# THE CHARACTERISTIC NUMBERS OF A REAL ALGEBRAIC PLANE CURVE.

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In the study of algebraic plane curves fundamental importance is attached to certain numbers, called PLÜCKER's numbers or characteristics, which are invariant when the curve is subjected to any projective transformation of the plane. In the present article we shall, for simplicity of statement, limit ourselves to those curves which have at most, point and tangent singularities of the second order. This limitation is one of form only, our main theorem deals with identities holding for all curves whatsoever. We shall denote the PLÜCKER characteristics by the following numbers

$n =$ the order of the curve
$m =$ the class of the curve
$p =$ the deficiency
$\delta =$ the number of double points
$k =$ the number of cusps
$\tau =$ the number of double tangents
$\iota =$ the number of inflections

These numbers are connected by certain equations, called PLÜCKER's equations, in virtue of which, if the values of three be known, the others may always be expressed rationally in terms of them. There is no account taken of the distinction between real and imaginary in defining these numbers.

Suppose, however, that we are especially interested in a real curve. In trying to draw such a curve, what is of first importance is not the total number of double points or inflections, real or imaginary, but the number of real singularities of one sort or another. There is a certain arbitrariness in the choice of those real characteristics which are worthy of attention. We give here the list which we shall follow in the present article, and which certainly seems to include all that could be desired

$n'$ = the apparent order of the curve, i. e. the maximum number of real intersections with any one real line, multiple intersections counting according to their multiplicity
$m'$ = the apparent class of the curve, definition corresponding to above
$\delta'_1$ = the number of real double points, each lying on two real branches
$\delta'_2$ = the number of conjugate or isolated double points, each lying on two conjugate imaginary branches
$k'$ = the number of real cusps
$\tau'_1$ = the number of real double tangents
$\tau'_2$ = the number of conjugate or isolated double tangents
$\iota'$ = the number of real inflections
$c'$ = the number of real circuits, open or closed.

Taking the two lists together we have sixteen characteristics. The first seven are connected by PLÜCKER's equations. Some of the first and some of the others are connected by a well known equation due to KLEIN, namely [1])

$$m + k' + 2\delta'_2 = n + \iota' + 2\tau'_2.$$

This equation has been put into a particularly elegant shape by SCHUH is a to little known article [2]). SCHUH considers a point as singular if every line in the plane through it have more than one intersection with the curve at that point, or if, even though it be not singular in this sense, it is yet the point of contact of a singular tangent. Corresponding definitions can be given for singular tangents. The order of a singular point is the minimum number of intersections which a straight line through that point will have with the curve there. A similar definition will hold for the order of a singular tangent. This premised, SCHUH's equation reads

$$m + \sum \nu' = n + \sum \mu'.$$

In words: *The class of a curve plus the sum of the orders of all the real singular points is equal to the order plus the sum of the orders of all real singular tangents.*

This relation is more elegant in form than KLEIN's and is readily extended to include imaginary curves, and higher singularities. Curiously enough in the case of a real curve it reduces to KLEIN's equation by cancellation.

We come now, naturally, to the highly interesting question « Are there any other equations of this sort connencting real or PLÜCKER characteristics, not derivable from those already given? » There are, certainly, new relations in the case of special types of curves. For instance, in the case of a non-singular curve we easily find

$$\iota = 3\iota' + 6\tau'_2.$$

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[1]) F. KLEIN, *Eine neue Relation zwischen den Singularitäten einer algebraischen Curve* [Mathematische Annalen, Bd. X (1876), pp. 199-209].

[2]) F. SCHUH, *Eene realiteitsvergelijking voor bestaanbare en onbestaanbare vlakke krommen met hoogere singulariteiten* [Koninklijke Akademie van Wetenschappen te Amsterdam. Verslag van de gewone Vergaderingen der wis-en natuurkundige Afdeeling, t. XII (1903-1904), pp. 845-854].

This amounts to saying that in the case of a real non-singular curve, not more than one third of the inflections can be real. Various relations have been established for curves of low order, notably the fourth. There are also certain relations valid in the case of unicursal curves [3]). With regard to the general question are there any other relations independent of these, valid for all curves, no pronouncement of a final sort has yet been forthcoming. JUEL seems to have believed that there were such, if we may judge by the following remark [4]). « *Ich möchte noch hinzufügen dass die Formel* (1), *selbstverständlich, nicht als die einzig mögliche Relation zwischen den reelen Singularitäten einer ebenen algebraischen Kurve anzusehen ist* ».

It is the object of the present paper to show that this opinion is not well founded by proving the following.

FUNDAMENTAL THEOREM. — *The only algebraic identities involving any combination of real or total singularities as here defined, which are valid for all real algebraic plane curves, are those which are deducible from the known equations of* PLÜCKER *and* KLEIN.

The proof is based upon the following.

ALGEBRAIC LEMMA. — *Given a polynomial in any number of variables, equal to zero. If it be possible to give to each variable in turn without altering the value of any of the others, a number of values greater than the degree of the polynomial with regard to that variable, then the polynomial is identically equal to zero.*

The lemma is certainly true in the case of a polynomial in one variable. Assume that it has been proved for one of $n - 1$ variables. Let us arrange according to the powers of the $n$th. variable. We have an equation in this variable with more roots than the degree allows. Hence the coefficients of each power of this variable vanish identically, for each is a polynomial in $n - 1$ variables vanishing for a large number of values of each variable independently.

Let us now suppose that we have a universally valid equation

$$\Phi(n, m, p, \delta, k, \tau, \iota, n', m', c', \delta'_1, \delta'_2, \tau'_1, \tau'_2, k', \iota') = 0.$$

Making use of the equations of PLÜCKER and KLEIN, we eliminate $m$, $p$, $\tau$, $\iota$, and $\tau'_2$

$$\mathfrak{F}(n, \delta, k, n', \delta'_2, k', \iota', \delta'_1, c', m', \tau'_1) = 0. \qquad (1)$$

We shall assume that the highest power to which any one variable appears is $N$. We proceed to construct a curve which can be altered in such a way that each of these arguments can be given more than $N$ values without altering the values of any of the other arguments.

[3]) Fr. MEYER, *Ueber Discriminanten und Resultanten der Gleichungen für Singularitäten von algebraische Raumcurven, mit Anwendung auf Realitätsverhältnisse* [Monatshefte für Mathematik und Physik, t. IV (1893), pp. 229-276, 331-363], p. 359.

[4]) C. JUEL, *Ueber einen neuen Beweis der* KLEIN*schen Relation zwischen den Singularitäten einer ebenen algebraischen Kurve* [Mathematische Annalen, Bd. LXI (1905), pp. 77-87], p. 86.

The equation of the curve in question shall be

$$f(x, y)\varphi(x, y) + \varepsilon(x - a)^3 1^3 \psi(x, y) = 0. \tag{2}$$

We proceed to describe it very carefully.

The curious expression $1^3$ indicates that the degree of the second part is less by 3 at least than $n$ the degree of the first. $\varepsilon$ is a real infinitessimal, and $\varphi$, though possessed of a real equation, has no real non-singular points. Hence the curve lies infinitely close to $f$ and follows the general shape of the latter. Moreover, by varying $\varepsilon$ we have a linear system of curves, and the general curve of a linear system has no singular point which is not a fixed singular point for all curves of the system. We shall assume that the compound curve $f\varphi$ has no singular point on $\psi$, the singular points of the general curve are thus the singular points of $f\varphi$ on the line

$$x - a = 0 \tag{3}$$

or on the line at infinity. Moreover, since (3) and the equation of the line at infinity appear to the third degree in (2) all curves of the system have the same tangents at each point of these two lines.

$\varphi$ and $\psi$ are taken as real in the sense that their equations are real, but they are not supposed to have any real non-singular points. It remains to describe $f$. This is supposed to consist in:

*a*) Finite quartic loops of three types to be described presently

*b*) Pairs of conjugate imaginary lines meeting in real points on (3) or on the line at infinity. We shall imagine that $n$ is so very large that we have these loops and lines in great profusion.

Types of quartic loop in $f$.

1° Elliptic loops or ovals. We get the equation of such a loop by multiplying together the equations of a real and a self-conjugate imaginary ellipse, the imaginary asymptotes of the two not being parallel. A goodly number of these ovals shall intersect in pairs on (3) and we shall have at least one nest of $N + 2$ of these ovals surrounding one another, all very small, and none meeting (3). There may be other elliptic loops scattered elsewhere in the plane.

2° Lima bean loops. We obtain one of these by an infinitesimal change in the coefficients in the equation of a cardioïd. Note that each of these has two real inflections, one real double tangent, but no singular point.

3° Moon shaped loops. Each of these has two real cusps, the horns; two real inflections, near the cusps on the limb, but only one double tangent, and no other real point inflection. We construct such a curve as follows. Starting with the equation

$$(xy + x + y)^2 - \varkappa y + \delta = 0$$

where $\delta$ is a real infinitessimal, we see that we have a quartic with a cusp at the end of each axis, but no other singular point besides. By turning the axes through an angle of 45° we easily find one conjugate tangent, and PLÜCKER's equations show that this is the only double tangent, and that the class is 6. Hence we find by KLEIN's equation that there are just two real inflections. A linear transformation of this curve will give us just what we require.

Let us next see what each of the arguments in (1) will depend upon in the case of a curve of type (2).

$n$. This dependes upon the total degree of $f\varphi$.

$\delta$ and $k$. These depend upon the total number of double points which $f$ and $\varphi$ have on (3) and on the line at infinity. They will not be altered by a change in $f$ which replaces a pair of real double points or cusps by a pair of conjugatei maginary ones.

$\delta'_2$. This depends upon the number of pairs of conjugate imaginary lines which go to make up $\varphi$ and which meet on (3) or on the line at infinity. It will not be affected by a change from one kind of pair to the other.

$n'$. This is the maximum number of real intersections with any one real line. The line (3) is supposed to meet as many loops in real points as does any other real line; in addition it is supposed to contain a number of conjugate points, so that the number $n'$ may be assumed to depend uniquely upon the number of points of our curve on (3).

$k'$. The only real cusps are pairs of horns on (3).

$\iota'$. The only real inflections are on the moon-shaped or the bean-shaped loops. An alteration in the curve which replaces one of these loops by another will not alter $\iota'$.

$\delta'_1$. This depends upon the number of pairs of loops intersecting on (3). It will not be altered by any alteration in the curve which leaves these intersections undisturbed.

$c'$. This depends upon the number of loops, as a conjugate point is not counted as a loop.

$m'$. This is the most troublesome of all the arguments. Its value will not be apparent at all from the form of the curve (2). The number of real tangents from any point will be made up of the lines through that point which touch the curve in real points, and those which touch it in pairs of conjugate imaginary points, i. e. conjugate tangents. The total number of these latter is determined by KLEIN's equation, not so their positions. We may imagine that $m$ so far overtops $n$ that $\tau'_2$ is well above $N+1$ and then require $f\varphi$ to touch the line at infinity in a large number of pairs of conjugate imaginary points, so that this line is a conjugate tangent of high order, and $m'$ will depend upon this multiplicity and upon the number of real tangents in any one direction.

$\tau'_1$. This will depend upon, first the number of lima-bean loops, and secondly upon the number of pairs of loops whereof the one is not everywhere concave to the other, the intersections of such loops, and the relations of their inflectional tangents.

At last we are able to put through our formal proof.

*A*) Independence of $\tau'_1$. We assume that $n$ is so very large that we have one nest of small ovals, separated from all other loops by such a considerable distance, that small alterations in their position will not affect the number of real tangents common to one of them and to a distant loop. Remove the innermost loop and place it near the nest outside. No loop shall meet (3) in real points after its removal. Going through our list of characteristics, we see that the only one to be altered is $\tau'_1$, wich has been increased by four times the number of loops left in the nest. We then take out a second, a third, etc. each time adding to $\tau'_1$ and leaving the other arguments unaltered. Thus $\tau'_1$ can take $N+1$ different values without altering the values of the other arguments. It appears then, that either (1) is independent of the argument $\tau'_1$ or else the coefficient of each power thereof vanishes for a curve of type (2). Under either hypothesis, if (1) subsist for the general curve, there must be in the case of a curve of type (2) one or more similar equations lacking the argument $\tau'_1$.

*B*) Independence of $m'$. We have pointed out that if we assume $m$ sufficiently greater than $n$, we may give to the line at infinity a high multiplicity as a conjugate tangent, and alter this multiplicity by $N+1$ different values, without altering the aspect of the curve in any visible way. It appears then, that $m'$ can be given $N+1$ different values without altering any of the other arguments in (1). We may reason on $m'$ exactly as we did on $\tau'_1$ and our conclusion is that if (1) subsist for every curve, then for a curve of type (2) there must be one or more such equations independent of the arguments $\tau'_1$ and $m'$. Our proof consists in showing, by means of the lemma, that these equations can only be $0 = 0$ so that the same must be true of (1).

*C*) Independence of $c'$. This argument is altered by replacing elliptic ovals which do not intersect (3) in real points, by self conjugate imaginary loops. The equation of such a loop is obtained by equating a definite quartic form to zero. It is true that this process alters $\tau'_1$ and $m'$, but we have just seen that they may be ignored, the other characteristics will be unaltered, and $c'$ may be given $N+1$ different values, hence it does not enter in the case of our present curves.

*D*) Independance of $\delta'_1$. This characteristic depends solely upon the number of pairs of loops meeting on (3). If we replace such a pair by one where each member meets (3) in two distinct real points, and make good the reduction in $\delta$ by giving to $f$ or $\varphi$ and extra pair of conjugate imaginary double points at infinity, then no characteristic has been altered but $\delta'_1$ which has been reduced by 2. Thus continuing the process, $\delta'_1$ can take $N+1$ values, and so does not enter. It is well to repeat that we assume $n$ so very large in comparison with $N$ that we may put $N+1$ pairs of extra double points on the line at infinity without overloading the latter with intersections.

*E*) Independence of $\iota'$. This is shown by replacing $N+1$ lima beans successively by elliptic ovals. Care must be taken that the bean and oval shall meet (3) in the same number of real points, so that the bean abolished must not be one that has four real intersections with (3). Alterations in $\tau'_1$ or $m'$ involved in this process have been shown to be immaterial, and $\iota'$ is the only other characteristic affected.

*F*) Independence of $k'$. The contribution of a moon loop to our curve is two cusps and two inflections. If we replace a moon by a lima bean having the same number of real intersections with (3) and restore $k$ to its pristine value by giving to $f$ or $\varphi$ an extra pair of conjugate imaginary cusps at infinity, then the only characteristic affected is $k'$ which has been reduced by 2. Remembering our remarks under *D*) about not overloading the line of infinity, we see that $k'$ can be given $N + 1$ values, so it too does not enter.

*G*) Independence of $\delta'_2$. This number depends upon the pairs of conjugate imaginary lines meeting on (3) or on the line at infinity. The former we shall leave unaltered as they affect $n'$, with regard to the latter, if we replace two pairs of conjugate imaginary lines meeting at infinity, by two imaginary ellipses with parallel asymptotes, but no finite intersection on (3), we have reduced $\delta'_2$ by 2, but left $\delta$ as it was. Thus $\delta'_2$ can take $N + 1$ independent values, and so does not enter into the equation.

*H*) Independence of $n'$. This characteristic depends upon the number of real intersections with (3) and can be altered by transferring conjugate points from the overworked line at infinity to (3). We easily see that $n'$ can thus be given $N + 1$ different values, without altering in the least any of the other characteristics in (1). Hence $n'$ does not enter.

*I*) Independence of $n$, $\delta$, $k$. The aspect of our curve is not in the least altered by giving to $\varphi$ extra definite quartic factors, or extra pairs of conjugate imaginary infinite double points or cusps, and these three operations are independent of one another. Hence each of the arguments $n$, $\delta$, $k$ can take $N + 1$ independent values, no one of them can enter.

Conclusion. We have shown that if any such equation as (1) hold in the case of the general curve, then for a curve of type (2) there must be an equation of this sort where the arguments $\tau'$ and $m'$ did not enter, and we have shown, by means of our lemma, that no such equation other than $0 = 0$ does exist for all curves of type (2). Hence (1) must also be illusory, and the only equations which do subsist are those which are derivable from the known equations of Plücker and Klein.

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