating factor of the deformation.

109. The value $m=2$ is that best adapted to the case of the particular Riccatian. I retain it here, and the results of the present are easily compared with those of my prior paper. Put, then, $w=x^2$.

110. The terordinal will be

$$y'''+py''-\left(\frac{p}{x}+\tfrac{1}{2}xr\right)y'+ry=0,$$

and, multiplied into x, will take the form
$$\Theta''+p\Theta'-\tfrac{1}{2}xr\Theta=0,$$
wherein
$$\Theta=xy'-2y.$$

111. Its deformation will be

$$Y'''-\left(p+2\frac{r'}{r}\right)Y''$$
$$+\left\{\left(p+\frac{r'}{r}\right)\frac{r'}{r}-\left(p+\frac{r'}{r}\right)'-\frac{p}{x}-\tfrac{1}{2}xr\right\}Y'-rY=0,$$

which, multiplied into $\dfrac{x}{r}$ and integrated, yields

$$\frac{x}{r}Y''-\left(\frac{x}{r}p+x\frac{r'}{r^2}+\frac{1}{r}\right)Y'-\tfrac{1}{2}x^2Y=\text{constant}$$

whence, if the constant be supposed to vanish,

$$Y''-\left(p+\frac{r'}{r}+\frac{1}{x}\right)Y'-\tfrac{1}{2}xrY=0.$$

112. The mixed integral of Art. 16 written in the form

$$y'\left\{Y''-\left(p+\frac{r'}{r}\right)Y'\right\}-y''Y'-ryY=cr,$$

therefore, becomes

$$y'\left(\frac{1}{x}Y'+\tfrac{1}{2}xrY\right)-y''Y'-ryY=cr,$$

or $$-\frac{1}{x}(xy''-y')\,Y'+\tfrac{1}{2}r(xy'-2y)\,Y=cr,$$

which is equivalent to

$$-\Theta'Y'+\tfrac{1}{2}xr\Theta Y=cxr.$$

113. The results are the same in form for the dotted as for the undotted letters. And by precisely similar steps we are led, in the correlate system, to the mixed integral

$$-\dot{\Theta}'Y'+\tfrac{1}{2}x\dot{r}\dot{\Theta}\dot{Y}=\dot{c}x\dot{r}.$$

114. The unsuffixed Θ of this paper is essentially different from the suffixed Θ's, say the Θ_m's. Thus, Θ_m means a function of x and of synthemes; while Θ is the dependent variable in the biordiual of Art. 110, whereof $\Theta=\theta$, and $\Theta=\vartheta$, are supposed to be independent particular integrals. And $\dot{\Theta}\ (=x\dot{y}'-2\dot{y})$ is the dependent variable in

$$\dot{\Theta}''+\dot{p}\dot{\Theta}'-\tfrac{1}{2}x\dot{r}\dot{\Theta}=0,$$

whereof $\dot{\Theta}=\dot{\theta}$, and $\dot{\Theta}=\dot{\vartheta}$, are to be taken as independent particular solutions.

115. Except in so far as it is necessary in particular cases to substitute appropriate particular values for p and r the process is, up to a certain point and so long as we keep to the value 2 of the exponent m of Art. 107, the same for all biordinals. The equations of Arts. 8—12 are the same for all. The meaning of θ, ϑ, $\dot{\theta}$, and $\dot{\vartheta}$ is the same for all. The quantities Θ_0, Θ_1, Θ_2, and Θ_3 are, when expressed in terms of x and synthemes, the same for all. But the expressions for Θ_0, Θ_1, Θ_2, and Θ_3 in terms of M and x are peculiar to the Riccatian discussed. So that while the relations of Art. 58 are always true those of Art. 68 are, so far at least as M and all which follows the " = " that precedes it are concerned, true only for the particular Riccatian.

116. The equation of Art. 16 holds for all values of Y and y.

Keeping to the same Y, let y_1, y_2, and y_3 be any three distinct values of y. Then the three equations

$$y_1'Y''-y_1''Y'-\left(p+\frac{r'}{r}\right)y_1'Y'-ry_1Y=c_1r,$$

$$y_2'Y''-y_2''Y'-\left(p+\frac{r'}{r}\right)y_2'Y'-ry_2Y=c_2r,$$

$$y_3'Y''-y_3''Y'-\left(p+\frac{r'}{r}\right)y_3'Y'-ry_3Y=c_3r,$$

always hold when proper values or ratios are given or assigned to or among the arbitrary constants c_1, c_2, and c_3.

117. Writing this system thus—

$$\begin{array}{lll} Y'', & Y', & Y, \\ y_1', & -y_1''-\left(p+\frac{r'}{r}\right)y_1', & -ry_1=c_1r, \\ y_2', & -y_2''-\left(p+\frac{r'}{r}\right)y_2', & -ry_2=c_2r, \\ y_3', & -y_3''-\left(p+\frac{r'}{r}\right)y_3', & -ry_3=c_3r, \end{array}$$

and dealing with the determinant on the sinister in the same way as the determinant δ was dealt with in Art. 21, we get

$$Y=e^{\int pdx}\left(-c_1\begin{vmatrix} y_2'', & y_2' \\ y_3'', & y_3' \end{vmatrix}-c_2\begin{vmatrix} y_3'', & y_3' \\ y_1'', & y_1' \end{vmatrix}-c_3\begin{vmatrix} y_1'', & y_1' \\ y_2'', & y_2' \end{vmatrix}\right).$$

118. Putting $c_1=0=c_2$, replacing Y, y_1, and y_2 by W, u, and v respectively, and merging $-c_3$ in the constant of integration, we get

$$W=e^{\int pdx}\begin{vmatrix} u'', & u' \\ v'', & v' \end{vmatrix},$$

and, by corresponding operations or by cyclical changes, we get the systems of Arts. 18 and 19.

119. If in Arts. 96, 97, we put $y_1, y_2, y_3=u, v, x^2$; then

$$u'Y''-u''Y'-\left(p+\frac{r'}{r}\right)u'Y'-ruY=c_1r,$$

$$v'Y''-v''Y'-\left(p+\frac{r'}{r}\right)v'Y'-rvY=c_2r,$$

$$Y''-\left(p+\frac{r'}{r}+\frac{1}{x}\right)Y'-\tfrac{1}{2}xrY=\tfrac{1}{2}c_3\frac{r}{x},$$

and if we suppose, in conformity with Art. 111, that the arbitrary constant in the integrated deformation vanishes, then $c_3 = 0$.

120. On these suppositions the formulæ of Art. 112 lead to

$$-\theta' Y' + \tfrac{1}{2}xr\theta Y = c_1 xr,$$

$$-\vartheta' Y' + \tfrac{1}{2}xr\vartheta Y = c_2 xr,$$

whence, eliminating Y', reducing and recalling Art. 56,

$$(\theta\vartheta' - \vartheta\theta')\, Y = 2\,(c_1\vartheta' - c_2\theta') = e^{-\int p\,dx}(c_1 U + c_2 V),$$

and $$e^{\int p\,dx}(\theta\vartheta' - \vartheta\theta') = \frac{c_1 U + c_2 V}{Y}.$$

121. The sinister is a constant, for

$$\{e^{\int p\,dx}(\theta\vartheta' - \vartheta\theta')\}'$$

vanishes identically when θ'' and ϑ'' are eliminated (for θ and ϑ are solutions of the biordinal of Art. 110); and $C_1 U + C_2 V$ being substituted for Y on the dexter, we have

$$c_1 : C_1 = c_2 : C_2 = k,$$

where $\log k$ is the constant of integration in $\int p\,dx$. Hence c_1 and c_2 cannot be supposed to vanish simultaneously without leading to a useless result. But either c_1 or c_2 may vanish separately, and in fact one (only) of them is supposed to vanish in deducing the systems of Arts. 18 and 19. This evanescence enabled us to give to the systems a shape which, though not the most general in form, is in substance general, and the simplest which can be constructed.

122. Let $Z = U$ or V; then by $\{3\}$ and $\{4\}$ of Art. 59, we have

$$Z = \pm\, 2e^{\int p\,dx}\,\Theta',$$

the positive sign $(+)$ and the value $\Theta = \vartheta$ being taken when $Z = U$, and the negative sign $(-)$ and the value $\Theta = \theta$ being taken when $Z = V$. Hence

$$Z' = pZ \pm 2e^{\int p\,dx}\,\Theta'' = pZ \pm 2e^{\int p\,dx}(-p\Theta' + \tfrac{1}{2}xr\Theta) = \pm\, e^{\int p\,dx}\,xr\Theta,$$

and $$\Theta' Z' = \pm\,\tfrac{1}{2}xr\,.\,2e^{\int p\,dx}\,\Theta'\Theta = \tfrac{1}{2}xr\Theta Z.$$

123. It follows that

$$-\vartheta' U' + \tfrac{1}{2}xr\vartheta U \quad\text{and}\quad -\theta' V' + \tfrac{1}{2}xr\theta V$$

both vanish identically. Hence, dealing with the first two equations of Art. 120, when we put $Y = U$, then we must put $c_2 = 0$, and

$$-\theta' U' + \tfrac{1}{2} xr\theta U = c_1 rx;$$

and when we put $Y = V$ then we must put $c_1 = 0$, and

$$-\vartheta' V' + \tfrac{1}{2} xr\vartheta V = c_2 rx.$$

124. When $c_2 = -c_1$ these last results coincide, for, multiplying both into $e^{\int p\,dx}$ and remembering Art. 122, each becomes

$$VU' - UV' = c_1 rx e^{\int p\,dx},$$

whence, since $R = -r$,

$$Re^{\int P dx}(VU' - UV') = -c_1 r^2 x e^{\int (p+P) dx} = -c_1 kKx,$$

a result which, when $-c_1 kK = 2$, coincides with the sixth relation of Art. 19. Here $\log K$ is supposed to be the constant in the integration $\int P dx$.

125. For the correlate system we get (since $\dot{R} = -\dot{r}$) a similar result

$$\dot{R}e^{\int \dot{P} dx}(\dot{V}\dot{U}' - \dot{U}\dot{V}') = -\dot{c}_1 \dot{r}^2 x e^{\int (\dot{p}+\dot{P}) dx} = -\dot{c}_1 \dot{k}\dot{K}x,$$

and a similar condition $-\dot{c}_1 \dot{k}\dot{K} = 2$; $\log \dot{k}$ and $\log \dot{K}$ being the constants of integration in $\int \dot{p} dx$ and $\int \dot{P} dx$ respectively.

126. Thus, when the arbitrary constants are properly adjusted, the mixed integrals introduce no new conditions.

§ XII. *On Certain Special Cases.*

127. Putting $rx = 2$, multiplying the first mixed integral of Art. 123 into λ, and substituting for λU a value given by $\{3\}$ of Art. 59, we get

$$\theta' \{x(1-x^2)\dot{\theta}' + x^2\dot{\theta}\}' - \theta \{x(1-x^2)\dot{\theta}' + x^2\dot{\theta}\} = 2c_1\lambda,$$

or

$$\theta' \{x(1-x^2)\dot{\theta}'' + (1-2x^2)\dot{\theta}' + 2x\dot{\theta}\} - \theta \{x(1-x^2)\dot{\theta}' + x^2\dot{\theta}\} = 2c_1\lambda.$$

But, in the case of the Riccatian, the biordinal of Art. 114 gives

$$\dot{\theta}'' + \left(\frac{1}{x} - x\right)\dot{\theta}' + \dot{\theta} = 0, \text{ and } x(1-x^2)\dot{\theta}'' = -(1-x^2)^2\dot{\theta}' - x(1-x^2)\dot{\theta},$$

and the mixed integral becomes

$$\theta' \left[\{1-2x^2-(1-x^2)^2\} \dot{\theta}' + x(1+x^2)\dot{\theta} \right] - \theta \{x(1-x^2)\dot{\theta}' + x^2\dot{\theta}\} = 2c_1\lambda,$$

or
$$-x^4\dot{\theta}'\theta' + x(1+x^2)\dot{\theta}\theta' - x(1-x^2)\dot{\theta}'\theta - x^2\dot{\theta}\theta = 2c_1\lambda.$$

128. Again, the corresponding mixed integral of the correlate system becomes
$$-\dot{\theta}'\dot{U}' - \dot{\theta}\dot{U} = -2\dot{c}_1,$$
since $x\dot{r} = -2$. Multiplying into $\dot{\lambda}$ and substituting for $\dot{\lambda}\dot{U}$ a value given by $\{1\}$ of Art. 59, we get in the same way

$$\dot{\theta}' \{x(1+x^2)\theta'' + (1+2x^2)\theta' - 2x\theta\} + \dot{\theta}\{x(1+x^2)\theta' - x^2\theta\} = -2\dot{c}_1\dot{\lambda},$$

whence, eliminating θ'' by means of

$$\theta'' + \left(\frac{1}{x} + x\right)\theta' - \theta = 0,$$

obtained from the biordinal of Art. 90, we get

$$\dot{\theta}' \{-x^4\theta' - x(1-x^2)\theta\} + \dot{\theta}\{x(1+x^2)\theta' - x^2\theta\} = -2\dot{c}_1\dot{\lambda}.$$

129. The two mixed integrals coincide if $2c_1\lambda = -2\dot{c}_1\dot{\lambda}$. But (Art. 54) $\dot{\lambda} = -\lambda$. Hence $\dot{c}_1 = c_1$, and recurring to Arts. 76 and 77, we have $2c_1\lambda = -2\dot{c}_1\dot{\lambda} = -c = \dot{c}$.

130. Unless we introduce transcendents these hybrid integrals, involving both $\dot{u}$ and u (and also their differential coefficients), can only be obtained in rare cases, of which the particular Riccatian is the most conspicuous example.

131. Suppose that the biordinal of Art. 105, put under Boole's form, is
$$D(D-b)z + x^2(D-a)z = 0,$$
then, proceeding as indicated in Art. 106, we have the terordinal

$$D(D-2)(D-b)y + x^2(D-2)(D-a)y = 0,$$

its deformation

$$D^2(D+b-2)Y - x^2(D+2)(D+a)Y = 0,$$

a correlate

$$D(D-2)(D-b)\dot{y} - x^2(D-2)(D-a)\dot{y} = 0,$$

and its deformation

$$D^2(D+b-2)\dot{Y} + x^2(D+2)(D+a)\dot{Y} = 0.$$

132. Following Boole and transforming the deformation into the correlate by the substitution

$$Y = P_2 \frac{D\,.\,D+a-2\,.\,D\,.\,D-2\,.\,D-b}{D^2\,.\,D+b-2\,.\,D-4\,.\,D-a-2}\dot{y} = P_2 \frac{D-2.D+a-2.D-b}{D-4.D-a.D+b-2}\dot{y},$$

we see that when, and only when, a is (an) even (integer) or when $b-a$ is even, or when $2a$ and $2b$ are both even, then a hybrid integral can be found without introducing transcendents. The like would hold if we compared the given equation with the deformation of its correlate. And the process of the prior paper admits of extension to all cases in which a is odd and b even. But the better course would be to transform at once to the case already discussed.

§ XIII. *On the Functions* Θ_0, Θ_1, Θ_2, *and* Θ_3.

133. Multiplying together the two identities

$$\theta\vartheta' - \vartheta\theta' = c_1 e^{-\int p\,dx} \text{ and } \dot{\theta}\dot{\vartheta}' - \dot{\vartheta}\dot{\theta}' = \dot{c}_1 e^{-\int \dot{p}\,dx},$$

we get

$$\begin{vmatrix} \dot{\theta}, & \dot{\vartheta} \\ \dot{\theta}', & \dot{\vartheta}' \end{vmatrix} \cdot \begin{vmatrix} \theta, & \vartheta \\ \theta', & \vartheta' \end{vmatrix} = \begin{vmatrix} \dot{\theta}\theta + \dot{\vartheta}\vartheta, & \dot{\theta}\theta' + \dot{\vartheta}\vartheta' \\ \dot{\theta}'\theta + \dot{\vartheta}'\vartheta, & \dot{\theta}'\theta' + \dot{\vartheta}'\vartheta' \end{vmatrix} = c_1\dot{c}_1 e^{-\int(\dot{p}+p)\,dx},$$

or

$$\Theta_0\Theta_3 - \Theta_1\Theta_2 = \dot{c}_1 c_1 e^{-\int(\dot{p}+p)\,dx} \quad \text{..........................(B).}$$

134. In verification make the substitutions appropriate to the Riccatian; then (Arts. 41, 43, and 68) this (B) becomes

$$M^2x^2 - M^2\left(x^2 - \frac{1}{x^2}\right) = c_1\dot{c}_1 e^{-\int(\dot{p}+p)\,dx} = \frac{c_1\dot{c}_1}{k\dot{k}x^2},$$

whence

$$M^2 = \frac{c_1\dot{c}_1}{k\dot{k}} = \frac{4c_1\dot{c}_1\lambda\dot{\lambda}}{4k\dot{k}\lambda\dot{\lambda}} = -(2c_1\lambda)(2\dot{c}\dot{\lambda}) = (2c_1\lambda)^2 = (-c)^2 = c^2,$$

as we see on turning to Arts. 67 and 109.

135. In further verification, differentiate. Then

$$(\Theta_0\Theta_3 - \Theta_1\Theta_2)' = \Theta_0\Theta_3' + \Theta_3\Theta_0' - \Theta_1\Theta_2' - \Theta_2\Theta_1' = -(\dot{p}+p)(\Theta_0\Theta_3 - \Theta_1\Theta_2),$$

as will be seen on substituting for the accented letters their respective values, viz.:

$$\begin{aligned} \Theta_3' &= -(\dot{p}+p)\,\Theta_3 + \tfrac{1}{2}x\dot{r}\,\Theta_2 + \tfrac{1}{2}xr\,\Theta_1, \\ \Theta_2' &= \Theta_3 - p\Theta_2 + \tfrac{1}{2}rx\,\Theta_0, \\ \Theta_1' &= \Theta_3 - \dot{p}\,\Theta_1 + \tfrac{1}{2}x\dot{r}\,\Theta_0, \\ \Theta_0' &= \Theta_2 + \Theta_1. \end{aligned}$$

136. By development

$$\Theta_0\Theta_3-\Theta_1\Theta_2 = x^4a+2x^3(b+c)+2x^2(2d+f+e)+4x(g+h)+4i,$$

and we have a decomposable form. Decomposing it, we get

$$\Theta_0\delta\Theta_3+\Theta_3\delta\Theta_0-\Theta_1\delta\Theta_2-\Theta_2\delta\Theta_1 = x^2\Theta_0\dot{u}''u''-(x^2\Theta_1+x\Theta_0)\dot{u}'u''$$

$$-(x^2\Theta_2+x\Theta_0)\dot{u}''u'+2x\Theta_1\dot{u}u''+2x\Theta_2\dot{u}''u+\{x^2\Theta_3+x(\Theta_1+\Theta_2)+\Theta_0\}\dot{u}'u'$$

$$-2(x\Theta_3+\Theta_1)\dot{u}u'-2(x\Theta_3+\Theta_2)\dot{u}'u+4\Theta_3\dot{u}u = c_1\dot{c}_1e^{-\int(\dot{p}+p)dx}.$$

137. In verification make the appropriate substitutions (Art. 68) and we get the mixed integral of Arts. 76 and 77; and, when this hybrid (Art. 130) integral is known, all the four Θ_m's are known.

138. Differentiation will elicit no new result from (B) nor, as Art. 95 shows, from the formulæ of Art. 85. And the last statement is confirmed when we follow the course of my prior paper. Six additional results are got by means of (2)′, (2)″, (3)′, and (3)″, or, in other words, by twice differentiating (2) and (3) of Art. 9, and then eliminating $\dot{u}'''$, u''', and the accented letters A', B', ... I' by means of the formulæ of Arts. 9, 11, and 12.

139. The relations so obtained will not, however, be in their simplest form. Remarking that (4) is (1)′, and that (5) of Art. 74 is in fact $(4)'-2(2)+(\dot{q}+q)(1)$, I form the several expressions

$$(2)'+(\dot{p}+p)(2), \text{ or say } (7);$$

$$(7)+(\dot{p}+p)(7)-2\{(3)+(\dot{q}+q)(2)+\dot{q}q(1)\}, \text{ or say } (8);$$

$$(3)'+(\dot{p}+p)(3), \text{ or say } (9);$$

and $$(9)'+(\dot{p}+p)(9)-2\dot{r}r(1), \text{ or say } (10);$$

and thus from that prior paper I get, putting

$$\dot{r}-\dot{p}\dot{q}-\dot{q}' = \dot{\rho} \text{ and } r-pq-q' = \rho,$$

$$i = \delta i, \quad d = \delta d, \quad a = \delta a \dots\dots (1, 2, 3),$$

$$g+h = \delta g+\delta h \dots\dots (4),$$

$$f+e+\dot{p}h+pg = \delta f+\delta e+\dot{p}\delta h+p\delta g \dots\dots (5),$$

$$-b-c+qh+\dot{q}g = -\delta b-\delta c+q\delta h+\dot{q}\delta g \dots\dots (7),$$

$$-2\dot{q}e-2qf-\rho h-\dot{\rho}g = -2\dot{q}\delta e-2q\delta f-\rho\delta h-\dot{\rho}\delta g \dots\dots (8),$$

$$r\dot{f}+\dot{r}e = r\delta\dot{f}+\dot{r}\delta e \quad\ldots\ldots(9),$$

$$-r\dot{c}-\dot{r}b+\pi r\dot{f}+\dot{\pi}\dot{r}e = -r\delta\dot{c}-\dot{r}\delta b+\pi r\delta\dot{f}+\dot{\pi}\dot{r}\delta e \quad\ldots\ldots(10),$$

π and $\dot{\pi}$ being already (Arts. 18, 41, 43) defined. If we proceed to another differentiation we get

$$-2p\dot{f}-2\dot{p}e-(\dot{p}^2-\dot{p}')\,h-(p^2-p')\,g$$
$$= -2p\delta\dot{f}-2\dot{p}\delta e-(\dot{p}^2-\dot{p}')\,\delta h-(p^2-p')\,\delta g,$$

and inasmuch as this can be put under the form

$$\lambda\,(4)+\mu\,(5)+\nu\,(8)+\sigma\,(9),$$

where λ, μ, ν, and σ are all free from any or either of the quantities f, e, h, and g, no new result is gained.

140. Whichever course we adopt, viz., whether we use the system of Art. 85 or that of Art. 139, we get nine equations, equivalent however to five independent equations only. Denote the six forms of J, taken in the order in which they occur in (the sinisters of) Art. 98, by (1, 2, 3, 4, 5, 6) respectively. Then, whether the small letters represent minors or decomposables, or whether they are regarded as arbitrary and independent, the identity

$$(1)+(4)+(6)-(2)-(3)-(5) = 0$$

subsists; and the relations

$$(1) = (2) = (3) = (4) = (5) = (6)$$

imply four conditions only. But they do imply four distinct conditions capable of being exhibited in various ways. For instance, strike out (6); then (1) = (2), (1) = (3), (1) = (4), (1) = (5), will give four distinct conditions, for Ff occurs in (1) = (2) only; Hh in (1) = (3) only; Aa in (1) = (4) only, and Ii in (1) = (5) only. Again, recurring to Art. 91, Gg, Bb, Ii, and Hh severally occur only in the first, second, third, and fourth respectively of the relations given at its close. But the system of Art. 91 is not really different from the one just considered.

Thus the nine equations which enter (explicitly) reduce themselves (implicitly) to (9 − 4, or) five conditions. And these conditions may be expressed in terms of A, B, ... I without imposing any additional restriction. For when (and if) the four O_m's are properly determined (say each in the form $O_m = X_m$), then the equation (B) of Art. 133 is necessarily satisfied, and the process of Art. 103 introduces no new relation. And, since each of the Θ_{m}'s is a linear function of some four of the nine quantities $A, B, \ldots I$, we have $(5+4$, or) nine relations, not all homogeneous, for determining the nine quantities $A, B, \ldots I$.

141. All this supposes that the four Θ_{m}'s can be finitely determined.

§ XIV. *On the Solution of the General Biordinal.*

142. Assume

$$Y = L\dot{y}'' + M\dot{y}' + N\dot{y},$$

then

$$Y' = L_1\dot{y}'' + M_1\dot{y}' + N_1\dot{y},$$

$$Y'' = L_2\dot{y}'' + M_2\dot{y}' + N_2\dot{y},$$

where

$$L_1 = -\dot{p}L + L' + M, \quad L_2 = -\dot{p}L_1 + L_1' + M_1,$$

$$M_1 = -\dot{q}L + M' + N, \quad M_2 = -\dot{q}L_1 + M_1' + N_1,$$

$$N_1 = -\dot{r}L + N', \qquad N_2 = -\dot{r}L_1 + N_1'.$$

143. Substituting these values of Y, Y', and Y'' in the biordinal of Art. 111, and putting for the moment $K = p + \frac{r'}{r} + \frac{1}{x}$, we get

$$(L_2 - KL_1 - \tfrac{1}{2}xrL)\dot{y}'' + (M_2 - KM_1 - \tfrac{1}{2}xrM)\dot{y}' + (N_2 - KN_1 - \tfrac{1}{2}xrN)\dot{y} = 0.$$

144. This will be satisfied if

$$L_2 - KL_1 - \tfrac{1}{2}xrL = 0, \quad M_2 - KM_1 - \tfrac{1}{2}xrM = 0, \quad N_2 - KN_1 - \tfrac{1}{2}xrN = 0.$$

145. An absolutely general solution of this system is not essential; but $L = 0$ will afford one of sufficient generality. I shall, however, for the present, defer the introduction of this condition ($L = 0$).

146. We see at once that

$$L_1 + xM_1 + \tfrac{1}{2}x^2N_1 = -(\dot{p} + x\dot{q} + \tfrac{1}{2}x^2\dot{r})\,L + (L + xM + \tfrac{1}{2}x^2N)'\,;$$

but inasmuch as $y = x^2$ is a solution of the terordinal of Art. 90, and $\dot{y} = x^2$ of its correlate, therefore

$$p + xq + \tfrac{1}{2}x^2r = 0 = \dot{p} + x\dot{q} + \tfrac{1}{2}x^2\dot{r},$$

and, in virtue of the last relation,

$$L_1 + xM_1 + \tfrac{1}{2}x^2N_1 = (L + xM + \tfrac{1}{2}x^2N)'.$$

147. So

$$L_2 + xM_2 + \tfrac{1}{2}x^2N_2 = (L_1 + xM_1 + \tfrac{1}{2}x^2N_1)' = (L + xM + \tfrac{1}{2}x^2N)''\,;$$

148. In the simplified mixed integral of Art. 112 substitute for Θ and Θ' (see Art. 110) and for Y and Y' (see Art. 142). We get

$$-xL_1\dot{y}''y''+(\tfrac{1}{2}x^2rL+L_1)\dot{y}''y'-xM_1\dot{y}'y''-xrL\dot{y}''y-xN_1\dot{y}y''$$
$$+(\tfrac{1}{2}x^2rM+M_1)\dot{y}'y'-xrM\dot{y}'y+(\tfrac{1}{2}x^2rN+N_1)\dot{y}y'-xrN\dot{y}y=cxr.$$

149. In the last relation of Art. 136 replace $\dot{u}$ and u by $\dot{y}$ and y respectively; and for shortness represent $e^{-\int(\dot{p}+p)\,dx}$ by ϕ. The result will be identical with that of Art. 148 if

$$-\frac{L_1}{cr}=x^2\frac{\Theta_0}{\phi};\quad -\frac{L}{c}=2x\frac{\Theta_2}{\phi};\quad -\frac{N_1}{cr}=2x\frac{\Theta_1}{\phi};\quad -\frac{N}{c}=4\frac{\Theta_3}{\phi};$$

$$\tfrac{1}{2}x\frac{L}{c}+\frac{L_1}{cxr}=-x^2\frac{\Theta_2}{\phi}-x\frac{\Theta_0}{\phi};\quad -\frac{M_1}{cr}=-x^2\frac{\Theta_1}{\phi}-x\frac{\Theta_0}{\phi};$$

$$-\frac{M}{c}=-2x\frac{\Theta_3}{\phi}-2\frac{\Theta_2}{\phi};$$

$$\tfrac{1}{2}x\frac{M}{c}+\frac{M_1}{cxr}=x^2\frac{\Theta_3}{\phi}+x\frac{\Theta_1+\Theta_2}{\phi}+\frac{\Theta_0}{\phi}\quad \tfrac{1}{2}x\frac{N}{c}+\frac{N_1}{cxr}=-2x\frac{\Theta_3}{\phi}-2\frac{\Theta_1}{\phi}.$$

150. All the conditions are satisfied if

$$L=-2cxe^{\int(\dot{p}+p)\,dx}\,\Theta_2;\ M=2ce^{\int(\dot{p}+p)\,dx}(x\Theta_3+\Theta_2);\ N=-4ce^{\int(\dot{p}+p)\,dx}\,\Theta_3,$$
$$L_1=-cx^2re^{\int(\dot{p}+p)\,dx}\,\Theta_0;\ M_1=cxre^{\int(\dot{p}+p)\,dx}(x\Theta_1+\Theta_0);$$
$$N_1=-2cxre^{\int(\dot{p}+p)\,dx}\,\Theta_1.$$

I have often, perhaps unnecessarily, retained arbitrary constants which might in strictness have been suppressed or merged. But the retention is, I think, convenient for purposes of reference.

151. We now get

$$L+xM+\tfrac{1}{2}x^2N=0;\quad L_1+xM_1+\tfrac{1}{2}x^2N_1=0,$$

and (see Art. 147) consequently,

$$L_2+xM_2+\tfrac{1}{2}x^2N_2=0.$$

152. Eliminating L_2, L_1, and L, we get

$$M''+m_1M'+m_2M+n_1N'+n_2N=0,$$
$$N''+\nu_1N'+\nu_2N+\mu_1M'+\mu_2M=0;$$

and here it will be convenient to introduce a symbol k defined by

$$k = \dot{p} + K = \dot{p} + p + \frac{r'}{r} + \frac{1}{x},$$

K having the same signification as in Art. 143.

153. This being so, we have

$$m_1 = 2x\dot{q} - K, \qquad \mu_1 = 2x\dot{r},$$

$$m_2 = (x\dot{q})' - k\dot{q} + x\,(\dot{r} - \tfrac{1}{2}r), \qquad \mu_2 = (x\dot{r})' - 2x\dot{r},$$

$$n_1 = x^2\dot{q} + 2, \qquad \nu_1 = x^2\dot{r} - K,$$

$$n_2 = (\tfrac{1}{2}x^2\dot{q} + 1)' - k\,(\tfrac{1}{2}x^2\dot{q} + 1), \quad \nu_2 = (\tfrac{1}{2}x^2\dot{r})' - k\,(\tfrac{1}{2}x^2\dot{r}) + x\,(\dot{r} - \tfrac{1}{2}r);$$

wherein it may be noticed that we have

$$m_1 + \nu_1 = -2k, \qquad \nu_1 - \tfrac{1}{2}x\mu_1 = -K,$$

$$m_1 - \nu_1 = 2x\dot{q} - x^2\dot{r}, \qquad \nu_2 - \nu_1' - \frac{x}{2}(\mu_2 - \mu_1') = K' + \tfrac{1}{2}x\,(\dot{r} - r),$$

$$m_2 - \nu_2 = \tfrac{1}{2}(m_1 - \nu_1)' - \tfrac{1}{2}k\,(m_1 - \nu_1), \quad \tfrac{1}{2}x\,(m_1 - \nu_1) - n_1 = -x^2\dot{r} - 2.$$

154. Multiply the first equation of Art. 152 into M, and the second into N, and subtract the last from the first result. Then $e^{-\int k\,dx}$ will be an integrating factor of the difference; and the integral may be written

$$MN' - NM' + \tfrac{1}{2}\mu_1 M^2 - \tfrac{1}{2}(m_1 - \nu_1)\,MN - \tfrac{1}{2}n_1 N^2 = e^{\int k\,dx}.$$

155. To verify this, substitute for N' and M' their values ($N' = N_1 + \dot{r}L$, $M' = M_1 + \dot{q}L - N$) obtained from Art. 142. The integral becomes, in virtue of Art. 153,

$$MN_1 - NM_1 + (\dot{r}M - \dot{q}N)\,L + x\dot{r}\,M^2 - (x\dot{q} - \tfrac{1}{2}x^2\dot{r})\,MN - \tfrac{1}{2}x^2\dot{q}N^2$$

$$= MN_1 - NM_1 + (\dot{r}M - \dot{q}N)\,(L + xM + \tfrac{1}{2}x^2N) = MN_1 - NM_1 = e^{\int k\,dx},$$

as appears from Art. 151. But from the formulæ of Art. 150, we get

$$MN_1 - NM_1 = 4c^2xr\,(\Theta_0\Theta_3 - \Theta_1\Theta_2)\,e^{2\int(\dot{p}+p)\,dx} = 4c^2\dot{c}_1c_1\,xr\,e^{\int(\dot{p}+p)\,dx}$$

$$= 4c^2\dot{c}_1c_1e^{\int k\,dx},$$

as follows from (B) of Art. 133, and from bringing xr under the exponential integral. Merging the arbitrary constant $4c^2\dot{c}_1c_1$, we get for $MN_1 - NM_1$ the same value as before. This verifies not only the particular calculations but also the relations, assigned in Art 150, between the Θ_m's and L, M, and N.

156. In further verification, we have

$$L_1 + \dot{p}L - L' - M = ce^{\int(\dot{p}+p)dx}\left\{-x^2 r\Theta_0 - 2x\dot{p}\Theta_2 + 2\left[1 + x(\dot{p}+p)\right]\Theta_2 + 2x\Theta_2' - 2(x\Theta_3 + \Theta_2)\right\}$$

$$= ce^{\int(\dot{p}+p)dx}\left\{-x^2 r\Theta_0 + 2xp\Theta_2 - 2x\Theta_3 + 2x\Theta_2'\right\},$$

which vanishes, as it ought, in virtue of Art. 135. Another easy test is obtained from $N_1 = -\dot{r}L + N'$.

157. A strong verification is had by applying the foregoing formulæ to the particular Riccatian discussed in my prior paper, for which we have

$$L = x^4 - x^2;\ \ M = x - 2x^3;\ \ N = 2x^2;\ \ L_1 = x^5;$$

$$M_1 = -2x^4 - 2x^2;\ \ N_1 = 2x^3 + 2x.$$

These values, being substituted in each of the three equations

$$-pL_1 + L_1' + M_1 - KL_1 - \tfrac{1}{2}xrL = 0 = -\dot{q}L_1 + M_1' + N_1 - KM_1 - \tfrac{1}{2}xrM$$

$$= -\dot{r}L_1 + N_1' - pN - \tfrac{1}{2}xrN$$

(which are obtained by a combination of results given in Arts. 143—7, 151), satisfy all three of them.

158. I now introduce a symbol λ defined by

$$e^{\int K\,dx}\int e^{-\int K\,dx}\,dx = \lambda, \text{ or by } \lambda' = K\lambda + 1,$$

and which will therefore be in general a transcendent. And I give to the coefficients of the correlate the following values, viz.,

$$\dot{p} = K + \frac{2}{\lambda};\ \ \dot{q} = K' - \frac{1}{x}K - \frac{2}{x\lambda} - \tfrac{1}{2}xr;\ \ \dot{r} = r - \frac{2K'}{x};$$

values which satisfy the condition of Art. 146. This being done, I say that

$$L = 0,\quad M = -\tfrac{1}{2}x\lambda,\quad N = \lambda$$

will fulfil all the conditions.

159. Put then

$$Y = L\dot{y}'' + M\dot{y}' + N\dot{y} = \lambda(\dot{y} - \tfrac{1}{2}x\dot{y}') = -\tfrac{1}{2}\lambda\dot{\Theta};$$

it follows that

$$Y' = -\tfrac{1}{2}(K\lambda+1)\,\dot{\theta}-\tfrac{1}{2}\lambda\dot{\theta}',$$
$$Y'' = -\tfrac{1}{2}\{K'\lambda+K(K\lambda+1)\}\,\dot{\theta}-(K\lambda+1)\,\dot{\theta}'-\tfrac{1}{2}\lambda\dot{\theta}''.$$

160. Hence

$$\begin{aligned} Y''-KY' &= -\tfrac{1}{2}K'\lambda\dot{\theta}-(\tfrac{1}{2}K\lambda+1)\,\dot{\theta}'-\tfrac{1}{2}\lambda\dot{\theta}'' \\ &= -\tfrac{1}{2}K'\lambda\dot{\theta}-(\tfrac{1}{2}K\lambda+1)\dot{\theta}'-\tfrac{1}{2}\lambda\,(\tfrac{1}{2}x\dot{r}\dot{\theta}-\dot{p}\dot{\theta}'),\ \text{[Art. 114.]} \\ &= -\tfrac{1}{2}\lambda\,\{(K'+\tfrac{1}{2}x\dot{r})\,\dot{\theta}+(K-\dot{p})\,\dot{\theta}'\}-\dot{\theta}' \\ &= -\tfrac{1}{2}\lambda\left\{\tfrac{1}{2}xr\dot{\theta}-\frac{2}{\lambda}\,\dot{\theta}'\right\}-\dot{\theta}', \\ &= -\tfrac{1}{4}xr\lambda\dot{\theta} = \tfrac{1}{2}xrY; \end{aligned}$$

therefore $Y''-KY'-\tfrac{1}{2}xrY = 0 = Y''-\left(p+\frac{r'}{r}+\frac{1}{x}\right)Y'-\tfrac{1}{2}xrY,$

and the equation of Art. 111 is satisfied.

161. By way of example: in the biordinal of Art. 105 put $p = 0$, $m = 2$, $r = x^n$, and it becomes $z''+\frac{2}{x}z'-\tfrac{1}{2}x^{n+1}z = 0$, which is transformed into $\zeta''-\tfrac{1}{2}x^{n+1}\zeta = 0$, by the substitution $xz = \zeta$. The terordinal (Art. 110) is

$$y'''+0\,.\,y''-\tfrac{1}{2}x^{n+1}y'+x^n y = 0,$$

the deformation is

$$Y'''-\frac{2n}{x}Y''+\left\{\frac{n(n+1)}{x^2}-\tfrac{1}{2}x^{n+1}\right\}Y'-x^nY = 0,\ \text{[Art. 111.]}$$

and the correlate is

$$\dot{y}'''+\frac{1-n}{x}\dot{y}''-\left(\frac{2}{x^2}+\tfrac{1}{2}x^{n+1}\right)\dot{y}'+\left(2\frac{n+1}{x^3}+x^n\right)\dot{y} = 0,$$

for $$K = \frac{1+n}{x},\quad \lambda = -\frac{x}{n},$$

and, by Art. 158,

$$\dot{p} = \frac{1-n}{x},\quad \dot{q} = -\frac{2}{x^2}-\tfrac{1}{2}x^{n+1},\quad \dot{r} = x^n+2\frac{n+1}{x^3}.$$

162. Hence $$Y = -\frac{x}{n}\left(\dot{y}-\frac{x}{2}\dot{y}'\right);$$

or, multiplying, as we may do, into $-2n$, and then replacing $-2nY$ by Y, we get $\qquad Y = x^2\dot{y}' - 2x\dot{y}, \quad Y' = x^2\dot{y}'' - 2\dot{y},$
and

$$Y'' = x^2\dot{y}''' + 2x\dot{y}'' - 2\dot{y}' = (n+1)\,x\dot{y}'' + \tfrac{1}{2}x^{n+3}\dot{y}' - \left(x^{n+2} + 2\,\frac{n+1}{x}\right)\dot{y}$$

$$= \frac{n+1}{x}\,(x^2\dot{y}'' - 2\dot{y}) + \tfrac{1}{2}x^{n+1}\,(x^2\dot{y}' - 2x\dot{y}) = \frac{n+1}{x}\,Y' + \tfrac{1}{2}x^{n+1}\,Y.$$

163. And this is right; for if, as in Art. 111, we integrate the deformation and suppose the arbitrary constant to vanish, we are led

to $$Y'' - \frac{n+1}{x}\,Y' - \tfrac{1}{2}x^{n+1}\,Y = 0,$$

and so to a verification of preceding results. In dealing with this example no transcendent has been introduced.

Thursday, April 12th, 1888.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

Mr. A. R. Johnson, M.A., Fellow of St. John's College, Cambridge, was elected a Member.

The following communications were made:—

Continuation of Former Paper on Simplicissima: W. J. C. Sharp, M.A.

Synthetical Solutions in the Conduction of Heat: E. W. Hobson, M.A.

Continuation of paper on Symmetric Functions: R. Lachlan, M.A.

On a Law of Attraction which might include both Gravitation and Cohesion: G. S. Carr, M.A.

The following presents were received:—

"Proceedings of the Royal Society," Vol. XLIII., No. 263.

"Educational Times," for April.

"Proceedings of the Manchester Literary and Philosophical Society," Vols. XXV., 1885—86, and XXVI., 1886—87.

"Memoirs of the Manchester Literary and Philosophical Society," Third Series, Tenth Volume.

"Annals of Mathematics," Vol. III., No. 6 (University of Virginia); Dec., 1887.

"Bulletin des Sciences Mathématiques," Tome XII.; March and April, 1888.

"Bulletin de la Société Mathématique de France," Tome XVI., No. 1.

"Journal de l'École Polytechnique," 57e cahier; Paris, 1887.

"Annales de l'École Polytechnique de Delft," Tome III., 4me Livraison; 1888.

"Beiblätter zu den Annalen der Physik und Chemie," Band XII., Stück 3; Leipzig, 1888.

"Jahrbuch über die Fortschritte der Mathematik," Band XVII., Heft 2, Jahrgang 1885; Berlin, 1888.

"Berichte über die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig," Math.-Phys. Classe, 1887, I., II.; Leipzig, 1888.

"Mittheilungen der Mathematischen Gesellschaft in Hamburg," No. 8; März, 1888.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. III., Fasc. 10—13, Nov. 20—Dic. 18, 1887.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Nos. 52—54; Febb. 29—Marzo 31, 1888.

"Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin," XL.—LIV., Oct. 20—Dec. 22, 1887, with Title, Index, &c.

"Memorias de la Sociedad Científica—'Antonio Alzate,'" Tomo I., No. 8; México, 1888.

"Calendar, for the year 1887—88, of the Imperial University of Japan," 8vo; Tokio, 1888.

"Sur la détermination d'une Courbe algébrique par des Points donnés," par H. G. Zeuthen. (Excerpt from "Mathematische Annalen," Bd. XXXI.)

"Die Rationalen ebenen Kurven 4. Ordnung und die binäre Form 6ter Ordnung," von Ernst Meyer, 8vo pamphlet. (Inaugural Dissertation zu Königsberg i. Pr., März 3, 1888.)

"On Systems of Circles and Spheres," by R. Lachlan, B.A. ("Philosophical Transactions," Vol. 177, Pt. II., 1886); from the Author.

Synthetical Solutions in the Conduction of Heat.

By E. W. Hobson, M.A.

[*Read April* 12*th*, 1888.]

The object of the present communication is to give the solutions expressed as definite integrals of certain problems in the variable motion of heat in two and three dimensions, in which the boundaries of the conducting body are straight edges or planes. The solutions are obtained by a method which has been applied by Sir W. Thomson,* to some cases of conduction; this method consists in superimposing

* See "Collected Works," Vol. II.