ART. LI.—*On Electrical Oscillations of Low Frequency and their Resonance;* by M. I. PUPIN, Ph.D., Columbia College.

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PART II. THEORETICAL DISCUSSION WITH SPECIAL REFERENCE TO THE THEORY OF RISE OF POTENTIAL BY RESONANCE.

I. *Introduction.*

A very faithful mechanical picture of the periodically varying flow in an electrical circuit possessing *localized** capacity and self-induction is obtained by considering the motion of a torsional pendulum, that is a heavy bar, say of cylindrical form, suspended on a stiff elastic wire. The moment of inertia of the bar and the elasticity of the suspension wire correspond to the coefficient of self-induction and the capacity of the circuit. The frictional resistance of the air corresponds to ohmic resistance, internal friction in the bar and the elastic suspension correspond to magnetic and dielectric hysteresis; angular displacement of the torsional pendulum corresponds to the electrical charge of the condenser, and therefore torsional reaction of the suspension to difference of potential between the condenser plates. Angular velocity in the one case stands for the current in the other, kinetic energy for electrokinetic energy, potential energy of the torsional forces stands for the electrostatic energy of the condenser charge.

In slow mechanical vibrations the decrement of the kinetic energy is chiefly due to external and internal frictional resistances. But as the frequency of the vibration increases other losses causing this decrement become more prominent; so the losses due to radiation in form of sound waves. Similarly in electrical oscillations of very high frequency; the decrement of the electrokinetic energy due to radiation in form of electromagnetic waves becomes considerably larger than that due to dissipation in consequence of ohmic resistance, magnetic and dielectric hysteresis. The analogy, therefore, supplied by mechanical vibrations is by no means a poor guide in the study of *even very rapid* electrical oscillations. *For slow vibrations the analogy is very striking and instructive.* To return to the torsional pendulum:—

Let I = moment of inertia of the bar,
θ = angle of displacement at any moment.

* The term *localized* is employed to distinguish the circuits considered in this paper from those electrical circuits in which self-induction and capacity are more or less uniformly distributed over the whole circuit, as, for instance, in the case of a Herzian Resonator.

Let the torsional force be as ordinarily assumed proportional to angle of displacement and the frictional resistance to angular velocity. An impulse having set the pendulum in motion it is required to describe the motion. The differential equation of motion is obtained by writing down the symbolical statement of the principle of moments, viz:

Rate at which the moment of momentum about the line of suspension varies	=	Moment of all the forces about the same line.

That is

$$-\frac{d}{dt}\left(I\frac{d\theta}{dt}\right)=\alpha\frac{d\theta}{dt}+\beta\theta \quad \ldots\ldots \qquad (1)$$

or

$$I\frac{d^2\theta}{dt^2}+\alpha\frac{d\theta}{dt}+\beta\theta=0 \quad \ldots\ldots \qquad (2)$$

Certain well known conditions being fulfilled the following integral is readily obtained:

$$\theta = Ae^{-\frac{a}{2I}t}\sin\frac{2\pi}{T}t \qquad (3)$$

where T = *natural period* of the pendulum $= \dfrac{2\pi}{\sqrt{\dfrac{\beta}{I}-\dfrac{\alpha^2}{4I^2}}}$

The arbitrary constant A depends on the energy of the impulse and can be easily determined by well known rules. When $\frac{\alpha^2}{4I^2}$ is small in comparison to $\frac{\beta}{I}$ then

$$T=2\pi\sqrt{I\cdot\frac{1}{\beta}} \quad \ldots\ldots\ldots \qquad (4)$$

that is, *the natural period of the pendulum is independent of the frictional resistance.*

I venture to discuss briefly this rather familiar mechanical problem; for, the discussion seems to throw a strong light upon some of the electrical problems which form the subject of this paper.

Let T_1 = natural period calculated by (3)
T_2 = " " " " (4)

By a simple transformation it is easily shown that

$$T_1 = T_2(1+\frac{1}{8\pi^2}r^2 - \ldots\ldots + \ldots) \quad \ldots \qquad (5)$$

where $r = \dfrac{\frac{T_2}{2}\alpha}{L}$

$=$ ratio (approx.) of frictional loss during any half period to the amplitude of the kinetic energy during the same half period. I shall call it the *dissipation ratio.*

It follows therefore that whenever the dissipation ratio is smaller than $\frac{1}{4}$ then T_2 differs from T_1 by less than $\frac{1}{10}$ of one per cent. But since on the other hand

$$e^{-\frac{R}{2L}t} = e^{-r\frac{2t}{T_2}}$$

It follows that when the dissipation ratio $r = \frac{1}{4}$ then the pendulum will be practically reduced to rest after 16 complete oscillations. *This simple calculation shows, therefore, that even in very damped oscillations the period can and in most cases will be practically independent of the frictional resistance.*

The following observations are too well understood to need a mathematical commentary:—*a.* If a periodically varying force is applied to a torsional pendulum the oscillations will be *free oscillations* if the period of the force is the same as the natural period of the pendulum, that is if the force and the pendulum are in *resonance* to each other. When this resonance does not exist the oscillations are forced.

b. Of two periodically varying forces of the same mean intensity the one which is in resonance with the pendulum will produce the largest maximum elongation. The maximum elongation is reached when the work done by the resonant force during a complete period is equal to the frictional losses during that time.

c. The torsional force of the suspension varies periodically, its period being the same as that of the impressed resonant force, but differing from it in phase by a quarter of a period. The *amplitude of the torsional force can be much larger than the amplitude of the impressed force, especially when the frictional resistances are small, the moment of inertia large and the oscillations rapid,* that is the torsional coefficient large. For in this case that part of the work of the impressed force which is stored up in the kinetic energy of the pendulum will become large before the maximum elongation has been reached. But since this large kinetic energy has to be stored up in the potential energy of the torsional forces once during each half oscillation it is evident that a large torsional force will be called into action. *The amplitude of the tor-*

sional force is evidently an accumulative effect of the impressed force, and can easily be made so large as to break the suspension. This is a complete analogy to the breaking down of condensers due to a great rise in potential produced by resonance described further below.

The analogy can be carried further by considering the motion of a torsional pendulum A which is acted upon by a periodically varying force F, not directly, but through another torsional pendulum B to which A is suitably connected. The study of the motion of this system under different conditions as regards resonance between A, B and F gives a complete mechanical picturé of the electrical flow in an electrical system consisting of a primary and a secondary circuit, each circuit having localized self-induction and capacity, when a periodically varying e. m. f. acts upon the primary circuit. An analytical discussion of the motion of this mechanical system would lead far beyond the limits of this paper. It seems sufficient to point out, that the analysis is almost identical with the following mathematical discussion of the electrical flow in resonant circuits and that it is possible to imitate in a mechanical model most of the electrical effects discussed below, by properly constructed torsional pendulums connected to each other in a suitable manner.

II. *On the Natural Period of an Electrical Circuit Possessing Localized Capacity and Self-induction.*

The circuit consists of a coil, whose coefficient of self-induction is L henrys, connected in series to a condenser of capacity C farads. Let the ohmic resistance be R ohms. An electrical impulse having started the electrical flow it is required to describe the flow. Let Q be the positive charge of the condenser in coulombs, at any moment, then the differential equation of the flow is obtained by writing down a symbolical expression of the generalized form of Ohm's law (disregarding losses due to magnetic and dielectric hysteresis)

$$-\frac{d}{dt}\left(L\frac{dQ}{dt}\right)=R\frac{dQ}{dt}+\frac{1}{C}Q \quad \ldots . \qquad (1^a)$$

or

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{C}Q=0 \quad \ldots \qquad (2^a)$$

Comparing these equations to (1) and (2) we see that certain well known conditions being fulfilled the familiar integral first discussed by Sir W. Thomson, can be written down as follows:

$$Q = Ae^{-\frac{R}{2L}t} \sin\frac{2\pi}{T}t$$

where T = natural period of the circuit

$$= \frac{2\pi}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}$$

When $\frac{R^2}{4L^2}$ is small in comparison to $\frac{1}{LC}$ then

$$T = 2\pi\sqrt{LC}$$

that is *the natural period of the circuit is independent of the ohmic resistance.*

To show that it is only under very exceptional circumstances that this condition is not fulfilled, I shall consider a circuit consisting of a large Bell telephone connected in series with a condenser of 1 microfarad capacity. The resistance of the telephone is 100 ohms, very large indeed, considering that its coefficient of self-induction is only about 0·5 henrys. Making this circuit a part of the secondary circuit of the small transformer excited by the electro-dynamic interrupter described in part I of this paper* it is found that the sound of the telephone is loudest when the frequency of the vibrator is about 225. The pitch of the sound is not sensibly altered by changing the resistance within very large limits; a result required by theory. For the period calculated from formula

$T = 2\pi\sqrt{LC}$ gives 224,4 vibrations per second.

Adding the correction given by formula (5) we get for the corrected period $T_1 = 224{,}9$ a difference of only about $\frac{1}{4}$ of one per cent. Since the dissipation ratio $r = \frac{1}{2{,}25}$ we get for the damping factor $e^{-\frac{1}{2{,}25}\frac{2t}{T}}$, that is to say the electrical oscillations would disappear almost completely after only 10 complete oscillations, which shows that the ohmic resistance produces a very strong damping and yet the period is practically independent of it.

In circuits consisting of well made coils with finely divided but split iron cores the dissipation ratio r is very small even for frequencies as low as 100 periods per second. The period, therefore, will be independent of the dissipation losses even if

* This Journal, April, 1893.

hysteresis and Foucault current-losses approach the order of magnitude of the losses due to ohmic resistance. The natural period of such circuits, especially when tuned up to a frequency of over 200 periods per second will be given *very accurately* by the formula

$$T=2\pi\sqrt{LC}$$

To such circuits only the following discussion refers.

III. *On the Electrical Flow in a Resonant Circuit.*

Let a simple harmonic e. m. f. of period T act upon a circuit having localized self-induction and capacity, coil and condenser being connected in series. By the generalized form of Ohm's law we have in the usual notation

$$L\frac{dx}{dt}+Rx+P=E\sin pt \tag{6}$$

The integral obtained by well-known rules is

$$x=\frac{p\,CE}{\sqrt{(1-p^2\,CL)^2+p^2\,C^2\,R^2}}\sin(pt-\varphi) \tag{7}$$

where $$\tan\varphi=\frac{1-p^2CL}{pRC}$$

which can also be written

$$x=\frac{E}{\sqrt{p^2L_{\prime}^2+R^2}}\sin(pt+\varphi_{\prime})$$

$$\tan\varphi_{\prime}=\frac{pL_{\prime}}{R}$$

The integral written in this last form shows, as Oliver Heaviside first pointed out, that a condenser of capacity C in series with a coil changes the impedance of the circuit in such a way as if the condenser had a negative coefficient of self-induction equal to $\frac{1}{p^2C}$.* It produces also a shifting of phase. *The impedance is reduced to ohmic resistance* when $L_1=0$ or $p^2LC=1$, that is when the period of the impressed e. m. f. is equal to the natural period of the circuit, or in other words, *when the two are in resonance.*

The current and therefore the amplitude of the charge of the condenser reach then their maximum value.

* It is well to observe here that later on in the analysis of more complicated circuits possessing localized self-induction and capacity, I simplify my calculations very much by substituting $L_1 = -L+\frac{1}{p^2C}$ for the coeffic. of self-induction and treating the circuit then as if it had no capacity.

The resonant flow consists in a conversion of electrokinetic into electrostatic energy, and *vice versa*, during each semi-oscillation, accompanied by a loss due to ohmic resistance which is the only work which the e. m. f. does. The amplitudes of the electrokinetic and electrostatic energies must therefore be equal to each other, hence

$$\tfrac{1}{2}\,L\left(\frac{E}{R}\right)^2 = \tfrac{1}{2}\,P_o^{\,2}\,C$$

where $P_o =$ amplitude of the potential difference in the condenser.

The last relation gives, remembering that owing to resonance $p^2LC = 1$,

$$P_o = \frac{E}{pCR} = \frac{pL}{R}\,E = \frac{\text{Inductance}}{\text{Resistance}} \times E \qquad (8)$$

If L and p are large and R small the rise in potential can be made as large as we please, or rather as large as the condenser will stand.

The analogy between this rise of potential due to resonance and the torsional reaction of the suspension in the resonant swinging of the torsion pendulum mentioned above is striking. In both cases the reaction is produced by an accumulative effect of the impressed force.

A rough experiment only, bearing on this point and which can be easily repeated in a few minutes in every electrical laboratory, will be briefly described here.

Two large choking coils and a Marshall condenser were connected in series with the secondary of a transformer. The core of the smaller of the two choking coils consisted of a removable bundle of soft iron wire. The condenser terminals were connected to a Thomson Electrostatic Voltmeter. The frequency of the impressed e. m. f. was about 100 periods per second. The capacity of the condenser was adjusted until the removal of the plug was accompanied by bright snapping sparks, which was a signal that resonance was near. Then the removable iron core of the smaller choking coil was moved up and down gradually until the Voltmeter gave the largest deflection. A rise from 60 volts (generated in the secondary and indicated by a Cardew Voltmeter) to about 900 volts in the condenser was easily obtained. When the impressed e. m. f. was raised to 80 the condenser indicated about 1200 volts, which showed that the rise in the condenser was proportional to the impressed e. m. f., as the theory requires.* The rise of potential is practically

* I feel that it is only just to mention here that Mr. Marshall's ordinary condensers stood these voltages very well indeed, considering that they are guaranteed to stand a 1000 volts as their upper limit.

confined to the condenser, for the voltage on the line, indicated by the Cardew Voltmeter, does not change sensibly when resonance is established. *There is a large and rapid change in the current with the approach of resonance* which can be studied in a rough way by the pull which the choking coil exerts upon the removable iron core when the core is moved up and down during the process of tuning. The variation of this pull indicates very plainly that the curve expressing the relation between the current and the self-induction (resistance, capacity and frequency being constant), has a very steep crest which is in perfect accordance with the carefully plotted curve of equation (7) in Bedell and Crehore's volume on alternating currents.*

There are, however, several large maxima in this curve, each corresponding to a different capacity and self-induction; the simple experiment just described shows their presence very forcibly. The maximum corresponding to the largest capacity with about the same self-induction being however considerably the highest. With the condensers that I had at my disposal at that time I did not dare to tune the circuit for the highest maximum. The existence of several maxima will be seen presently to be a necessary consequence of the theory.

IV. *Electrical Resonance in a Circuit with a Complex Harmonic Electromotive Force.*

By Fourier's theorem a complex harmonic alternating e. m. f, can always be represented by the following series:

$$E = a_1 \sin pt + a_2 \sin 2pt + \dots a_n \sin npt \dots$$
$$= \sum_1^{\infty}{}_a\, a_a \sin a \text{pt}$$

In this expression I shall call $a_1 \sin pt$, $a_2 \sin 2\,pt$, ... the *component harmonics*, $a_1 \sin pt$ is the *fundamental harmonic*, its frequency, the *fundamental frequency*. The other harmonics will be referred to as the *upper harmonics*. The order of magnitude of their amplitudes is $a_1 > a_2 > a_3 > \dots > a_n > \dots$

The symbolical expression of Ohm's law is this:

$$L\frac{dx}{dt} + Rx + P = \sum_1^{\infty}{}_a\, a_a \sin a\, pt$$

Comparing this to (6) it is seen from the integral in (7) that this differential equation has the following expression for its integral:

* See Bedell and Crehore's treatise: Alternating Currents, p. 138, published by W. J. Johnston Co., New York.

$$x = \sum_1^{\infty} a \frac{\alpha p C a_a}{\sqrt{(1-\alpha^2 p^2 CL)^2 + \alpha^2 p^2 C^2 R}} \sin(\alpha pt - \varphi_a)$$

Where $\tan \varphi_a = \dfrac{1-\alpha^2 p^2 CL}{\alpha p CR}$

If we make $1 - p^2CL = 0$, then the *circuit is brought in resonance with the fundamental harmonic* and the current is given by

$$x = \frac{a_1}{R} \sin pt + \sum_2^{\infty} a \frac{\alpha a_a}{\sqrt{p^2(1-\alpha^2)^2 L^2 + \alpha^2 R^2}} \sin(\alpha pt - \varphi_a).$$

If the coefficient of self-induction is large then it is perfectly evident that the amplitude of the fundamental harmonic current is by far the largest especially when the frequency of the fundamental harmonic of the impressed e. m. f. is high.

For instance, let $L = 2$, $R = 5$, $p = 2\pi \times 100$.

I select these values so as to be near the conditions under which the above experiment was performed. Under these conditions we should have for the amplitude of the next harmonic, supposing it to be an octave

$$\frac{a_2}{6\pi \times 10^2} \text{ (very nearly).}$$

The amplitude of the fundamental is therefore at least 360 times as large. In all probability this ratio is considerably larger, considering that a_1 is generally several times larger than a_2.

The higher harmonics have even much smaller amplitudes. *The rise of potential in the condenser is therefore just the same as if a simple harmonic e. m. f. of amplitude a_1 and pulsation p, acted upon the circuit.*

The tuning of the circuit produces therefore two distinct effects: 1st, It produces a rise of potential in the condenser, and 2nd, It weeds out the upper harmonics.

It may happen, however, that the circuit is tuned to one of the upper harmonics, as for instance when $\alpha^2 p^2 CL = 1$.

In this case the current is given by

$$x = \frac{a_a}{R} \sin \alpha pt + \Sigma \frac{a_\beta}{\sqrt{(\alpha^2 - \beta^2)^2 p^2 L^2 + \beta^2 R^2}} \sin(\beta p - \varphi_\beta)$$

β to take integral values from 1 to ∞ except the value α.

It is evident that now the fundamental harmonic with all the upper harmonics excepting the harmonic a is practically weeded out on account of the strengthening of the harmonic

$\frac{a_a}{R}$ *sin apt by resonance.* The rise of potential according to formula (8) is given by

$$P_a = \frac{apL}{R} a_a.$$

To show how this rise of potential compares to the rise obtained by resonance to the fundamental harmonic, let $a = 5$ and let the coefficient of self-induction be the same as before.*

$$P_1 = \frac{pL}{R} a_1$$

$$P_5 = \frac{5pL}{R} a_5$$

Hence $$\frac{P_1}{P_5} = \frac{a}{5a_5}.$$

It is a well known fact that well made alternators are constructed in such a way that a_1 is generally larger than $5a_5$; hence, P_1 will be generally considerably larger than P_5. This was confirmed by the above experiment.

(It is well to observe that this suggests a rather interesting method of analysing a complex harmonic e. m. f. into its component harmonics *and of determining the relative value of the amplitude of each component.*)

The bearing of this on the method of producing a simple harmonic current by electrical resonance, described in the first part of this paper (l. c.) needs, I venture to say, no further discussion.

The study of resonance in electrical systems consisting of a primary and a secondary circuit with localized self-induction and capacity presents several features which deserve careful attention; a brief discussion of these together with a description of several experiments bearing upon the theory of low frequency resonance will be given in my next paper.

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Columbia College, April 15th, 1893.

[To be continued.]

* In the experiment described above the capacity was the principal variable; for, the first approximation to resonance was obtained by plugging the condenser until the vicinity of resonance was reached. The maximum point was finally obtained by a, comparatively speaking, slight variation of the coefficient of self-induction.