# DIFFERENTIAL METHOD OF COMPUTING APPARENT PLACES OF STARS FOR LATITUDE WORK.

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When a number of stations have been occupied during a season for the determination of latitude, the necessary reductions of the stars from mean to apparent positions requires considerable time. With a view of accomplishing the task sooner, as well as making the work much less laborious, and at the same time having an accuracy fully equal to the requirements of the case, the following investigation was made. Although the superiority of this method is most marked when the observations only extend over three or four days, and when several stations with long star lists are to be reduced at one time, yet in any case it is considerably shorter than the usual logarithmic method. Little is gained by observing a star more than three times; and with the improved mean star places now available and allowing a probable error of observation of $0''\cdot50$ for an experienced observer, with good weather, three evenings work will reduce the uncertainty of the latitude to about ten feet. So that this method may be employed nearly always with great advantage.

The usual computation of the apparent places of stars for the dates of observation may be abridged in two ways: first, in using Crelle's tables instead of making the ordinary four-place logarithmic computation; and, second, after having one date getting a neighboring date by the application of differential quantities derived from the usual formulæ.

If we consider the tabular differences of the quantities that vary with the date as the differential coëfficients of the quantities with respect to the time at the date already computed, we have the following formulæ as representing the change in declination between the two dates:

$$-\,g\;d\,G\;sin\;(G+\alpha)+dg\;cos\;(G+\alpha)$$

$$[-\,h\;d\,H\;sin\;(H+\alpha)+dh\;cos\;(H+\alpha)]\;sin\;\delta$$

$$di\;cos\;\delta$$

The following relations exist between the independent star numbers:

$$G = tan^{-1}\,\frac{B}{K\,A} \qquad H = tan^{-1}\,\frac{C}{D}$$

$$g = \frac{B}{sin\;G} \qquad h = \frac{C}{sin\;H} \qquad i = C\,tan\,\omega$$

Where the letters have the signification given in the American ephemeris, $K$ is the precession constant $20''\cdot0533$.

The greatest departure from a uniform change for a five-day period in $B$ and $A$, is due to terms depending on the moon's longitude. The terms depending on the longitude of the sun, of the moon's ascending node, and on the longitude of the sun's and moon's perigees, being either quite regular for a five-day period, or else being extremely small. In 1887, $G$ does not change as much as $3°$; $g$ changes less than $0''\cdot5$.

The tangent of $H$ varies inversely as the tangent of the sun's mean longitude. Hence $H$ varies nearly uniformly throughout the year, changing about $1°$ daily. $h$ depends on the same quantity and has its maximum values at the solstices and its minimum ones at the equinoxes. For a five-day period it departs little from a uniform change. $i$ varies also with the sun's longitude and has its maximum with the minimum $h$ and *vice versa*. The greatest daily change in $i$ is not much more than $0''\cdot10$, while that of $h$ is very much less.

The value of $G$ is in general principally affected by changes in terms depending on the sun's and moon's longitude, and on the longitude of the moon's ascending node. The latter has a daily motion of 3′. The first two have a daily motion of about 1° and 13°, respectively. *tan G* varies directly as *B* and inversely as *A*, and since the former depends on the cosines and the latter on the sines of the above functions, they do not both change rapidly at the same time. At 90° *cos* ☊ changes 0·005 in five days and the change in *cos* 2 ☊ may be neglected. When ☉ = 45°, *cos* 2 ☉ has a change of 0·087. For an equal period and position *cos* 2 ☽ changes about one unit. The terms in which these quantities enter will therefore vary by 0″·05, 0″·05 and 0″·09, respectively. Hence the greatest change in *B* comes from the change in the moon's longitude. In case of all these changes having their maximum at the same time, and tending in the same direction, the value of *B* would only be changed by about $\frac{1}{50}$th the part of itself, and since

$$d\ tan^{-1}\ y = \frac{d\,y}{1 + y^2}$$

the change in *G* dependent on *B* will not be more than about 1°.

The longitude of the moon's ascending node does not pass through 90° until 1890, but its change is slow compared with that of the others, and in its relation to *B* we need not for the present consider its effect on *G*.

The above quantities enter *A* as a sine function with coëfficients about $\frac{1}{20}$th of those for *B*, but the precession factor appearing in the denominator of *tan G* makes the changes in numerator and denominator about equal for maximum values of the function. But *G* being determined by its tangent, the magnitude of its changes depends also on the absolute values of *B* and *A*—for when *B* is small a given change in *A* has very much more influence on the angle. In general, we may expect changes in *G* of less than a degree per day.

When we have very small values for *B*, as in 1890, and also very small values for *A*, as in May, a combination of these may give a change in *G* for five days, amounting to

30° or more; but, as will be shown later, this does not render the method inapplicable.

When $B$ has its largest value, $G$ does not change more than, say, 5°, which reduces the product of $dg$ by $d\ G$ to a quantity less than 0″·10, and when $d\ G$ is very large $dg$ is small enough to reduce the product considerably under 0″·10, so that, in general, we may estimate the neglected term to be less than 0″·10. The product of any two of these differences that actually occur together is usually only a few hundredths of a second, so that the method will satisfy all the requirements of latitude work.

These considerations show that the stars' position may be derived, with all necessary accuracy, by the application of differential quantities, when the difference between the two dates is not more than five days. The following two forms show the reduction by both methods. It will be noticed that the method by differences involves only about half the number of figures used in the logarithmic method, besides requiring very little mental labor.

STAR 289.—METHOD BY LOGARITHMS.

| | | | | |
|---|---|---|---|---|
| $\alpha_0$<br>$\delta_0$ = | *h. m. s*<br>3 11 40 | $\alpha$<br>° ′<br>47 55<br>33 48 | *sin* $\delta$<br>9·7453 | *cos* $\delta$<br>9·9196 |
| | $\delta' - \delta$ | $g \cos (G + a)$ | $h \cos (H + a)$ | $i \cos \delta$ |
| | *January 20th.*<br>″<br>— 0·45 | ° ′<br>151 5<br>9·9422<br>0·8751<br>— 7·50 | ° ′<br>19 9<br>9·9753<br>1·0219<br>+ 10·52 | 0·5408<br>— 3·4 |
| | *January 25th.*<br>″<br>— 0·34 | 148 59<br>9·9330<br>0·8499<br>— 7·08 | 14 15<br>9·9864<br>1·0301<br>+ 10·72 | 0·5995<br>— 3·98 |
| | <br>$G$<br>$H$<br>*log g*<br>*log h*<br>*log i* | *January 20th.*<br>° ′<br>103 10<br>331 14<br>0·9329<br>1·3013<br>$0·6212_n$ | *January 25th.*<br>° ′<br>101 4<br>326 20<br>0·9169<br>1·2984<br>$0·6799_n$ | |

STAR 289.—METHOD BY DIFFERENCES.

| | | | | |
|---|---|---|---|---|
| | *h. m. s.* | $\alpha$ ° ′ | | |
| $\alpha_0$ = | 3 11 40 | 47 55 | | |
| $\delta_0$ = | | 33 48 | | |
| | | ($G + \alpha$) | ($H + \alpha$) | |
| | | ° ′ | ° ′ | |
| | *January 20th.* | 151 5 | 19 9 | *January 25th.* |
| | *sin* ($G + \alpha$) | + ·483 | | + ·15 |
| | *cos* ($G + \alpha$) | — ·875 | — 7·50 | + ·27 |
| | *sin* ($H + \alpha$) | + ·328 | | + ·56 |
| | *cos* ($H + \alpha$) | + ·945 | 10·51 | — ·12 |
| | *sin* $\delta$ | + ·556 | | + ·24 |
| | *cos* $\delta$ | ·831 | — 3·47 | — ·51 |
| | | | — 0·46 | + ·15 |
| | | *January 20th.* ° ′ | | |
| | $G$ = | 103 10 | $d\,G$ = | — ·0367 |
| | $H$ = | 331 14 | $d\,H$ = | — ·0855 |
| | $g$ = | + 8·57 | $d\,g$ = | — ·31 |
| | $h$ = | 20·01 | $d\,h$ = | — ·13 |
| | $i$ = | — 4·18 | $d\,i$ = | — ·61 |
| | | | $-g\,d\,G$ = | + ·314 |
| | | | $-h\,d\,H$ = | + 1·71 |

EXPLANATION OF COMPUTATION.

In both methods the quantities below the double line are the same for all stars, varying only with the date, and are therefore written but once for each station. The first computation is the usual logarithmic one, and needs no explanation. The second is by Crelle's tables and differences. In the first column are the natural trigonometric functions. In the second are the quantities $g\ cos\ (G + \alpha)$, $h\ cos\ (H + \alpha)\ sin\ \delta$ and $i\ cos\ \delta$, the sum of which is the reduction to apparent place for January 20th. The proper motion of the star is not considered in comparing the two methods. The third column contains the products of the constant multipliers by the corresponding sines and cosines to obtain the following quantities of the differential equations:

$$-g\ d\ G\ sin\ (G + \alpha) + dg\ cos\ (G + \alpha)$$

$$[-h\ d\ H\ sin\ (H + \alpha) + dh\ cos\ (H + \alpha)]\ sin\ \delta$$

$$di\ cos\ \delta$$

It should be stated that $d\ G$ and $d\ H$ are first reduced to linear quantities. The sum of this last column omitting the two middle values, gives the quantity to be applied to the reduction for January 20th to obtain that for January 25th, and will in most cases be found to be correct within one or two hundredths of a second.

The method by differences is considered to be a saving of about one-half the usual time, besides being very much easier, as many as thirty pairs being computed for two dates, in about seven hours, by a person familiar with the method. After the computation of the first date, the corrections to be applied to these to get those for the second date were found in two hours. But in order to work advantageously each step is taken up systematically and carried through the entire number of pairs, and often two steps may be carried along simultaneously where the multipliers are single or when the tables may be kept open two places at once. Care should be exercised to avoid using more places than are necessary. For example, in the direct computation for the first date, three figures are sufficient, except where $h$ enters. It is not considered essential to secure exactly the fourth place here, but it may be done with Crelle's tables mentally, and with very little labor, by taking the nearest unit in the third place and applying to the product the algebraic sum of the unit's place by the thousandths, one or two places at most only being considered. In forming the products for the differences two places generally need only be retained.

The difference of 0''·03 between that calculated rigorously for January 25th and that derived by the formulæ is due to the fact that the differences have been treated as differentials and not as finite differences. The neglected product, $dh$, $d\,H$, $\sin\,(H + a)$, does not amount to more than 0'·003 and need not be regarded when $\sin\,\delta$ is as much as 0·90, for, as a rule, stars are not observed above 65° declination.

If we had treated the difference in the cosine of $(H + a)$ as a finite difference, using the formula,

$$-\,2\,\sin\,(\gamma + \tfrac{1}{2}\,\Delta\,\gamma)\,\sin\,\tfrac{1}{2}\,\Delta\,\gamma$$

instead of $-\,\sin\,\gamma\,d\,\gamma$, the agreement would, of course, have been perfect; the essential points in the method being that the differences are considered as differentials, and the term involving the product of the differences is neglected.

It might be supposed that if we have a difference of 0''·03 in the position of a star for a difference in $H$ of, say, 5°, that this discrepancy would amount to a quantity entirely inadmissible in the case of $G$, in May, 1890, where the difference is upwards of 30°; but, since, when these excessive changes in $G$ occur, $B$ is necessarily quite small, because the longitude of the moon's ascending node is near 90°, the discrepancy between the values of $g\ cos\ (G + a)$, calculated by the differential formulæ, and those by actual multiplication, does not much exceed that in the present case; in fact, they only differ by 0''·05. Indeed, the large discrepancy in the present case is due to the fact that the error committed in neglecting the formula for finite differences must be multiplied by $h$, which increases it twenty-fold. $g$ in the extreme case of 1890 is 0''·8, hence only $\frac{1}{25}$th of $h$ for this case. But the discrepancy for the values of May, 1890, comes from another source, viz: From the product of the two differentials $dg$ and $d\ G$, and even then will only occur for a few pairs where $(G + a)$ is near 90°, and where the sine is large. It will be noticed that, assuming a value for $(G + a)$, which gives the most rapid change in the cosine, also gives a large value for the sine, and hence increases the value of the term $dg\ sin\ (G + a)\ d\ G$, there is a combination of circumstances tending to increase the discrepancy to 0''·08. This must be regarded, therefore, as a very exceptional case. When we consider that the probable errors of the declinations of the individual stars are several times as large, this may be neglected.

In general, the errors introduced by this method are quite insignificant, even admitting the declinations to be absolutely true, for errors of observation will much exceed these. Besides, for the extreme case of 1890, we have assumed a value for $(G + a)$, which would give the greatest possible change in the cosine for the change in $G$ under consideration; that is, a value extending from about 85° to 105°. Moreover, since this term depends on the star's right ascension, for any station this extreme case would only apply to a few pairs which involved values of $(G + a)$ passing through the points 90° or 270°; no night's work ever lasting long enough to pass through or even near them both.

When the observations do not extend beyond five days, the last date is derived from the first by differences. For work extending over a period from five to fifteen days, the middle date is actually computed, and the first and last obtained by differences. Where on account of bad weather observations are very much scattered, it is better to make separate computations for each date. Under ordinary circumstances, three successive nights are all that are required, which involves differences in the star numbers for only two days. In this case, the result by differences will be found to be identical with that of a rigorous calculation. For where $d\ G$ and $d\ H$ are about 2°, and $d\ g$ and $d\ h$ one or two-tenths, their product does not affect the hundredths place; and the change in the cosine of an arc, whether computed as a differential or a finite difference, is practically the same for differences of arc of 2°, the discrepancy never amounting to a unit in the hundredths place.

Assuming the probable error of observation to be 0″·50, which is about the usual experience, and the probable error of one declination to be 0″·30, we find the following relations between the number of nights, number of pairs, and the probable error of the mean result.

| No. of Nights. | Number of Pairs. | | | | | |
|---|---|---|---|---|---|---|
| | 5 | 10 | 15 | 20 | 25 | 30 |
| | *Resulting Probable Error of Mean Result.* | | | | | |
| | ″ | ″ | ″ | ″ | ″ | ″ |
| 3 | 0·17 | 0·12 | 0·09 | 0·08 | 0·07 | 0·07 |
| 5 | ·15 | ·10 | ·08 | ·07 | ·06 | ·06 |
| 7 | ·13 | ·09 | ·07 | ·06 | ·06 | ·05 |