stant being $\infty S . \infty H$. The points S, H are foci of all the loci, and the loci of Q, Q′ have double foci at A, A′ respectively; so that for the oval

$$(\sqrt{2}-1)\,SQ - (\sqrt{2}+1)\,HQ = 2AQ,$$

and for the infinite branch,

$$(\sqrt{2}+1)\,HQ - (\sqrt{2}-1)\,SQ = 2AQ.$$

Other properties perhaps worth mentioning are—(1) the straight line bisecting UR at right angles (which passes through Q, since QU=QR), and that bisecting U′R at right angles, intersect on the circle and on the conjugate axis of the hyperbola; (2) the angle Q′RP exceeds the angle PRQ by a right angle.

On Determinants of Alternate Numbers. By WILLIAM SPOTTISWOODE, M.A., F.R.S.

[*Read February* 10*th*, 1876.]

Some very interesting unpublished notes on determinants and other functions of alternate numbers, kindly communicated to me by Professor Clifford, were the occasion of the following investigation of the leading properties of such determinants. With his permission I have incorporated his theorems in the present paper, specifying in each case that which is due to him.

If $\lambda_1, \lambda_2, \ldots, \mu_1 \mu_2, \ldots, \ldots$ be alternate numbers, their laws of combination are thus expressed:

$$\lambda_1^2 = 0, \ldots \lambda_1\lambda_2 + \lambda_2\lambda_1 = 0 \ldots\ldots \;\ldots\ldots\ldots\ldots \quad (1).$$

Then, for determinants of alternate numbers, Hankel's fundamental theorem takes the following form:—If $a_1, a_2, \ldots$ be any other set of alternate numbers, then the continued product

$$(a_1\lambda_1 + a_2\lambda_2 + \ldots)\,(a_1\mu_1 + a_2\mu_2 + \ldots)\,\ldots = -a_1 a_2 \ldots \begin{matrix} \lambda_1, & \lambda_2, \ldots \\ \mu_1, & \mu_2 \,.. \\ \ldots & \ldots \end{matrix} \quad \text{(I.)}$$

It will be convenient generally in this paper to adopt Sylvester's umbral notation, and whenever no ambiguity is likely to arise to drop the enclosing brackets; in accordance with which we shall generally write

$$\begin{matrix} \lambda, \mu, \ldots & \text{for} & \lambda_1, \lambda_2, \ldots \\ 1, 2, \ldots & ,, & \mu_1, \mu_2, \ldots \\ & & \ldots \ldots \ldots \end{matrix}$$

Now in the development of such a determinant, the factors entering into each term may be arranged either in the natural order of the letters λ, μ, ..., or in that of the suffixes 1, 2, ... But as these two arrangements give results differing in form, it is necessary to understand beforehand that the normal form of development will be that according to letters. Thus, for the degrees 2 and 3, we should have

$$\begin{matrix}\lambda, \mu \\ 1, 2\end{matrix} = \lambda_1\mu_2 - \lambda_2\mu_1 = \lambda_1\mu_2 + \mu_1\lambda_2,$$

$$\begin{matrix}\lambda, \mu, \nu \\ 1, 2, 3\end{matrix} = \lambda_1(\mu_2\nu_3 - \mu_3\nu_2) + \lambda_2(\mu_3\nu_1 - \mu_1\nu_3) + \lambda_3(\mu_1\nu_2 - \mu_2\nu_1)$$

$$= \lambda_1(\mu_2\nu_3 + \nu_2\mu_3) + \mu_1(\nu_2\lambda_3 + \lambda_2\nu_3) + \nu_1(\lambda_2\mu_3 + \mu_2\lambda_3);$$

and generally, if on developing according to lines (letters, λ, μ, ...) the terms have alternate signs on the rule of ordinary determinants; then, on developing according to columns (suffixes 1, 2, ...), the terms will all have the same sign. This result may be expressed by the following formulæ

$$\begin{matrix}\lambda, \mu, \ldots \\ 1, 2, \ldots\end{matrix} = \Sigma \pm \begin{matrix}\lambda, \mu \ldots \\ 1, 2\end{matrix} = \Sigma + \begin{matrix}1, 2, \ldots \\ \lambda, \mu\end{matrix} \qquad \ldots\ldots(\text{II.}),$$

in which last two expressions it is to be understood that the order of symbols in the upper line is to be retained, and that in the lower permuted.

In the case of odd-degreed determinants, there is yet a third way in which the development may be arranged; *e.g.*,

$$\begin{matrix}\lambda, \mu, \nu \\ 1, 2, 3\end{matrix} = \lambda_1(\mu_2\nu_3 - \mu_3\nu_2) + \mu_1(\nu_2\lambda_3 - \nu_3\lambda_2) + \nu_1(\lambda_2\mu_3 - \lambda_3\mu_2),$$

in which neither the order of the letters, nor that of the suffixes, is retained. The possibility of this depends upon the fact that the signs of all the terms in the first stage of the development, whether according to lines or columns, are alike positive.

Beside the alternate numbers proper, there is a set of numbers complementary to them, or "concurrent numbers" as they may be called, whose laws of combination are expressed by the following formulæ,

$$\alpha'^2 = 0, \;\; \beta'^2 = 0, \;\; \therefore \; \alpha'\beta' = \beta'\alpha', \;\ldots$$

In fact, any even-degreed products of alternate numbers will form such a set, and in the same way that alternate numbers α_1, α_2,... serve to resolve a determinant $\begin{matrix}\lambda, \mu, \ldots \\ 1, 2, \ldots\end{matrix}$ into a line product

$$(\alpha_1\lambda_1 + \alpha_2\lambda_2 + \ldots)(\alpha_1\mu_1 + \alpha_2\mu_2 + \ldots)\ \ldots,$$

so do the complementary numbers α', β',... serve to resolve it into a column product

$$(\alpha'\lambda_1 + \beta'\mu_1 + \ldots)(\alpha'\lambda_2 + \beta'\mu_2 + \ldots).$$

In fact,

$$(\alpha'\lambda_1+\beta'\mu_1)\,(\alpha'\lambda_2+\beta'\mu_2) = \alpha'\lambda_1\beta'\mu_2+\beta'\mu_1\alpha'\lambda_2 = \alpha'\beta'\,(\lambda_1\mu_2-\lambda_2\mu_1),$$

$$\begin{aligned}
&(\alpha'\lambda_1+\beta'\mu_1+\gamma'\nu_1)(\alpha'\lambda_2+\beta'\mu_2+\gamma'\nu_2)(\alpha'\lambda_3+\beta'\mu_3+\gamma'\nu_3)\\
&\quad= (\alpha'\lambda_1+\ldots)\,\{\ \beta'\mu_2\gamma'\nu_3+\gamma'\nu_2\beta'\mu_3\\
&\qquad\qquad +\gamma'\nu_2\alpha'\lambda_3+\alpha'\lambda_2\gamma'\nu_3\\
&\qquad\qquad +\alpha'\lambda_2\beta'\mu_3+\beta'\mu_2\alpha'\lambda_3\}\\
&\quad= (\alpha'\lambda_1+\ldots)\,\{\ \beta'\gamma'\,(\mu_2\nu_3+\mu_3\nu_2)\\
&\qquad\qquad +\gamma'\alpha'\,(\nu_2\lambda_3-\nu_3\lambda_2)\\
&\qquad\qquad +\alpha'\beta'\,(\lambda_2\mu_3-\lambda_3\mu_2)\}\\
&\quad= \alpha'\beta'\gamma'\,\{\lambda_1\,(\mu_2\nu_3-\mu_3\nu_2)+\mu_1\,(\nu_2\lambda_3-\nu_3\lambda_2)+\nu_1\,(\lambda_2\mu_3-\lambda_3\mu_2)\}\\
&\quad= \alpha'\beta'\gamma'\,\{\lambda_1\,(\mu_2\nu_3-\mu_3\nu_2)+\lambda_2\,(\mu_3\nu_1-\mu_1\nu_3)+\lambda_3\,(\mu_1\nu_2-\mu_2\nu_1)\}\\
&\quad= \alpha'\beta'\gamma'\ \begin{matrix}\lambda, & \mu, & \nu,\\ 1, & 2, & 3,\end{matrix}
\end{aligned}$$

and so on.

This being so, it will be found, as remarked by Professor Clifford, that an interchange of two consecutive columns will, but that one of two consecutive rows will not, change the sign of the determinant. This may be expressed thus—

$$\begin{matrix}\lambda, & \mu,\ldots\\ 1, & 2,\ldots\end{matrix} = \begin{matrix}\mu, & \lambda,\ldots\\ 1, & 2,\ldots\end{matrix}\qquad \begin{matrix}\lambda, & \mu,\ldots\\ 2, & 1,\ldots\end{matrix} = -\begin{matrix}\lambda, & \mu,\ldots\\ 1, & 2,\ldots\end{matrix}\qquad\ldots\ldots\text{(III.)}$$

As a corollary to (III.) we have, when two columns are identical, in other words, when two suffixes are the same,

$$\begin{matrix}\lambda, & \mu, & \nu,\ldots\\ 1, & 1, & 3,\ldots\end{matrix} = \begin{matrix}\lambda_1, & \lambda_1, & \lambda_2,\ldots\\ \mu_1, & \mu_1, & \mu_2,\ldots\\ \ldots & \ldots & \ldots\end{matrix} = 0\ \ldots\ldots\ldots\ldots\ \text{(III. 1)};$$

also, when two lines are identical, say $\lambda=\mu$,

$$\begin{matrix}\lambda, & \lambda, & \nu,\ldots\\ 1, & 2, & 3,\ldots\end{matrix} = 2\lambda_1\lambda_2\ \begin{matrix}\nu, & \rho,\ldots\\ 3, & 4\ldots\end{matrix} - 2\lambda_1\lambda_3\ \begin{matrix}\nu, & \rho,\ldots\\ 1, & 4,\ldots\end{matrix} + \ldots\qquad\ldots\text{(III. 2)}.$$

For the addition and subtraction of determinants we have the ordinary formulæ:

$$\begin{matrix}\lambda_1, & \lambda_2\\ \mu_1, & \mu_2\end{matrix} + \begin{matrix}\lambda_1, & \lambda_2\\ \nu_1, & \nu_2\end{matrix} = \lambda_1\mu_2-\lambda_2\mu_1+\lambda_1\nu_1-\lambda_2\nu_1 = \begin{matrix}\lambda_1, & \lambda_2,\\ \mu_1+\nu_1, & \mu_2+\nu_2,\end{matrix}$$

$$\begin{matrix}\lambda_1, & \lambda_2\\ \mu_1, & \mu_2\end{matrix} + \begin{matrix}\lambda_1, & \lambda_3\\ \mu_1, & \mu_3\end{matrix} = \lambda_1\mu_2-\lambda_2\mu_1+\lambda_1\mu_3-\lambda_3\mu_1 = \begin{matrix}\lambda_1, & \lambda_2+\lambda_3;\\ \mu_1, & \mu_2+\mu_3\end{matrix}$$

and, generally,

$$\left.\begin{matrix}
\begin{matrix}\lambda, & \mu, & \ldots & \rho, & \ldots\\ 1, & 2, & \ldots & k, & \ldots\end{matrix} + \begin{matrix}\lambda, & \mu, & \ldots & \rho', & \ldots\\ 1, & 2, & \ldots & k, & \ldots\end{matrix} = \begin{matrix}\lambda, & \mu, & \ldots & \rho+\rho', & \ldots\\ 1, & 2, & \ldots & k, & \ldots\end{matrix}\\
\begin{matrix}\lambda, & \mu, & \ldots & \rho, & \ldots\\ 1, & 2, & \ldots & k, & \ldots\end{matrix} + \begin{matrix}\lambda, & \mu, & \ldots & \rho, & \ldots\\ 1, & 2, & \ldots & k', & \ldots\end{matrix} = \begin{matrix}\lambda, & \mu, & \ldots & \rho, & \ldots\\ 1, & 2, & \ldots & k+k', & \ldots\end{matrix}
\end{matrix}\right\}\ \ldots\text{(IV)}.$$

For multiplication, consider in the first instance the elementary case

$$\begin{matrix}\lambda, & \mu\\ 1, & 2\end{matrix}\ .\ \begin{matrix}\lambda', & \mu'\\ 1, & 2\end{matrix} = \lambda_1\mu_2\lambda'_1\mu'_2-\lambda_1\mu_2\lambda'_2\mu'_1-\lambda_2\mu_1\lambda'_1\mu'_2+\lambda_2\mu\,\lambda^1\mu'_1.$$

On the other hand, if we develope the ordinary formula, viz.,

$$\begin{matrix} \lambda_1\lambda'_1 + \lambda_2\lambda'_2, & \lambda_1\mu'_1 + \lambda_2\mu'_2, \\ \mu_1\lambda'_1 + \mu_2\lambda'_2, & \mu_1\mu'_1 + \mu_2\mu'_2, \end{matrix}$$

in the usual way, and then change the sign of the last term, we obtain

$$(\lambda_1\lambda'_1+\lambda_2\lambda'_2)\,(\mu_1\mu'_1+\mu_2\mu'_2) + (\lambda_1\mu'_1+\lambda_2\mu'_2)\,(\mu_1\lambda'_1+\mu_2\lambda'_2),$$

$$\lambda_1\lambda'_1\mu_1\mu'_1+\lambda_1\lambda'_1\mu_2\mu'_2+\lambda_2\lambda'_2\mu_1\mu'_1+\lambda_2\lambda'_2\mu_2\mu'_2 = -\begin{matrix}\lambda, \mu & \lambda', \mu \\ 1, 2 & 1, 2\end{matrix}$$

$$-\lambda_1\lambda'_1\mu_1\mu'_1-\lambda_1\lambda'_2\mu_2\mu'_1-\lambda_2\lambda'_1\mu_1\mu'_2-\lambda_2\lambda'_2\mu_2\mu_2.$$

In fact, if we write for brevity

$$\lambda_1\lambda'_1 + \ldots = (\lambda\lambda'), \quad \lambda_1\mu'_1 + \ldots = (\lambda\mu'), \ldots,$$

we should have

$$-\begin{matrix}\lambda, \mu \\ 1, 2\end{matrix} \,.\, \begin{matrix}\lambda', \mu' \\ 1, 2\end{matrix} = (\lambda\lambda')(\mu\mu') + (\lambda\mu')(\mu\lambda'),$$

and the change of sign in the last term might have been provided for by writing the formula thus, $(\lambda\lambda')\,(\mu\mu') - (\lambda\mu')\,(\lambda'\mu)$; but this hardly expresses the meaning of the change of sign in a manner applicable to all cases. The reason of it is this: in the original product the terms are all understood to be arranged in the order $\lambda\mu\lambda'\mu'$ (with their proper suffixes); the term $(\lambda\lambda')(\mu\mu')$ must therefore undergo a change of sign due to the transposition of $\lambda'\mu$; while the term $(\lambda\mu')\,(\mu\lambda')$ will retain its sign. In other words, the two terms, instead of having different, must have the same sign.

This being so, the ordinary formula for the multiplication of determinants may be applied, viz.,

$$-\begin{matrix}\lambda, \mu, \ldots & \lambda', \mu', \ldots \\ 1, 2, \ldots & 1, 2, \ldots\end{matrix} = \left.\begin{matrix}(\lambda\lambda'), & (\lambda\mu'), & \ldots \\ (\mu\lambda'), & (\mu\mu'), & \ldots \\ \ldots & \ldots & \ldots\end{matrix}\right\} \quad \ldots\ldots\ldots \text{(V.)},$$

if it be understood that, after developing the right-hand side of the equation according to the ordinary rule, those terms which require an odd number of changes to bring them into the form $\lambda\mu \ldots \lambda'\mu' \ldots$ are to have their signs altered, while those which require an even number of changes are to retain their signs.

When the determinants to be multiplied together become equal, in other words, when the power of a determinant is required, the question is much simplified; for then

$$(\lambda\lambda') = 0, \ldots, (\lambda\mu)+(\mu\lambda) = 0, \ldots;$$

and consequently, if we develope the expression

$$-\begin{pmatrix}\lambda, \mu, \ldots \\ 1, 2, \ldots\end{pmatrix}^2 = \begin{matrix}0, & (\lambda\mu), & \ldots \\ (\lambda\mu), & 0, & \ldots \\ \ldots & \ldots & \ldots\end{matrix} = \left.\begin{matrix}0, & (\lambda\mu), & \ldots \\ -(\lambda\mu), & 0, & \ldots \\ \ldots & \ldots & \ldots\end{matrix}\right\} \ldots\ldots \text{(V. 1)}$$

according to the usual rule, the terms will be already arranged in the form $\lambda\mu \dots \lambda\mu \dots$ as in the power originally proposed. The expression for the square of a determinant is in fact a skew symmetrical determinant; and the laws appertaining to such determinants in ordinary algebra will apply to those in alternate numbers.

One very important consequence of this is, that the powers of odd-degreed determinants of alternate numbers necessarily vanish.

Thus,

$$-\begin{pmatrix}\lambda, \mu, \nu \\ 1, 2, 3\end{pmatrix}^2 = \begin{matrix} 0, & (\lambda\mu), & -(\nu\lambda) \\ -(\lambda\mu), & 0, & (\mu\nu) \\ (\nu\mu), & -(\mu\nu), & 0, \end{matrix} = 0 \dots\dots\dots (\text{V. } 2).$$

Odd-degreed determinants of alternate numbers are in fact themselves alternate numbers.

For even degrees we have

$$-\begin{pmatrix}\lambda, \mu \\ 1, 2\end{pmatrix}^2 = -(\lambda\mu)^2 = -2\lambda_1\mu_1\lambda_2\mu_2 = 2\lambda_1\lambda_2\mu_1\mu_2,$$

$$-\begin{pmatrix}\lambda, \mu, \nu, \rho \\ 1, 2, 3, 4\end{pmatrix}^2 = \begin{matrix} 0, & (\lambda\mu), & (\lambda\nu), & (\lambda\rho) \\ -(\lambda\mu), & 0, & (\mu\nu), & (\mu\rho) \\ -(\lambda\nu), & -(\mu\nu), & 0, & (\nu\rho) \\ -(\lambda\rho), & -(\mu\rho), & -(\nu\rho), & 0 \end{matrix}$$

$$= (\mu\nu)^2(\lambda\rho)^2 + (\nu\lambda)^2(\mu\rho)^2 + (\lambda\mu)^2(\nu\rho)^2$$
$$+ 2\{(\nu\lambda)(\lambda\mu)(\mu\rho)(\nu\rho) + (\lambda\mu)(\mu\nu)(\nu\rho)(\lambda\rho) + (\mu\nu)(\nu\lambda)(\lambda\rho)(\mu\rho)\}$$
$$= \{(\mu\nu)(\lambda\rho) + (\nu\lambda)(\mu\rho) + (\lambda\mu)(\nu\rho)\}^2.$$

Before proceeding further, I quote a remark by Professor Clifford: "Alternate numbers may be considered as given in *sets* of n at a time (like the coordinates of a point in n-fold space); and in that case it is convenient to regard the product of all the numbers in any set as equal to unity. Hence all the products of all but one of the numbers make a new set, the *reciprocal* numbers." In what follows this view will be adopted, viz., it will be supposed that $\lambda_1\lambda_2 \dots \lambda_n = 1$, $\lambda_2\lambda_3 \dots \lambda_n = \lambda_1^{-1}, \dots$

Beside the method used above there is another, depending upon Hankel's fundamental theorem, which may be conveniently applied to the powers of determinants. If $\alpha_1, \alpha_2, \dots, \beta_1, \beta_2, \dots, \dots$ be any new sets of alternate numbers, then

$$\begin{Bmatrix}\lambda, \mu, \dots \\ 1, 2, \dots\end{Bmatrix}^2 = (\alpha_1\lambda_1 + \alpha_2\lambda_2 + \dots)(\alpha_1\mu_1 + \dots) \dots (\beta_1\lambda_1 + \dots)(\beta_1\mu_1 + \dots) \dots$$
$$= (\alpha_1\lambda_1 + \dots)(\beta_1\lambda_1 + \dots)(\alpha_1\mu_1 + \dots)(\beta_1\mu_1 + \dots) \dots$$
$$= (\lambda_1\lambda_2 \underset{1, 2}{\alpha, \beta} + \dots)(\mu_1\mu_2 \underset{1, 2}{\alpha, \beta} + \dots) \dots$$

Similarly,

$$\begin{Bmatrix}\lambda, \mu, \ldots\\ 1, 2, \ldots\end{Bmatrix}^3 = (\lambda_1\lambda_2\lambda_3\,\underset{1,\,2,\,3}{\alpha, \beta, \gamma} + \ldots)\,(\mu_1\mu_2\mu_3\,\underset{1,\,2,\,3}{\alpha, \beta, \gamma} + \ldots)\ldots,$$

and so on. Consequently

$$\begin{Bmatrix}\lambda, \mu\\ 1, 2\end{Bmatrix}^2 = \lambda_1\lambda_2\mu_1\mu_2\begin{Bmatrix}\alpha, \beta\\ 1, 2\end{Bmatrix}^2 = -2\lambda_1\lambda_2\mu_1\mu_2 = -2,$$

$$\begin{Bmatrix}\lambda, \mu, \nu\\ 1, 2, 3\end{Bmatrix}^2 = (\lambda_2\lambda_3\underset{2,3}{\alpha,\beta} + \lambda_3\lambda_1\underset{3,1}{\alpha,\beta} + \lambda_1\lambda_2\underset{1,2}{\alpha,\beta})(\mu_2\mu_3\underset{2,3}{\alpha,\beta} + \ldots)(\nu_2\nu_3\underset{2,3}{\alpha,\beta} + \ldots)$$

Now, from the value of $\begin{Bmatrix}\alpha, \beta\\ 2, 3\end{Bmatrix}^2$ it will easily be seen, as will be more fully discussed below, that,

$$\begin{Bmatrix}\alpha, \beta\\ 2, 3\end{Bmatrix}^3 = 0, \quad \begin{Bmatrix}\alpha, \beta\\ 2, 3\end{Bmatrix}^2\begin{Bmatrix}\alpha, \beta\\ 3, 1\end{Bmatrix} = 0;$$

also

$$\underset{2,\,3}{\alpha, \beta}\,.\,\underset{3,\,1}{\alpha, \beta}\,.\,\underset{1,\,2}{\alpha, \beta} = -\underset{2,\,3}{\alpha, \beta}\,\alpha_1\beta_1(\alpha_2\beta_3 + \alpha_3\beta_2)$$

$$= -\alpha_1\beta_1(\alpha_3\beta_2\,\alpha_2\beta_3 - \alpha_2\beta_3\,\alpha_3\beta_2) = 0;$$

and consequently, as found before,

$$\begin{Bmatrix}\lambda, \mu, \nu\\ 1, 2, 3\end{Bmatrix}^3 = 0, \quad \begin{Bmatrix}\lambda, \mu, \nu\\ 1, 2, 3\end{Bmatrix}^4 = 0, \ \ldots\ldots$$

A similar process will apply to the case of the fifth degree. But as the theorem has been already proved, I will merely indicate the principal steps, as an exemplification of the present process.

$$\begin{Bmatrix}\lambda, \mu, \nu, \rho, \sigma\\ 1, 2, 3, 4, 5\end{Bmatrix}^2 = (\alpha_1\lambda_1 + \ldots)(\alpha_1\mu_1 + \ldots)(\alpha_1\nu_1 + \ldots)(\alpha_1\rho_1 + \ldots)(\alpha_1\sigma_1 + \ldots)$$

multiplied by similar factors in β

$$\begin{aligned} = \ & (\lambda_2\lambda_3\underset{2,\,3}{\alpha, \beta} + \lambda_3\lambda_1\underset{3,\,1}{\alpha, \beta} + \lambda_1\lambda_2\underset{1,\,2}{\alpha, \beta} \\ & + \lambda_4\lambda_5\underset{4,\,5}{\alpha, \beta} + \lambda_5\lambda_3\underset{5,\,3}{\alpha, \beta} + \lambda_3\lambda_4\underset{3,\,4}{\alpha, \beta} \\ & + \lambda_1\lambda_4\underset{1,\,4}{\alpha, \beta} + \lambda_1\lambda_5\underset{1,\,5}{\alpha, \beta} + \lambda_2\lambda_4\underset{2,\,4}{\alpha, \beta} + \lambda_2\lambda_5\underset{2,\,5}{\alpha, \beta}), \end{aligned}$$

multiplied by similar factors in μ, ν, ρ, σ

$= (A_\lambda + B_\lambda + C_\lambda)$ multiplied by similar factors in μ, ν, ρ, σ, suppose.

Dropping the suffixes for a moment, and writing $A^2, A^3, \ldots$ for AA, AAA, ... with any suffixes whatever, it will be seen from the combinations of $\underset{2,\,3,}{\alpha, \beta,}\ \underset{3,\,1,}{\alpha, \beta,}\ \ldots$ therein involved, that terms of the following forms must vanish, viz.,

$$A^3 = 0, \quad A^4 = 0, \quad A^5 = 0, \quad B^3 = 0, \quad B^4 = 0, \quad B^5 = 0.$$

The terms which remain will consequently be of the forms C^5, C^4A, C^4B, C^3A^2, C^3B^2, C^3AB. Now in C^3 it will be found that the only terms which do not vanish will have as their $\alpha_1\beta_1$ coefficients the following combinations, viz.,

$$\alpha_2\alpha_3\alpha_4\alpha_5\beta_2\beta_3\beta_4\beta_5,$$
$$\alpha_1\alpha_3\alpha_4\alpha_5\beta_1\beta_3\beta_4\beta_5,$$
$$\alpha_1\alpha_2\alpha_3\alpha_5\beta_1\beta_2\beta_3\beta_5,$$
$$\alpha_1\alpha_2\alpha_3\alpha_4\beta_1\beta_2\beta_3\beta_4;$$

and consequently terms of the following forms vanish, viz.,

$$C^5 = 0,\ C^4A = 0,\ C^4B = 0.$$

The same reasoning will show that the terms

$$C^3A^2 = 0,\quad C^3B^2 = 0;$$

so that the only terms remaining are of the form C^3AB, and of these all obviously vanish excepting those which have as factors the following expressions (in which for brevity 12 is written for $\underset{1,\,2}{\alpha,\,\beta}$):

$$(14^2 . 25 + \tfrac{1}{2} 15 . 24 . 25)\, 23 . 53 = (14^2 + \tfrac{1}{2} 15 . 24)\, 25 . 23 . 53,$$
$$(15^2 . 24 + \tfrac{1}{2} 24 . 25 . 14)\, 23 . 34 = (15^2 + \tfrac{1}{2} 25 . 14)\, 24 . 23 . 34,$$
$$(24^2 . 15 + \tfrac{1}{2} 25 . 14 . 15)\, 31 . 53 = (24^2 + \tfrac{1}{2} 25 . 14)\, 15 . 31 . 53,$$
$$(25^2 . 14 + \tfrac{1}{2} 14 . 15 . 24)\, 31 . 34 = (25^2 + \tfrac{1}{2} 15 . 24)\, 14 . 31 . 34,$$

all of which vanish in virtue of the outside factor. Hence finally, as before, the square and higher powers of the determinant of the fifth degree vanish.

It is hardly necessary to go through the calculations for the square of the determinant of the fourth degree; but the cube of the same determinant gives an important result. Proceeding as before,

$$\begin{Bmatrix} \lambda, \mu, \nu, \rho \\ 1, 2, 3, 4 \end{Bmatrix}^3 = (\lambda_2\lambda_3\lambda_4 \underset{2,\,3,\,4}{\alpha, \beta, \gamma} + \ldots) \times \text{ terms in } \mu, \nu, \rho.$$

Now, if we write

$$\underset{2,\,3,\,4}{\alpha, \beta, \gamma} = A_1,\quad \underset{3,\,4,\,1}{\alpha, \beta, \gamma} = A_2, \ldots,$$

then will $\quad A_1^2 = 0,\ A_2^2 = 0,\ \ldots\ A_1A_2 + A_2A_1 = 0,\ \ldots;$

in other words, A_1, A_2, ... are alternate numbers; and consequently we have

$$\begin{Bmatrix} \lambda, \mu, \nu, \rho^3 \\ 1, 2, 3, 4 \end{Bmatrix} = (A_1\lambda_1^{-1} + \ldots)\,(A_1\mu_1^{-1} + \ldots)\,(A_1\nu_1^{-1} + \ldots)\,(A_1\nu_1^{-1} + \ldots)$$
$$= \underset{1,\ \ 2,\ \ 3,\ \ 4}{\lambda^{-1}, \mu^{-1}, \nu^{-1}, \rho^{-1}};$$

in other words, the cube of the determinant of the fourth degree is equal to the determinant of the reciprocal numbers.

Replacing the reciprocals by the products to which they are severally equal, we have

$$\begin{Bmatrix} \lambda, \mu, \nu, \rho \\ 1, 2, 3, 4 \end{Bmatrix}^4 = (\alpha_1\lambda_1 + \ldots)\,(\beta_1\lambda_2\lambda_3\lambda_4 + \ldots) \times \text{terms in } \mu, \nu, \rho.$$
$$= -(\alpha_1\beta_1 + \alpha_2\beta_2 + \ldots)^4$$
$$= -1\,.\,2\,.\,3\,.\,4.$$

And the same process enables us to generalise these results. In fact,

$$\begin{Bmatrix} \lambda, \mu, \ldots \\ 1, 2, \ldots \end{Bmatrix}^{n-1} = (\lambda_1^{-1}\ \begin{matrix} \alpha, \beta, \ldots \\ 2, 3, \ldots \end{matrix} + \ldots) \times \text{terms in } \mu, \ldots;$$

and if the quantities $\lambda, \mu, \ldots$ are even in number, the quantities $\alpha, \beta, \ldots$ will be odd; so that the coefficients of $\lambda_1^{-1}, \ldots$ are themselves alternate numbers (in virtue of the properties of odd-degreed determinants), and the result is

$$\begin{Bmatrix} \lambda, \mu, \ldots \\ 1, 2, \ldots \end{Bmatrix}^{n-1} = \begin{Bmatrix} \lambda^{-1}, \mu^{-1}, \ldots \\ 1, \quad 2, \quad \ldots \end{Bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\text{(VI.)};$$

and similarly
$$\begin{Bmatrix} \lambda, \mu, \ldots \\ 1, 2, \ldots \end{Bmatrix}^{n} = -1\,.\,2 \ldots n \ldots\ldots\ldots\ldots\ldots\ldots \text{(VII.)}$$

These results are due to Professor Clifford.

Compound determinants of alternate numbers present some peculiarities which distinguish them from ordinary compound determinants. Beginning with the simplest case, viz. those formed from, $\begin{matrix} \lambda, \mu, \nu \\ 1, 2, 3 \end{matrix}$ we have

$$\begin{matrix} \nu, \lambda & \nu, \lambda \\ 3, 1 & 1, 2 \\ \lambda, \mu & \lambda, \mu \\ 3, 1 & 1, 2 \end{matrix} = -\begin{matrix} \nu_1, & \nu_2, & \nu_3, & . \\ \lambda_1, & \lambda_2, & \lambda_3, & . \\ . & \lambda_2, & \lambda_3, & \lambda_1 \\ . & \mu_2, & \mu_3, & \mu_1 \end{matrix} = -\begin{matrix} \lambda_1, & \lambda_1, & \lambda_2, & \lambda_3 \\ \nu_1, & \nu_1, & \nu_2, & \nu_3 \\ . & \lambda_1, & \lambda_2, & \lambda_3 \\ . & \mu_1, & \mu_2, & \mu_3 \end{matrix} = \begin{matrix} \lambda_1, & \lambda_1, & \lambda_2, & \lambda_3. \\ \mu_1, & \mu_1, & \mu_2, & \mu_3 \\ . & \lambda_1, & \lambda_2, & \lambda_3 \\ . & \nu_1, & \nu_2, & \nu_3 \end{matrix}$$

These results may be expressed by the following formulæ:

$$= -\left\|\begin{matrix} \lambda, \nu \\ 1, . \end{matrix}\right\| . \left\|\begin{matrix} \lambda, \nu, \lambda, \mu \\ 1, \ .\ \ 2, 3 \end{matrix}\right\| \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(VIII.)},$$
$$= \left\|\begin{matrix} \lambda, \mu \\ 1, . \end{matrix}\right\| . \left\|\begin{matrix} \lambda, \mu, \lambda, \nu \\ 1, \ .\ \ 2, 3 \end{matrix}\right\|$$

if it be understood that these expressions are to be thus developed (*e.g.*, taking the second)

$$= \lambda_1 \begin{matrix} \mu, \lambda, \nu \\ 1, 2, 3 \end{matrix} + \mu_1 \begin{matrix} \lambda, \lambda, \nu \\ 1, 2, 3 \end{matrix}$$
$$= \lambda_1 \begin{matrix} \lambda, \mu, \nu \\ 1, 2, 3 \end{matrix} + 2\mu_1 (\nu_1 : \lambda_1 + \nu_2 : \lambda_2 + \nu_3 : \lambda_3).$$

If the λ's had been ordinary instead of alternate numbers, the second term of this expression would have vanished, and we should have had the ordinary formula of compound determinants.

Again, if we form the determinant of all the first minors of the given determinant, we shall find by a similar process the following expression:

$$\begin{matrix}\mu, \nu \\ 2, 3\end{matrix} \; \begin{matrix}\mu, \nu \\ 3, 1\end{matrix} \; \begin{matrix}\mu, \nu \\ 1, 2\end{matrix} \quad \begin{matrix}\nu, \lambda \\ 2, 3\end{matrix} \; \begin{matrix}\nu, \lambda \\ 3, 1\end{matrix} \; \begin{matrix}\nu, \lambda \\ 1, 2\end{matrix} \quad \begin{matrix}\lambda, \mu \\ 2, 3\end{matrix} \; \begin{matrix}\lambda, \mu \\ 3, 1\end{matrix} \; \begin{matrix}\lambda, \mu \\ 1, 2\end{matrix} = \begin{matrix} \mu_1, & \mu_2, & \mu_3 & . & . & . \\ \nu_1, & \nu_2, & \nu_3 & . & . & . \\ . & . & . & \nu_1, & \nu_2, & \nu_3 \\ . & . & . & \lambda_1, & \lambda_2, & \lambda_3 \\ \lambda_1, & \lambda_2, & \lambda_3, & \lambda_1, & \lambda_2, & \lambda_3 \\ \mu_1, & \mu_2, & \mu_3, & \mu_1, & \mu_2, & \mu_3 \end{matrix}$$

$$= \left\{ \begin{matrix}\lambda, \mu, \nu \\ 1, 2, 3\end{matrix} \right\}^2 + \begin{matrix}\mu, \mu, \nu \\ 1, 2, 3\end{matrix} \; \begin{matrix}\lambda, \lambda, \nu, \\ 1, 2, 3\end{matrix}$$

or, resuming the notation adopted above,

$$= \left\| \begin{matrix}\lambda, \mu, \mu, \nu \\ 1, \;.\; 2, 3\end{matrix} \right\| \cdot \left\| \begin{matrix}\lambda, \mu, \lambda, \nu \\ 1, \;.\; 2, 3\end{matrix} \right\|$$

The first term of this expression, being the square of an odd-degreed determinant, vanishes, as was proved above; and the second

$$= 4\,(\nu_1 : \mu_1 + \nu_2 : \mu_2 + \nu_3 : \mu_3)\,(\nu_1 : \lambda_1 + \nu_2 : \lambda_2 + \nu_3 : \lambda_3)$$

$$= 4 \; \begin{matrix}\lambda^{-1}, \mu^{-1}, \nu^{-1}. \\ 1, \quad 2, \quad 3\end{matrix}$$

Hence the determinant of all the first minors of a determinant of the third degree is equal to four times the determinant of the reciprocals of the original constituents.

Proceeding to the fourth degree, and for brevity representing the coefficients of $\lambda_1, \lambda_2, \ldots$ in the development of $\begin{matrix}\lambda, \mu, \nu, \rho, \\ 1, 2, 3, 4\end{matrix}$ by $[\lambda_1], [\lambda_2], \ldots$

we shall find

$$\begin{matrix}[\nu], [\rho] \\ 3, \;\; 4\end{matrix} = \begin{matrix} \lambda_1 & \lambda_2 & \lambda_4 & . & . & \lambda_3 \\ \mu_1 & \mu_2 & \mu_4 & . & . & \mu_3 \\ \rho_1 & \rho_2 & \rho_4 & . & . & \rho_3 \\ . & . & \lambda_4 & \lambda_1 & \lambda_2 & \lambda_3 \\ . & . & \mu_4 & \mu_1 & \mu_2 & \mu_3 \\ . & . & \nu_4 & \nu_1 & \nu_2 & \nu_3 \end{matrix} = - \begin{matrix} \lambda_1 & \lambda_2 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \mu_1 & \mu_2 & \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \nu_1 & \nu_2 & \nu_1 & \nu_2 & \nu_3 & \nu_4 \\ . & . & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ . & . & \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ . & . & \rho_1 & \rho_2 & \rho_3 & \rho_4 \end{matrix}$$

$$= - \left\| \begin{matrix}\lambda, \mu, \nu \\ 1, 2 \;.\end{matrix} \right\| \cdot \left\| \begin{matrix}\lambda, \mu, \nu, \lambda, \mu, \rho \\ 1, \;.\; .\; 2, 3, 4\end{matrix} \right\|$$

And similarly, by permuting letters and suffixes, we should obtain expressions for all other compound determinants of the same order. Passing to the next stage, we should find, after certain permutations,

$$\begin{matrix}[\mu], [\nu], [\rho] \\ 2, \;\; 3, \;\; 4\end{matrix} = \begin{matrix} \lambda_1 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \mu_1 & \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \nu_1 & \nu_1 & \nu_2 & \nu_3 & \nu_4 & \nu_1 & \nu_2 & \nu_3 & \nu_4 \\ . & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & . & . & . & . \\ . & \mu_1 & \mu_2 & \mu_3 & \mu_4 & . & . & . & . \\ . & \rho_1 & \rho_2 & \rho_3 & \rho_1 & . & . & . & . \\ . & . & . & . & . & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ . & . & . & . & . & \nu_1 & \nu_2 & \nu_3 & \nu_4 \\ . & . & . & . & . & \rho_1 & \rho_2 & \rho_3 & \rho_4 \end{matrix}$$

$$= \left\| \begin{matrix}\lambda, \mu, \nu \\ 1, \;.\; .\end{matrix} \right\| \cdot \left\| \begin{matrix}\lambda, \mu, \nu, \nu, \lambda, \rho \\ 1, \;.\; .\; 2, 3, 4\end{matrix} \right\| \cdot \left\| \begin{matrix}\lambda, \mu, \nu, \lambda, \mu, \rho \\ 1, \;.\; .\; 2, 3, 4\end{matrix} \right\|$$

which is to be developed in the same way as before.

From this expression we may, as in the case of the third degree, deduce the expression for the determinant of all the first minors of the given determinant, viz.,

$$= -\left\|\begin{matrix}\lambda, & \mu, & \nu, & \mu, & \nu, & \rho \\ 1, & . & . & 2, & 3, & 4\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, & \mu, & \nu, & \nu, & \lambda, & \rho \\ 1, & . & . & 2, & 3, & 4\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, & \mu, & \nu, & \lambda, & \mu, & \rho \\ 1, & . & . & 2, & 3, & 4\end{matrix}\right\|$$

The sign of the expression will in all cases be determined by the number of permutations of columns necessary to bring the same product on the principal diagonal in this as in the proposed expression, say $[\lambda_1]\,[\mu_2]\,[\nu_3]\,[\rho_4]$.

It is, perhaps, unnecessary to pursue these formulæ, which are quite general, further, except perhaps as a final example to write down the system of expressions for the compound determinants formed from two, three, four, and all of the first minors of a determinant of the fifth degree, viz.,

$$-\left\|\begin{matrix}\lambda, \mu, \nu, \rho \\ 1, 2, 3, .\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, \mu, \nu, \rho, & \lambda, \mu, \nu, \sigma \\ 1, \;.\;\;.\;\;. & 2, 3, 4, 5\end{matrix}\right\|$$

$$\left\|\begin{matrix}\lambda, \mu, \nu, \rho \\ 1, 2, .\;\;.\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, \mu, \nu, \rho, & \lambda, \mu, \nu, \rho \\ 1, \;.\;\;.\;\;. & 2, 3, 4, 5\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, \mu, \nu, \rho, & \lambda, \mu, \rho, \sigma \\ 1, \;.\;\;.\;\;. & 2, 3, 4, 5\end{matrix}\right\|$$

$$-\left\|\begin{matrix}\lambda, \mu, \nu, \rho \\ 1, \;.\;\;.\;\;.\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, \mu, \nu, \rho, & \lambda, \mu, \nu, \sigma \\ 1, \;.\;\;.\;\;. & 2, 3, 4, 5\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, \mu, \nu, \rho, & \lambda, \mu, \rho, \sigma \\ 1, \;.\;\;.\;\;. & 2, 3, 4, 5\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, \mu, \nu, \rho, & \lambda, \nu, \rho, \sigma \\ 1, \;.\;\;.\;\;. & 2, 3, 4, 5\end{matrix}\right\|,$$

$$\left\|\begin{matrix}\lambda, \mu, \nu, \rho, & \mu, \nu, \rho, \sigma \\ 1, \;.\;\;.\;\;. & 2, 3, 4, 5\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, \mu, \nu, \rho, & \lambda, \nu, \rho, \sigma \\ 1, \;.\;\;.\;\;. & 2, 3, 4, 5\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, \mu, \nu, \rho, & \lambda, \mu, \rho, \sigma \\ 1, \;.\;\;.\;\;. & 2, 3, 4, 5\end{matrix}\right\| \cdot \left\|\begin{matrix}\lambda, \mu, \nu, \rho, & \lambda, \mu, \nu, \sigma \\ 1, \;.\;\;.\;\;. & 2, 3, 4, 5\end{matrix}\right\|$$

and it is to be borne in mind that, in developing these expressions, the squares and higher powers of $\begin{matrix}\lambda, \mu, \nu, \rho, \sigma, \\ 1, 2, 3, 4, 5\end{matrix}$ being of an odd degree, will vanish.

In the case of even-degreed determinants the evaluation of the determinant of all the first minors admits of a special solution. For example, that of the fourth degree

$$\begin{matrix}\mu, \nu, \rho & \mu, \nu, \rho & \mu, \nu, \rho & \mu, \nu, \rho \\ 2, 3, 4 & 3, 4, 1 & 4, 1, 2 & 1\;\;2\;\;3 \\ \nu, \rho, \lambda & & & \\ 2, 3, 4 & & & \\ \vdots & & & \end{matrix}$$

$$= \left(\begin{matrix}\alpha_1\, \mu, \nu, \rho + \ldots \\ 2, 3, 4\end{matrix}\right)\left(\begin{matrix}\alpha_1\, \nu, \rho, \lambda + \ldots \\ 2, 3, 4\end{matrix}\right)\left(\begin{matrix}\alpha_1\, \rho, \lambda, \mu + \ldots \\ 2, 3, 4\end{matrix}\right)\left(\begin{matrix}\alpha_1\, \lambda, \mu, \nu + \ldots \\ 2, 3, 4\end{matrix}\right)$$

$$= \left(\alpha_2^{-1}\alpha_3^{-1}\alpha_4^{-1}\, \begin{matrix}\mu, \nu, \rho \\ 2, 3, 4\end{matrix} + \ldots\right) \times \text{similar terms in } \nu\rho\lambda \times \ldots$$

But since $a_1^{-1} = a_2a_3a_4, \ldots$ it follows that $a_1^{-1}, \ldots$ are alternate numbers, and consequently the first factor written above

$$= (a_1^{-1}\mu_1+a_2^{-1}\mu_2+\ldots)\,(a_1^{-1}\nu_1+a_2^{-1}\nu_2+\ldots)\,(a_1^{-1}\rho_1+a_2^{-1}\rho_2+\ldots),$$

and the whole expression

$$\begin{aligned} &= (a_1^{-1}\lambda_1+\ldots)^3\,(a_1^{-1}\mu_1+\ldots)^3\,(a_1^{-1}\nu_1+\ldots)^3\,(a_1^{-1}\rho_1+\ldots)^3 \\ &= 6^4\,(a_1\lambda_1^{-1}+\ldots)\,(a_1\mu_1^{-1}+\ldots)\,(a_1\nu_1^{-1}+\ldots)\,(a_1\rho_1^{-1}+\ldots) \\ &= 6^4\,\underset{1,\ 2,\ 3,\ 4}{\lambda^{-1}\,\mu^{-1}\,\nu^{-1}\,\rho^{-1}}, \end{aligned}$$

and this method is obviously general.

Determinants sometimes occur as factors of a product having one or more lines common to more than one. When that is the case the product presents some peculiarities. This was pointed out by Prof. Clifford, from whose notes the following is a quotation:

$$\text{“}\,(\lambda_1\mu_2-\lambda_2\mu_1)\,(\mu_1\nu_2-\mu_2\nu_1) \left.\begin{aligned} &= \lambda_1\mu_2\mu_1\nu_2+\lambda_2\mu_1\mu_2\nu_1 \\ &= \mu_1\mu_2\,(\lambda_2\nu_1-\lambda_1\nu_2 \\ &= -\,(\lambda_1\nu_2-\lambda_2\nu_1) \end{aligned}\right\} \ldots\ldots \text{(X.)}$$

Hence, generally,

$$\left.\underset{1,\,2}{\lambda,\mu}\ .\ \underset{1,\,2}{\mu,\nu}\ \ldots\ \underset{1,\,2}{\sigma,\tau}\ (n\text{-factors}) = (-)^{n-1}\,\underset{1,\,2}{\lambda,\tau}\right\} \ldots\ldots\ \text{(XI.)}$$

An analogous theorem holds for determinants of the nth order [when each determinant contains only two sets of numbers, but with repetitions]; viz., if we denote a determinant with r rows of λ, and s rows of μ, by (λ^r, μ^s), where $r+s = n$, then

$$\begin{aligned} |\,\lambda^r; \mu^s\,|\ \ldots\ |\,\mu^r, \nu^s\,|\ \ldots\ |\,\sigma^r; \tau^s\,| &= (-)^{s(n+r)}\,(\lambda^r, \tau^s) \\ &= (-)^s\,(\lambda^r, \tau^s)\ \ldots\ldots\ \text{(XII.)}, \end{aligned}$$

and since $n+r = 2r+s = s$ (mod. 2); and so $s\,(n+r) \equiv s^2 = s$ (mod. 2).”

The following is a theorem supplementary to that first quoted:

$$\left.\underset{1,\,2}{\mu,\nu}\ .\ \underset{1,\,2}{\nu,\lambda}\ .\ \underset{1,\,2}{\lambda,\mu} = -\,2\right\} \ldots\ldots\ldots\ldots\ldots\ \text{(XIII.)}$$

This admits of extension to higher degrees, thus

$$\underset{1,\,2,\,3}{\lambda,\mu,\nu}\ .\ \underset{1,\,2,\,3}{\mu,\nu,\rho}\ .\ \underset{1,\,2,\,3}{\nu,\rho,\lambda}\ .\ \underset{1,\,2,\,3}{\rho,\lambda,\mu} = \underset{1,\,2,\,3}{\lambda,\mu,\nu}\ .\ \rho_1\rho_2\rho_3\ .\ \underset{1}{[\lambda]},\ \underset{2}{[\mu]},\ \underset{3}{[\nu]}$$

$$\begin{aligned} = \lambda_1[\lambda_1]+\ldots,\ \ &\lambda_1[\mu_1]+\ldots,\ \ \lambda_1[\nu_1]+\ldots = \lambda\,[\lambda]\ \lambda\,[\mu]\ \lambda\,[\nu] \\ \mu_1[\lambda_1]+\ldots,\ \ &\mu_1[\mu_1]+\ldots,\ \ \mu_1[\nu_1]+\ldots \qquad \mu\,[\lambda] \\ \nu_1[\lambda_1]+\ldots,\ \ &\nu_1[\mu_1]+\ldots,\ \ \nu_1[\nu_1]+\ldots \qquad \nu\,[\lambda]. \end{aligned}$$

But $\lambda\,[\lambda] = \mu\,[\mu] = \nu\,[\nu] = \underset{1,\,2,\,3}{\lambda,\mu,\nu} = \nabla$, and consequently the products of these expressions vanish. Hence the whole expression,

$$\begin{aligned} &= \nabla\,(-\mu\,[\nu]\,.\,\nu\,[\mu]-\lambda\,[\mu]\,.\,\mu\,[\lambda]-\lambda\,[\nu]\,.\,\nu\,[\lambda]) \\ &\quad +\lambda\,[\mu]\,\mu\,[\nu]\,\nu\,[\lambda] + \lambda\,[\nu]\,\mu\,[\lambda]\,\nu\,[\mu]. \end{aligned}$$

But since ∇ is a factor of the whole expression, it follows that, if we put $\lambda\,[\lambda] = 0$, $\mu\,[\mu] = 0$, $\nu\,[\nu] = 0$, the whole must vanish; in other words, the last two terms must vanish, and the whole

$$= -\nabla\,(\mu\,[\nu]\,.\,\nu\,[\mu] + \lambda\,[\mu]\,.\,\mu\,[\lambda] + \lambda\,[\nu]\,.\,\nu\,[\lambda]).$$

But
$$-\mu\,[\nu] = 2\,(\lambda_1 : \mu_1 + \lambda_2 : \mu_2 + \lambda_3 : \mu_3)$$
$$-\nu\,[\mu] = 2\,(\lambda_1 : \nu_1 + \lambda_2 : \nu_2 + \lambda_3 : \nu_3),$$

and consequently

$$\mu\,[\nu]\,.\,\nu\,[\mu] = 4\begin{matrix}\lambda^{-1}, & \mu^{-1}, & \nu^{-1}. \\ 1, & 2, & 3\end{matrix}$$

Hence the whole expression,

$$\begin{aligned} &= -4\begin{matrix}\lambda, \mu, \nu \\ 1, 2, 3\end{matrix}\ \begin{matrix}\lambda^{-1}, & \mu^{-1}, & \nu^{-1} \\ 1, & 2, & 3\end{matrix} \\ &= -4\,(\alpha_1\lambda_1 + \ldots)\,(\alpha_1\mu_1 + \ldots)\,(\alpha_1\nu_1 + \ldots)\,(\beta_1\lambda_2\lambda_3 + \ldots) \\ &\qquad\qquad (\beta\,\mu_2\mu_3 + \ldots)\,(\beta\,\nu_2\nu_3 + \ldots) \\ &= -4\,(\alpha_1\lambda_1 + \ldots)\,(\beta_1\lambda_2\lambda_3 + \ldots)\,\ldots \\ &= -4\,(\alpha_1\beta_1 + \ldots)\,\lambda_1\lambda_2\lambda_3\,\ldots \\ &= -4\,(\alpha_1\beta_1 + \ldots)^3 \\ &= -4\,.\,3\,.\,2\,.\,1. \end{aligned}$$

It may be observed that the formula

$$\begin{matrix}\lambda, \mu, \ldots \sigma \\ 1, 2, \ldots n\end{matrix}\ \begin{matrix}\mu, \nu, \ldots \tau \\ 1, 2, \ldots n\end{matrix} \ldots = \begin{matrix}\lambda, \mu, \ldots \sigma \\ 1, 2, \ldots \nu\end{matrix}\ \begin{matrix}[\lambda] & [\mu] & \ldots & [\sigma] \\ 1, & 2, & \ldots & n\end{matrix}$$

is perfectly general; and any conclusions at which we can arrive about the determinant formed from the first minors of a given determinant will assist the evaluation. If we carry the process a step further, viz.,

$$\begin{matrix}\lambda\,[\lambda], & \lambda\,[\mu] \ldots \\ \mu\,[\lambda], & \mu\,[\mu] \ldots \\ \ldots & \ldots\ \ldots\end{matrix}$$

then, in the case of odd-degreed functions, we may put

$$\begin{matrix}0 & \lambda\,[\mu] & \ldots \\ \mu\,[\lambda] & 0 & \ldots \\ \ldots & \ldots & \ldots\end{matrix} = 0;$$

and afterwards we need retain only the terms having $\lambda\,[\lambda]$, $\mu\,[\mu]$, ... respectively as factors.

Sometimes determinants occur as factors of a product, such that their n columns are severally cyclical arrangements of $n-1$ out of n columns. For instance, as noticed by Prof. Clifford,

$$\begin{matrix}\lambda, \mu \\ 1, 2\end{matrix}\ \begin{matrix}\lambda, \mu \\ 2, 3\end{matrix} = -\lambda_2\mu_2\,(\lambda_1\mu_3 + \lambda_3\mu_1),$$

and, as a consequence,

$$\begin{matrix}\lambda, \mu \\ 2, 3\end{matrix}\ \begin{matrix}\lambda, \mu \\ 3, 1\end{matrix}\ \begin{matrix}\lambda, \mu \\ 1, 2\end{matrix} = 0.$$

For the third degree we should have, writing for brevity, 12 instead of $\begin{matrix}\mu, \nu \\ 1, 2\end{matrix}$

$$\begin{matrix}\lambda, \mu, \nu & \lambda, \mu, \nu & \lambda, \mu, \nu & \lambda, \mu, \nu \\ 2, 3, 4 & 3, 4, 1 & 4, 1, 2 & 1, 2, 3\end{matrix}$$

$$\begin{array}{llll} =(& . \quad \lambda_2 34+\lambda_3 42+\lambda_4 23) & = \quad . & 34, -24, 23 = -1.2.3.4 \\ & (\lambda_1 34 \quad . \quad +\lambda_3 41+\lambda_4 13) & \quad -34, \quad . & 14, 31 \\ & (\lambda_1 24+\lambda_2 41 \quad . \quad +\lambda_4 12) & \quad 24, -14, & . \quad 12 \\ & (\lambda_1 23+\lambda_2 31+\lambda_3 12 \quad . \quad) & \quad -23, -31, & -12, \quad . \end{array}$$

$$= (23 \,.\, 14 + 31 \,.\, 24 + 12 \,.\, 34)^2.$$

From this it is easily seen that such a product can always be reduced to a skew symmetrical determinant; and consequently that, when the number (n) of columns is odd, the product will vanish. When the number is even, the product will

$$= -1 \,.\, 2 \,\ldots\, n. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(XV.)}$$

On the Transformation of Gauss' Hypergeometric Series into a Continued Fraction. By Thomas Muir, M.A., F.R.S.E.

[*Read* 10*th February*, 1876.]

Gauss, in his *Disquisitio circa seriem infinitam**, viz., the series

$$1+\frac{\alpha\beta}{\gamma}x+\frac{\alpha(\alpha+1)}{1\,.\,2}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{1\,.\,2\,.\,3}\cdot\frac{\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}x^3+\ldots,$$

or $F(\alpha, \beta, \gamma, x)$, established a simple proposition regarding it, and from this was able to express

$$\frac{F(\alpha, \beta+1, \gamma+1, x)}{F(\alpha, \beta, \gamma, x)} \text{ in the form } \cfrac{1}{1-\cfrac{a_1x}{1-\cfrac{a_2x}{1-\ldots}}}$$

where $a_1, a_2, \ldots$ are functions of α, β, γ; and the continued fraction so obtained for any particular series was found to be quite different in form from that got by using the previously established general method of Euler. The object of the present short paper is to place on as firm

* *Vide* Abhandl. der Götting. Gesellsch. d. Wissensch. II., 1812, and Werke, t. iii., p. 125.