On the Existence of a Root of a Rational Integral Equation.

By E. B. Elliott. Read and received March 8th, 1894.

1. It may be that a proof, not depending on the theory of functions of a complex variable, of the theorem that every rational integral algebraic equation has a root is still a desideratum. It is at all events worth while to examine whether such proofs as have been given are or can be made sound. I have recently been studying two simple apparent proofs depending on the theory of elimination, one by the late Professor Clifford (*Math. Papers*, p. 20; *Camb. Proceedings*, Vol. II.), and one by Mr. J. C. Malet (*Transactions of the Royal Irish Academy*, Vol. XXVI.), and find both to be wanting in completeness. I have also, and I hope successfully, endeavoured to construct a proof of the same character free from the corresponding defects.*

Clifford's method is to show that a quadratic divisor x^2+px+q of

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n$$

can be found if a root q exists of a certain equation of degree $\frac{n(n-1)}{2}$, the result of eliminating p between $M=0$ and $N=0$, where $Mx+N$ is the remainder when the n-ic is divided by the quadratic, the corresponding p being the common root of the equations $M=0$, $N=0$, with that value inserted for q in them. Now, n being m times even, *i.e.*, of the form $2^m\times$ an odd number, $\frac{n(n-1)}{2}$ is only $m-1$ times even. Thus the argument is that, if every equation of degree $m-1$ times even has a root, every equation of degree m times even has a pair of roots. Now every equation of odd degree with real coefficients has a root. Hence every equation with real coefficients of degree once even has a root. It is then concluded by induction that every equation whatever has a root.

The successs of the induction is considerably interfered with by the question of imaginary roots and coefficients. There is, however, a far more fundamental objection to the validity of the method. Everything depends on the uniqueness of the value of p found as corresponding to a known q. There is no reason to assume that, when the q-eliminant of M and N vanishes, the G.C.M. of M and N is linear. What is proved as a basis for mathematical induction is at

* *Cf.* Gordan, *Math. Ann.*, x., pp. 573, &c., for a proof in which the essential argument is similar. [*June*, 1894.]

most that an equation of degree $n = 2^m \times$ odd number has a root, if one of degree $2^{m-1} \times$ odd number has one, and if every equation of degree less than $n-1$ has.

The idea that the vanishing of the eliminant of u and v is sufficient to ensure that $u = 0$, $v = 0$ have a common root is one altogether subsequent to and dependent upon that of the fact that an equation has roots. The eliminant approached without previous idea of roots is, as will be seen later, the criterion only for a common factor of unknown degree of u and v. To assume that such a common factor implies a common root or roots is to assume the theorem of which a proof is desired.

2. Mr. Malet's argument is almost identical, though different in analytical form. His method is practically to show that p is determined if an equation of degree $\frac{n(n-1)}{2}$ can be solved, and that x^2, where x is a root, then follows as the common root of two equations. His induction proceeds exactly as Clifford's. His tacit assumption which needs justification is that of the determinateness of a common root of two equations when p is known, just as Clifford's was that of the determinateness of p when q is known. He does not ignore the question of the imaginary.

In the following articles I do not endeavour to perfect either of the two proofs criticised in the form in which it stands, but find it convenient to adopt a somewhat different (and in one respect more cumbrous) analysis leading to the same essential argument.

3. Since the ordinary theory of the order of eliminants is based on the assumption that an equation of the n^{th} degree has n roots, it is in the first place necessary to have a clear idea of what is necessitated by the vanishing of the dialytic determinant of two forms when we are not entitled to make any such assumption.

For simplicity's sake, I write down only

$$\begin{vmatrix} a & b & c & d & e & & \\ & a & b & c & d & e & \\ & & a & b & c & d & e \\ a' & b' & c' & d' & & & \\ & a' & b' & c' & d' & & \\ & & a' & b' & c' & d' & \\ & & & a' & b' & c' & d' \end{vmatrix}$$

the dialytic determinant of the quartic and cubic

$$ax^4+bx^3+cx^2+dx+e,$$

$$a'x^3+b'x^2+c'x+d',$$

the argument for this case applying generally.

It is convenient to consider the determinant from Euler's rather than Sylvester's point of view. Its vanishing necessitates that y_1, y_2, y_3, y_4 and z_1, z_2, z_3, not all zero, exist, such that

$$\begin{aligned}
az_1 &= a'y_1,\\
bz_1+az_2 &= b'y_1+a'y_2,\\
cz_1+bz_2+az_3 &= c'y_1+b'y_2+a'y_3,\\
dz_1+cz_2+bz_3 &= d'y_1+c'y_2+b'y_3+a'y_4,\\
ez_1+dz_2+cz_3 &= d'y_2+c'y_3+b'y_4,\\
ez_2+dz_3 &= d'y_3+c'y_4,\\
ez_3 &= d'y_4,
\end{aligned}$$

i.e., that an auxiliary cubic $y_1x^3+\dots$ and quadratic $z_1x^2+\dots$ exist, such that

$$(z_1x^2+z_2x+z_3)(ax^4+bx^3+cx^2+dx+e)$$

$$\equiv (y_1x^3+y_2x^2+y_3x+y_4)(a'x^3+b'x^2+c'x+d');$$

and this implies that

$$ax^4+bx^3+cx^2+dx+e,$$

$$a'x^3+b'x^2+c'x+d'$$

have a common measure of the first or some higher degree in x.

Thus in all cases the vanishing of the dialytic determinant of two binary forms expresses that those forms have a common factor.

In case the two forms can be expressed as products of linear factors, the product of all differences between roots of the one and roots of the other, made integral by the smallest adequate powers of a and a' as factors, is, by the usual theory, of the same order in the coefficients as the dialytic determinant, and expresses by its vanishing the same property. The two are then in such a case identical.

4. Now consider the dialytic determinant

$$\Delta \equiv \begin{vmatrix} a_0\rho^n, & a_1\rho^{n-1}, & \dots & a_{n-1}\rho, & a_n & & & \\ & a_0\rho^n, & \dots & a_{n-2}\rho^2, & a_{n-1}\rho, & a_n & & \\ \dots & \dots & \dots & \dots & \dots & & & \\ & & & a_0\rho^n, & a_1\rho^{n-1}, & a_2\rho^{n-2}, & \dots & a_n \\ a_n, & a_{n-1}\rho, & \dots & a_1\rho^{n-1}, & a_0\rho^n & & & \\ & a_n, & \dots & a_2\rho^{n-2}, & a_1\rho^{n-1}, & a_0\rho^n & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ & & & a_n, & a_{n-1}\rho, & a_{n-2}\rho^2, & \dots & a_0\rho^n \end{vmatrix}$$

of

$$a_0\rho^n \,.\, z^n + a_1\rho^{n-1} \,.\, z^{n-1} + \dots + a_n$$

and

$$a_n \,.\, z^n + a_{n-1}\rho \,.\, z^{n-1} + \dots + a_0\rho^n.$$

The condition $\Delta = 0$, if it can be satisfied by a value of ρ, will necessitate that these two forms with that value of ρ inserted in them have a common factor involving z.

Δ is of degree $2n^2$ in ρ. To examine its form, let us for a moment take for $a_0, a_1, \dots a_n$ the coefficients $a_0', a_1', \dots a_n'$ in an equation

$$f(x) \equiv a_0'x^n + a_1'x^{n-1} + \dots + a_n' \equiv a_0'(x-x_1)(x-x_2) \dots (x-x_n)$$
$$= 0,$$

which has been so formed as to have n roots. This imposes no relation on $a_0', a_1', \dots a_n'$. Otherwise $a_0', x_1, x_2, \dots x_n$, of which they are functions, must be connected by a relation, whereas they may be taken absolutely independent of each other. Thus the form of Δ is not affected by the substitutions.*

* [This must not be misunderstood. To say that no *relation* connects $a_0', a_1', \dots a_n'$ is not to say that there is no *restriction* upon the values of those letters. That there is no *restriction* is what we are about to prove.

The distinction may be illustrated by reference to other theories. Thus, for instance, no *relation* connects the coefficients in $ax^2+2bx+c=0$ when its roots, supposed to exist, are real. Otherwise three perfectly arbitrary real quantities, a and x_1, x_2 the two roots, are connected by a relation. But there is a *restriction* on the values of a, b, c; viz., their values must be such that $ac-b^2$ is negative. Any function of a, b, c and other letters, which we may call ρ, σ, τ, ..., will have its algebraical form in all the letters perfectly independent of any such *restriction* on the ranges of values to which we may consider them open, though a *relation* in them might make the form special. The question in the text is one of algebraical form in certain letters, and not of arithmetical form when numbers are substituted for those letters.]

Now, when two equations have numbers of roots indicated by their degrees, their dialytic determinant is the product of all the differences between a root of one and a root of the other, made integral by a product of the lowest adequate powers of their leading coefficients.

The dialytic determinant Δ' of

$$a_0'\rho^n z^n + a_1'\rho^{n-1}z^{n-1} + \dots + a_n'$$

and

$$a_n' z^n + a_{n-1}'\rho z^{n-1} + \dots + a_0'\rho^n$$

is then equivalent, but perhaps for a numerical and sign multiplier, to

$$(a_0'\rho^n)^n a_n'^n \Pi\left(\frac{x_r}{\rho} - \frac{\rho}{x_s}\right),$$

r and s having separately given them all values from 1 to n inclusive, *i.e.*, to

$$a_0'^{2n} \Pi(x_r x_s - \rho^2).$$

Now, in this product, a factor $x_r^2 - \rho^2$ in which $r = s$ occurs once, but a factor in which r and s are different occurs twice, once as $x_r x_s - \rho^2$, and once as $x_s x_r - \rho^2$. Thus Δ' is equivalent to

$$a_0'^2 \Pi(x_r^2 - \rho^2) \left\{a_0'^{n-1} \Pi(x_r x_s - \rho^2)\right\}^2,$$

i.e., to

$$f(\rho) f(-\rho) \times \text{perfect square},$$

the squared function being of degree $\frac{n(n-1)}{2}$ in ρ^2, with coefficients which, by the theory of symmetric functions, are rational and integral in $a_0', a_1', a_2', \dots a_n'$.

This form is preserved when for $a_0', a_1', \dots a_n'$ are written $a_0, a_1, \dots a_n$, as above explained. Thus a factor of Δ is a rational integral function in ρ^2 of degree $\frac{n(n-1)}{2}$, its coefficients being rational integral functions of $a_0, a_1, \dots a_n$.

Consequently, if a certain rational integral equation of degree $\frac{n(n-1)}{2}$ in ρ^2 has a root, the two forms

$$a_0\rho^n . z^n + a_1\rho^{n-1} . z^{n-1} + \dots + a_n,$$

$$a_n . z^n + a_{n-1}\rho . z^{n-1} + \dots + a_0\rho^n,$$

with that value of ρ^2 substituted in them, have a common factor.

5. Now, every equation of odd degree with real coefficients has certainly a real root. We proceed to consider an equation,

$$a_0x^n+a_1x^{n-1}+\ldots+a_n=0,$$

with real coefficients whose degree n is twice an odd number, $=2(2m+1)$, say. For this $\frac{n(n-1)}{2}=(2m+1)(4m+1)$ is odd.

By the above a real ρ^2, and consequently a real or purely imaginary ρ, exists, which makes

$$a_0\rho^n z^n+a_1\rho^{n-1}z^{n-1}+\ldots+a_n,$$

$$a_n z^n+a_{n-1}\rho z^{n-1}+\ldots+a_0\rho^n,$$

have a common factor. The two cases must be regarded separately.

Firstly, if ρ^2 be positive, and so ρ real, the two expressions in z have real coefficients. Their G.C.M. (proved to exist) has then real coefficients, as the ordinary process for finding it necessitates. Call it $P(z)$, and let $Q(z)$ be the complementary factor of $a_0\rho^n z^n+\ldots$. Then, writing x for ρz, the equation

$$a_0x^n+a_1x^{n-1}+\ldots+a_n=0$$

is equivalent to $$P\left(\frac{x}{\rho}\right)Q\left(\frac{x}{\rho}\right)=0,$$

or, say, to $$P'(x)\,Q'(x)=0,$$

the coefficients in P' and Q' being all real.

Secondly, if ρ^2 be negative, and so ρ a pure imaginary $r\sqrt{-1}$, the two forms in z may be written, multiplying the second by $(\sqrt{-1})^n$, *i.e.*, by -1, and putting ζ for $z\sqrt{-1}$,

$$a_0r^n\zeta^n+a_1r^{n-1}\zeta^{n-1}+a_2r^{n-2}\zeta^{n-2}+\ldots+a_n,$$

$$a_n\zeta^n-a_{n-1}r\zeta^{n-1}+a_{n-2}r^2\zeta^{n-2}-\ldots+a_0r^n.$$

These two expressions in ζ with real coefficients have a common factor which can be found by the G.C.M. process. Call it $P(\zeta)$ and let $Q(\zeta)$ be the complementary factor of the first form. The coefficients in these are found as real quantities. Thus, writing $\frac{x}{r}$ for ζ, the equation

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n=0$$

is the same as $$P\left(\frac{x}{r}\right) Q\left(\frac{x}{r}\right) = 0,$$

or, say, as $$P'(x)\, Q'(x) = 0,$$

where the coefficients in the factors are real.

6. Two cases again arise. Either P' may be of degree n, *i.e.*, be the whole form, and Q' a constant, or P' may be of degree between 1 and $n-1$ inclusive.

The former case would mean that the two equations

$$a_0\rho^n z^n + a_1\rho^{n-1}z^{n-1} + \ldots + a_n = 0,$$

$$a_n z^n + a_{n-1}\rho z^{n-1} + \ldots + a_0\rho^n = 0,$$

or the two equations

$$a_0 r^n \zeta^n + a_1 r^{n-1}\zeta^{n-1} + \ldots + a_n = 0,$$

$$a_n\zeta^n - a_{n-1} r\zeta^{n-1} + \ldots + a_0 r^n = 0,$$

as the case may be, are identical. This being so, each of the first pair would be

$$a_0\rho^n (z^n+1) + a_1\rho^{n-1}(z^{n-1}+z) + \ldots + 2a_{\frac{1}{2}n} z^{\frac{1}{2}n} = 0,$$

which is an equation of degree $\frac{n}{2}$, *i.e.* odd degree, in $z+\frac{1}{z}$; or else each of the second pair would be

$$a_0 r^n (\zeta^n+1) + a_1 r^{n-1}\zeta(\zeta^{n-2}-1) + a_2 r^{n-2}\zeta^2(\zeta^{n-4}+1) + a_3 r^{n-3}\zeta^3(\zeta^{n-6}-1)$$
$$+ \ldots + a_{\frac{1}{2}n-1}\zeta^{\frac{1}{2}n-1}(\zeta^2+1),$$

which vanishes when $$\zeta = \pm\sqrt{-1},$$

since $n, n-4, n-8, \ldots 2$ are twice odd numbers, and $n-2, n-6, n-10, \ldots 4$ are twice even numbers. In this case our equation of degree n has the roots $\pm r\sqrt{-1}$. In the former case, a real value of $z+\frac{1}{z}$ is given by an equation of odd degree $\frac{1}{2}n$, and consequently two values, real or of the form $\alpha+\beta\sqrt{-1}$, of z, *i.e.*, two roots ρz, real or of that imaginary form, of our equation in x.

In the more general case, P' and Q' have complementary degrees both between 1 and $n-1$ inclusive. Now, these degrees cannot both be divisible by 4. Otherwise their sum n would be, as by supposition it is not. Either then one of the two degrees must be odd, that of Q', say, in which case $Q'=0$ has a real root, or one at least, that of Q, say again, must be twice an odd number less than the odd number $\frac{1}{2}n$.

Thus an equation $$a_0x^n + \ldots + a_n = 0,$$

whose coefficients are real and whose degree is twice an odd number $2m+1$, has certainly a root, real or of the form $\alpha+\beta\sqrt{-1}$, if every equation $Q'=0$ whose coefficients are real and whose degree is twice an odd number less than $2m+1$ has. Now a quadratic, whose degree is twice the smallest odd number, can be solved, its two roots being real or of the form $\alpha+\beta\sqrt{-1}$. Thus induction establishes that every equation with real coefficients whose degree is twice an odd number has a root, real or of the form $\alpha+\beta\sqrt{-1}$.

7. The next step in the argument is to prove that every equation of odd degree whose coefficients are of the form $a+b\sqrt{-1}$ has a root, real or of that form.

If $$f(x)+\sqrt{-1}\,\phi(x) = 0$$

be such an equation, where the coefficients of $f(x)$ and $\phi(x)$ are real, then $$\{f(x)+\sqrt{-1}\,\phi(x)\}\{f(x)-\sqrt{-1}\,\phi(x)\} = 0$$

is an equation of degree twice an odd number with real coefficients. It has then a root, real or of the form $\alpha+\beta\sqrt{-1}$. This root must make one of the two factors vanish, for the product of two non-vanishing quantities cannot vanish even though both be of the form $A+B\sqrt{-1}$.

If the root be real and make

$$f(x)-\sqrt{-1}\,\phi(x) = 0,$$

it must make $f(x)=0$ and $\phi(x)=0$ separately, and so also be a root of $$f(x)+\sqrt{-1}\,\phi(x) = 0.$$

If, on the other hand, it be of the form $\alpha+\beta\sqrt{-1}$, there must also be a conjugate root $\alpha-\beta\sqrt{-1}$; and if

$$f(x)-\sqrt{-1}\,\phi(x)$$

be the factor which $\alpha+\beta\sqrt{-1}$ makes vanish, then $\alpha-\beta\sqrt{-1}$ satisfies

$$f(x)+\sqrt{-1}\,\phi(x)=0,$$

for, if $$A+B\sqrt{-1}-\sqrt{-1}\,(C+D\sqrt{-1})=0,$$

then $$A-B\sqrt{-1}+\sqrt{-1}\,(C-D\sqrt{-1})=0.$$

Thus every equation of odd degree with coefficients of the form $a+b\sqrt{-1}$ has a root, real or of the form $\alpha+\beta\sqrt{-1}$.

8. Now, let it be assumed that every equation whose degree is $m-1$ or fewer times even, *i.e.*, contains the factor 2, if at all, not more than $m-1$ times, and whose coefficients are real or of the form $a+b\sqrt{-1}$ has a root, real or of the form $\alpha+\beta\sqrt{-1}$.

Consider the equation

$$a_0x^n+a_1x^{n-1}+\ldots+a_n=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

in which a_0, a_1, ... a_n are real, or of the form $a+b\sqrt{-1}$, and in which

$$n=2^m(2p+1),$$

i.e., is m times even.

By § 4 and our assumption, a value of ρ^2, and consequently two values of ρ, real or of the form $\alpha+\beta\sqrt{-1}$, given as a root of an equation of degree

$$\frac{n(n-1)}{2}=2^{m-1}(2p+1)\left\{2^m(2p+1)-1\right\},$$

which is only $m-1$ times even, exists, the substitution of which in

$$a_0\rho^nz^n+a_1\rho^{n-1}z^{n-1}+\ldots+a_n,$$

$$a_nz^n+a_{n-1}\rho z^{n-1}+\ldots+a_0\rho^n,$$

makes them have a common factor. This common factor, found by the ordinary G.C.M. process, will have coefficients real or of the form $a+b\sqrt{-1}$.

If this G.C.M. be of degree n, the two expressions are identical, but for a factor free from z, with

$$a_0\rho^n(z^n+1)+a_1\rho^{n-1}(z^{n-1}+z)+a_2\rho^{n-2}(z^{n-2}+z^2)+\ldots,$$

and this equated to zero is an equation of degree $\frac{1}{2}n$, *i.e.*, $m-1$ times even only, in $z+\dfrac{1}{z}$, and is accordingly by our present assumption satisfied by a value of $z+\frac{1}{z}$, real or of the form $\alpha+\beta\sqrt{-1}$, and consequently by two values of z real or of that form. In other words, the equation (1) is satisfied by two values ρz of x, real or of the form $\alpha+\beta\sqrt{-1}$.

On the other hand, if the G.C.M. be of degree less than n, call it $P(z)$. Then the left-hand side of equation (1) has the factor $P\left(\frac{x}{\rho}\right)$, or, say $P'(x)$. Let $Q'(x)$ be the complementary factor. We have thus (1) resolved into

$$P'(x)\, Q'(x) = 0,$$

the coefficients in P' and Q' being real or of the form $a+b\sqrt{-1}$.

Now the degrees of P' and Q' cannot both be divisible by 2^{m+1}, for their sum n is only divisible by 2^m. One or the other of them, the degree of Q', say, must then be m times even at most.

On the assumption therefore that every equation of degree $m-1$ or fewer times even, and that every equation of degree m times even and less than n, has a root, it is proved that any equation of degree n, which is m times even, has a root.

Take now 2^m the smallest number which is m times even. An equation of this degree has, by the same argument, a root if one of degree $2^{m-1}(2^m-1)$, which is $m-1$ times even, has a root, and if every equation $Q'(x)=0$ of degree less than 2^m has. On our assumption this is the case, no such degree being so many as m times even. We thus proceed to degrees $2^m.3$, $2^m.5$, ... $2^m(2p+1)$, ..., so that the following general statement is accurate.

"If every equation with coefficients real or of the form $a+b\sqrt{-1}$, whose degree is $m-1$ or fewer times even, has a root, real or of the form $\alpha+\beta\sqrt{-1}$, then every equation with such coefficients and of degree m times even has such a root."

In this m may be any positive integer, unity included. Now, the last article has established the existence of a root for odd degrees, *i.e.*, for the case $m=1$. It follows, then, successively for the cases $m=2, 3, 4, \ldots$, *i.e.*, for equations of degrees once, twice, three times, and generally any number of times, even.

[*April* 17*th*, 1894.

As some doubt has been thrown on the argument of § 4, I proceed to show otherwise that

$$a_0 \rho^n . z^n + a_1 \rho^{n-1} . z^{n-1} + \dots + a_n \equiv F(\rho z, 1)$$

and

$$a_n z^n + a_{n-1} \rho . z^{n-1} + \dots + a_0 \rho^n \equiv F(\rho, z)$$

can be made to have a common factor if ρ^2 can be determined so as to satisfy an equation of degree $\frac{1}{2} n(n-1)$.

As in the applications required n is always even, I, for simplicity, confine myself to this case.

The two will have a common factor if, and only if, their sum and difference have.

Now, the difference of $F(\rho z, 1)$ and $F(\rho, z)$ is divisible by $z^2 - 1$, n being even. The complete condition that a factor of z^2-1 be the common factor in question is

$$F(\rho, 1)\, F(-\rho, 1) = 0,$$

i.e., is $(a_0 \rho^n + a_2 \rho^{n-2} + \dots + a_n)^2 - (a_1 \rho^{n-1} + a_3 \rho^{n-3} + \dots + a_{n-1}\rho)^2 = 0,$

of which the left-hand side is a function of ρ^2.

Also the sum of $F(\rho z, 1)$ and $F(\rho, z)$ may be written

$$(a_0 \rho^n + a_n)(z^2+1)^{\frac{1}{2}n} + B\,(z^2+1)^{\frac{1}{2}n-1} z + \dots + K z^{\frac{1}{2}n},$$

and the difference, after the removal of the factor z^2-1, may be written

$$(a_0 \rho^n - a_n)(z^2+1)^{\frac{1}{2}n-1} + B'\,(z^2+1)^{\frac{1}{2}n-2} z + \dots + K' z^{\frac{1}{2}n-1},$$

where $B, \dots K, B', \dots K'$ are of degree n in ρ. Thus the remaining factor of the criterion required is the dialytic determinant of

$$(a_0 \rho^n + a_n)\, t^{\frac{1}{2}n} + B t^{\frac{1}{2}n-1} + \dots + K$$

and

$$(a_0 \rho^n - a_n)\, t^{\frac{1}{2}n-1} + B' t^{\frac{1}{2}n-2} + \dots + K',$$

whose degree in ρ is at most $n\left(\frac{n}{2} + \frac{n}{2} - 1\right)$, *i.e.*, $n(n-1)$, and will be seen below to be exactly this number. Call this dialytic determinant δ.

Accordingly the complete condition expressed by $\Delta = 0$ of § 4, whose degree is $2n^2$ in ρ, is expressed also by the alternatives

$$F(\rho, 1)\, F(-\rho, 1) = 0, \quad \delta = 0,$$

whose degrees are $2n$, and at most $n(n-1)$, respectively.

Moreover, Δ is a function of ρ^2, for the result of changing ρ into $-\rho$ in it is to alter the signs of alternate columns, and then the signs of alternate rows, *i.e.*, is to multiply it by $(-1)^{2n}$, *i.e.*, not to alter it.

Also $F(\rho, 1)\, F(-\rho, 1)$ is a function of ρ^2. Consequently so is δ.

Now the degree of δ in ρ, seen above to be not greater than $n(n-1)$, cannot be less than $n(n-1)$. This will certainly be true, $a_0, a_1, a_2, \ldots a_n$ being unrestricted, if it is true even when particular restrictions are imposed on them, for a function of any degree in ρ cannot have that degree raised by supposing its coefficients made special. Now, when $a_0, a_1, a_2, \ldots a_n$ are so chosen that the equation

$$F(x, 1) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$$

has n roots, as may certainly be done by taking it to be

$$a_0(x-x_1)(x-x_2)\ldots(x-x_n) = 0,$$

there are $\frac{1}{2}n(n-1)$ values of ρ^2 which must satisfy $\delta = 0$, namely, the $\frac{1}{2}n(n-1)$ products of two and two of $x_1, x_2, \ldots x_n$. [For instance, the value $\sqrt{x_1x_2}$ of ρ makes $F(\rho z, 1)$ and $F(\rho, z)$ have the common factors $z - \sqrt{\frac{x_1}{x_2}}$, $z - \sqrt{\frac{x_2}{x_1}}$.] [These $\frac{1}{2}n(n-1)$ products with the squares $x_1^2, x_2^2, \ldots x_n^2$, *i.e.*, the values of ρ^2 which make

$$F(\rho, 1)\, F(-\rho, 1) = 0,$$

make up all the solutions of the equation in ρ^2, $\Delta = 0$.]

The condition $\delta = 0$ is then, in the general case, no less than in the one which has at present to be taken as special, one of degree exactly $\frac{1}{2}n(n-1)$ in ρ^2.

δ is, of course, the square root of the quotient $\Delta / F(\rho, 1)\, F(-\rho, 1)$.

It is unfortunate for the simplicity of the argument of this paper that the property of such a determinant as Δ, that, after division by its obvious factors,

$$F(\rho, 1) \equiv a_0\rho^n + a_1\rho^{n-1} + a_2\rho^{n-2} + \ldots + a_n$$

and $F(-\rho, 1)$, it leaves a perfect square as quotient, is one which direct algebraic methods have as far as I know not yet supplied. For low values of n the proof is, of course, easy, but I have not yet succeeded in giving a general form to such proofs as I have obtained and had given me by my friends.]