10.

De transformatione integralis $\iint\frac{\partial\varphi\,\partial\psi}{\sqrt{(\sin^2\nu-\sin^2\varphi\cos^2\psi)}}$.

(Auctore *Haedenkamp*, Hamm. Guestph.)

Integrale duplex indefinitum

$$1.\quad \iint\frac{\partial\varphi\,\partial\psi}{\sqrt{(\sin^2\nu-\sin^2\varphi\cos^2\psi)}}$$

per simplicem transformationem hujus formae

$$\tang\tfrac{1}{2}\psi = \tang\tfrac{1}{2}\varphi'\sqrt{\left(\frac{\cos\nu+\cos\varphi}{\cos\nu-\cos\varphi}\right)}$$

ad formam

$$\iint\frac{\partial\varphi\,\partial\varphi'}{\sqrt{(1-\cos^2\nu-\cos^2\varphi-\cos^2\varphi'+2\cos\nu\cos\varphi\cos\varphi')}}$$

revocare licet. Ex aequatione enim

$$\tang\tfrac{1}{2}\psi = \tang\tfrac{1}{2}\varphi'\sqrt{\left(\frac{\cos\nu+\cos\varphi}{\cos\nu-\cos\varphi}\right)}.$$

provenit

$$\cos\psi = \frac{\cos\nu\cos\varphi'-\cos\varphi}{\cos\nu-\cos\varphi\cos\varphi'},$$

$$\sin\psi = \frac{\sin\varphi'\sqrt{(\cos^2\nu-\cos^2\varphi)}}{\cos\nu-\cos\varphi\cos\varphi'},$$

$$\partial\psi = \frac{\sqrt{(\cos^2\nu-\cos^2\varphi)}}{\cos\nu-\cos\varphi\cos\varphi'}\partial\varphi';$$

quibus valoribus in formulam (1.) substitutis, nanciscimur formam integralis enunciatam

$$2.\quad \iint\frac{\partial\varphi\,\partial\varphi'}{\sqrt{(\sin^2\nu-\sin^2\varphi\cos^2\psi)}}$$

$$=\iint\frac{\partial\varphi\,\partial\varphi'}{\sqrt{(1-\cos^2\nu-\cos^2\varphi-\cos^2\varphi'+2\cos\nu\cos\varphi\cos\varphi')}} = \iint\frac{\partial\varphi\,\partial\varphi'}{\triangle(\varphi,\varphi')};$$

ubi per $\triangle(\varphi,\varphi')$ expressio sub radicali compendii causa significetur.

Per secundam transformationem hujusmodi:

$$3.\quad \begin{cases}\cos\varphi = k\cos u\,\triangle(k',u')-k'\sin u'\,\triangle(k,u),\\ \cos\varphi' = k\cos u\,\triangle(k',u')+k'\sin u'\,\triangle(k,u),\end{cases}$$

ubi $k^2=\cos^2\frac{1}{2}\nu$, $k'^2=\sin^2\frac{1}{2}\nu$

$$\triangle(k,u)=\sqrt{(1-k^2\cos^2u)},\qquad \triangle(k',u')=\sqrt{(1-k'^2\sin^2u')},$$

fit

$$\partial \Phi \, \partial \Phi' = \frac{2 k k' \sin u \cos u' \, \partial u \, \partial u'}{\Delta(k, u) . \Delta(k', u')}$$

et

$$\Delta(\Phi, \Phi') = 2 k k' \sin u \cos u';$$

qua substitutione facta, variabilia in integrali $\frac{\partial \varphi \, \partial \varphi}{\Delta(\varphi, \varphi')}$ separata sunt, atque integrale propositam in hoc abire videmus

$$\iint \frac{\partial \varphi \, \partial \psi}{\sqrt{(\sin^2 \nu - \sin^2 \varphi \cos^2 \psi)}} = \int \frac{\partial u}{\Delta(k, u)} \int \frac{\partial u'}{\Delta(k', u')} = F(k, u) \, F(k', u'),$$

ubi $F(k, u)$, $F(k, u)$ volito more integralia elliptica primae speciei significant, quorum moduli k et k'. Collegimus igitur integrale $\iint \frac{\partial \varphi \, \partial \psi}{\sqrt{(\sin^2 \nu - \sin^2 \varphi \cos^2 \psi)}}$ esse productum integralium ellipticorum primae speciei, quorum moduli alter alterius complementum.

Id transformationem in (3.) adhibitam hac ratione pervenire potes. Expressionem

$$1 - \cos^2 \nu - \cos^2 \Phi - \cos^2 \Phi' + 2 \cos \nu \cos \Phi \cos \Phi'$$

ut notam est, in hos factores resolvere licet:

$$\sin\left(\frac{\varphi + \varphi' + \nu}{2}\right) \sin\left(\frac{\varphi + \varphi' - \nu}{2}\right) \sin\left(\frac{\nu + \varphi - \varphi'}{2}\right) \sin\left(\frac{\nu - \varphi - \varphi'}{2}\right)$$

vel etiam

$$\left[\cos^2 \frac{\nu}{2} - \cos^2\left(\frac{\varphi + \varphi'}{2}\right)\right] \left[\sin^2 \frac{\nu}{2} - \sin^2\left(\frac{\varphi - \varphi'}{2}\right)\right].$$

Posito jam

$$\frac{\varphi + \varphi'}{2} = x, \qquad \frac{\varphi - \varphi'}{2} = x'$$

integrale duplex

$$\iint \frac{\partial \varphi \, \partial \varphi'}{\Delta(\varphi, \varphi')}$$

in hoc abire videmus

$$2 \iint \frac{\partial x \, \partial x'}{\sqrt{((k^2 - \cos^2 x)(k'^2 - \sin^2 x')}};$$

quo in integrali fluit ex substitutione

$$\cos x = k \cos u, \qquad \sin x = k' \sin u'$$

transformatio supra (3.) adhibita.

Inveni, generalioris formae integrale duplex indefinitum.

$$\iint \frac{\partial \varphi \, \partial \psi \cos 2 n \varphi}{\sqrt{(\sin^2 \nu - \sin^2 \varphi \cos^2 \psi)}},$$

si n numeros quicunque integer, ad productum integralium ellipticorum primae et secundae speciei revocari posse, ita quidem ut easdem ut supra transformationes adhibeas.

Etenim quod

$$\Phi = x + x', \qquad \Phi' = x - x',$$

facile invenitur

$$4. \quad \iint \frac{\partial\varphi\,\partial\psi\cos 2n\varphi}{\sqrt{(\sin^2\nu - \sin^2\varphi\cos^2\psi)}}$$

$$\iint \frac{\partial x\,\partial x'\cos 2nx\cos 2nx'}{\sqrt{((k^2-\cos^2 x)(k'^2-\sin^2 x')}} - \iint \frac{\partial x\,\partial x'\sin 2nx\sin 2nx'}{\sqrt{((k^2-\cos^2 x)(k'^2-\sin^2 x'))}}.$$

Altera pars integralis duplicis;

$$5. \quad \iint \frac{\partial x\,\partial x'\sin 2nx\sin 2nx'}{\sqrt{((k^2-\cos^2 x)(k'^2-\sin^2 x'))}}$$

est algebraica, quae unacum $\Delta(\Phi, \Phi') = 0$ vel etiam pro limitibus ipsorum u et u': 0 et $\frac{1}{2}\pi$, evanescit. Qua parte integralis hoc loco objecta, nanciscimur igitur

$$\iint \frac{\partial\varphi\,\partial\psi\cos 2n\varphi}{\sqrt{(\sin^2\nu - \sin^2\varphi\cos^2\psi)}} = \int \frac{\cos 2nx\,\partial u}{\Delta(k,u)} \int \frac{\cos 2nx'\,\partial x'}{\Delta(k',u')}.$$

Quod $\cos 2nx$ et $\cos 2nx'$ secundum pares dignitates ipsorum $\cos u$ et $\sin u'$ evolvi possant, integrale

$$\int \frac{\partial x\cos 2nx}{\Delta(k,u)}$$

est, ut ex integralibus ellipticis constat, functio linearis ipsorum $E(k,u)$, $F(k,u)$ et integrale

$$\iint \frac{\cos 2nx'\,\partial u'}{\Delta(k',u')}$$

eodem modo functio linearis ipsorum $E(k',u)$ et $F(k',u')$. Est igitur integrale

$$\iint \frac{\partial\varphi\,\partial\psi\cos 2n\varphi}{\sqrt{(\sin^2\nu - \sin^2\varphi\cos^2\psi)}}$$

productum duorum integralium ellipticorum primae et secundae speciei.

Si integralia $F(k,u)$, $E(k',u')$, $F(k',u')$, $E(k,u)$ inter limites 0 et $\frac{1}{2}\pi$ sumimus, pars algebraica (5.) evanescit, et nanciscimur, posito $n=0$:

$$\iint \frac{\partial\varphi\,\partial\psi}{\sqrt{(\sin^2\nu - \sin^2\varphi\cos^2\psi)}} = F(k)F(k'),$$

$n=1$:

$$\iint \frac{\partial\varphi\,\partial\psi\cos 2\varphi}{\sqrt{(\sin^2\nu - \sin^2\varphi\cos^2\psi)}} = \pi + F(k).F(k') - 4E(k).E(k')$$
$$= (F(k) - 2E(k)).(F(k') - 2E(k')),$$

$n = 2$:

$$\iint \frac{\partial \varphi \, \partial \psi \cos 4\varphi}{\sqrt{(\sin^2 \nu - \sin^2 \varphi \cos^2 \psi)}} = (\tfrac{1}{3})^2 [8 \cos \nu \, E(k) + (1 - 4 \cos \nu) F(k)] . [8 \cos \nu \, E(k') - (1 + 4 \cos \nu) F(k')]$$

etc. etc.

Posito pro casu speciliali $\nu = \frac{1}{2}\pi$, integralia abeunt

$$\iint \frac{\partial \varphi \, \partial \psi}{\sqrt{(\sin^2 \nu - \sin^2 \varphi \cos^2 \psi)}} = (F \sqrt{\tfrac{1}{2}})^2,$$

$$\iint \frac{\partial \varphi \, \partial \psi \cos 2\varphi}{\sqrt{(\sin^2 \nu - \sin^2 \varphi \cos^2 \psi)}} = \frac{\pi^2}{4 (F \sqrt{\frac{1}{2}})^2},$$

$$\iint \frac{\partial \varphi \, \partial \psi \cos 4\varphi}{\sqrt{(\sin^2 \nu - \sin^2 \varphi \cos^2 \psi)}} = (\tfrac{1}{3})^2 (F \sqrt{\tfrac{1}{2}})^2.$$

Per substitutionem

$$\cos \psi = \frac{\sin \nu}{\sin \varphi} \cos \psi$$

propositum integrale in hoc transmutare licet:

$$\iint \frac{\partial \varphi \, \partial \psi'}{\sqrt{(1 - \cos^2 \varphi - \sin^2 \nu \cos^2 \psi')}};$$

quod ut ad formam $\iint \frac{\partial u \, \partial u'}{\Delta(k, u) \Delta(k', u')}$ revocetur, primum ponendum est:

$$\tan \tfrac{1}{2} \psi' = \tan \tfrac{1}{2} \varphi' \sqrt{\left(\frac{\cos \varphi + \cos \nu}{\cos \varphi - \cos \nu}\right)},$$

deinde substitutio (3.) adhibenda.

Hamm in Guestph. Mens. Aprilis 1839.