ON THE DIFFRACTION OF A SOLITARY WAVE

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IN a former communication* a simplified proof was given of Sommerfeld's results relating to the diffraction of an infinite train of plane waves of simple-harmonic type by the straight edge of a plane perfectly reflecting screen. Since a plane wave of arbitrary character can always be resolved, by Fourier's theorem, into simple-harmonic trains, the formulæ appropriate to the case of a solitary wave can theoretically be deduced. It was in this way that the particular cases to be hereafter considered were in the first instance worked out; but in the paper as presented I have somewhat varied and simplified the procedure. I find that, in the case of one or two special types which present themselves quite naturally for consideration, the results assume a comparatively simple form, so that conclusions as to the whole history of the disturbance in various parts of the field can be drawn without much difficulty. The more interesting points are illustrated by means of figures drawn to scale.

It should be added that the diffraction of a solitary wave has also been discussed by Sommerfeld,† but chiefly in relation to the theory of Röntgen rays, where the boundary condition is different; the method employed is moreover quite distinct.

1. The screen, whose thickness is neglected, is supposed to occupy that half of the xz-plane for which x is positive. For simplicity the case of normal incidence is alone considered, although there is no difficulty in extending the results in this respect, in the manner explained in the former paper. The primary wave is accordingly taken to be of the type

$$\phi = F(ct+y). \tag{1}$$

* "On Sommerfeld's Diffraction Problem," *Proceedings*, Vol. 4, p. 190 (1906).

† "Theoretisches über die Beugung der Röntgenstrahlen," *Zeitschrift für Math. u. Physik*, Bd. 46, p. 11 (1901).

In the complete solution, representing both the primary and the diffracted waves, we have to satisfy the equation

$$\frac{\partial^2\phi}{\partial t^2} = c^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right), \tag{2}$$

subject to the condition either that the value of ϕ, or that the value of its normal derivative, shall vanish over both faces of the screen. We begin with the latter form of the problem, the boundary condition being

$$\frac{\partial\phi}{\partial y} = 0. \tag{3}$$

This corresponds to the case of sound waves, or of electric waves polarized in the plane xy. In the latter case, the components of magnetic and electric force are connected with ϕ by the relations

$$\left.\begin{array}{lll} \alpha = 0, & \beta = 0, & \gamma = \phi, \\ \dot{X} = c\,\partial\phi/\partial y, & \dot{Y} = -c\,\partial\phi/\partial x, & \dot{Z} = 0. \end{array}\right\} \tag{4}$$

It is easily seen that the function

$$\chi = \frac{\partial\phi}{\partial x}, \tag{5}$$

which is also a solution of (2), must vanish over the negative half of the plane xz, whilst its normal derivative $\partial\chi/\partial y$ must vanish at both faces of the screen. If we write

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

where θ is supposed to range from 0 to 2π, these conditions are satisfied by

$$\chi = f(ct-r)\,.\,r^{-\frac{1}{2}}\cos\tfrac{1}{2}\theta, \tag{6}$$

which is an obvious generalization of the assumption (7) of the former paper.

To deduce a value of ϕ, we introduce the parabolic coordinates

$$\xi = r^{\frac{1}{2}}\cos\tfrac{1}{2}\theta, \qquad \eta = r^{\frac{1}{2}}\sin\tfrac{1}{2}\theta, \tag{7}$$

in terms of which we have

$$x = \xi^2-\eta^2, \quad y = 2\xi\eta, \quad r = \xi^2+\eta^2. \tag{8}$$

"The curves $\xi = \text{const.}$, $\eta = \text{const.}$ form a system of confocal parabolas. The coordinate η is everywhere positive, and the line $\eta=0$

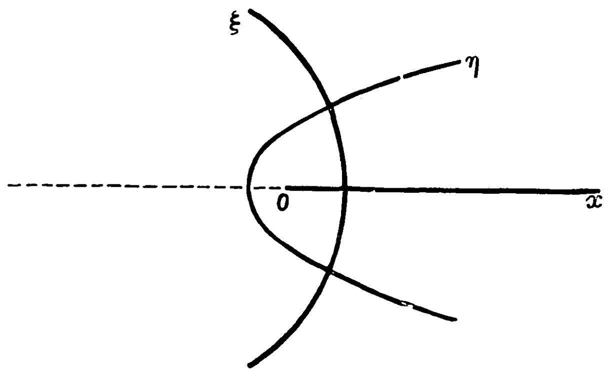

FIG. 1.

represents the section of the screen. The coordinate ξ has opposite signs on the two sides of the axis of x, and the line $\xi=0$ represents the free portion of this axis."

In terms of these variables we have

$$\left.\begin{aligned}\frac{\partial\phi}{\partial x}&=\frac{1}{2r}\left(\xi\frac{\partial\phi}{\partial\xi}-\eta\frac{\partial\phi}{\partial\eta}\right),\\ \frac{\partial\phi}{\partial y}&=\frac{1}{2r}\left(\eta\frac{\partial\phi}{\partial\xi}+\xi\frac{\partial\phi}{\partial\eta}\right).\end{aligned}\right\}\tag{9}$$

Hence, from (5) and (6),

$$\xi\frac{\partial\phi}{\partial\xi}-\eta\frac{\partial\phi}{\partial\eta}=2\xi f(ct-\xi^2-\eta^2).\tag{10}$$

The integration of this equation follows exactly the same lines as in the special case treated in the former paper (Vol. 4, p. 193). If we adjust the complementary function so as to make $\partial\phi/\partial y=0$, or $\partial\phi/\partial\eta=0$, at the screen, where $\eta=0$, we obtain

$$\phi=\int_0^{\xi+\eta}f(ct+y-\zeta^2)\,d\zeta+\int_0^{\xi-\eta}f(ct-y-\zeta^2)\,d\zeta+\tfrac{1}{2}F(ct+y)+\tfrac{1}{2}F(ct-y).\tag{11}$$

The functions f and F are so far arbitrary and independent. The relation between them in the present problem is supplied by the consideration that for large negative values of x the result must tend to the form (1). Now when θ is nearly equal to π, and r is very great, the upper limits of the integrals in (11) are $+\infty$ and $-\infty$, respectively, and the limiting form is accordingly

$$\phi = \int_0^\infty f(ct+y-\zeta^2)\,d\zeta - \int_0^\infty f(ct-y-\zeta^2)\,d\zeta + \tfrac{1}{2}F(ct+y) + \tfrac{1}{2}F(ct-y). \quad (12)$$

This becomes identical with (1), provided

$$\int_0^\infty f(y-\zeta^2)\,d\zeta = \tfrac{1}{2}F(y). \quad (13)$$

The complete solution of the problem of diffraction of a plane wave of arbitrary type is thus reduced to that of the integral equation (13).

By way of verification we may put

$$F(y) = e^{iky}. \quad (14)$$

Since

$$\int_0^\infty e^{-ik\zeta^2}\,d\zeta = \tfrac{1}{2}\frac{\sqrt{\pi}}{\sqrt{k}}\,e^{-\frac{1}{4}i\pi}, \quad (15)$$

the solution is

$$f(y) = \frac{\sqrt{k}}{\sqrt{\pi}}\,e^{\frac{1}{4}i\pi}\,e^{iky}. \quad (16)$$

This agrees with the known result (Vol. 4, p. 194), if regard be had to the slightly altered definition of ξ, η.

This case leads at once to a general solution of (13). Writing Fourier's theorem in the form

$$F(y) = \frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^\infty F(a)\,e^{ik(y-a)}\,da, \quad (17)$$

with the convention that the real part alone is to be retained, we have

$$f(y) = \frac{1}{\pi^{\frac{3}{2}}}\int_0^\infty k^{\frac{1}{2}}\,dk \int_{-\infty}^\infty F(a)\,e^{i\,[k(y-a)+\frac{1}{4}\pi]}\,da, \quad (18)$$

subject to the same convention.*

2. There is one type of function expressing a more or less concentrated disturbance which has been especially affected by mathematical physicists from the time of Cauchy downwards, on account of the facility of analytical treatment, viz.,

$$F(y) = \frac{b}{b^2+y^2}. \quad (19)$$

* Mr. H. Bateman, to whom I submitted the question, has obtained a simpler solution in the form

$$f(y) = \frac{1}{\pi}\int_{-y}^\infty \frac{F'(-z)\,dz}{\sqrt{(z+x)}}.$$

This makes $$\int_{-\infty}^{\infty} F(y)\,dy = \pi, \tag{20}$$

independently of the value of b; but the disturbance is the more concentrated the smaller this value. The familiar graphical representation of this function is given, for purposes of subsequent comparison, in Fig. 2.

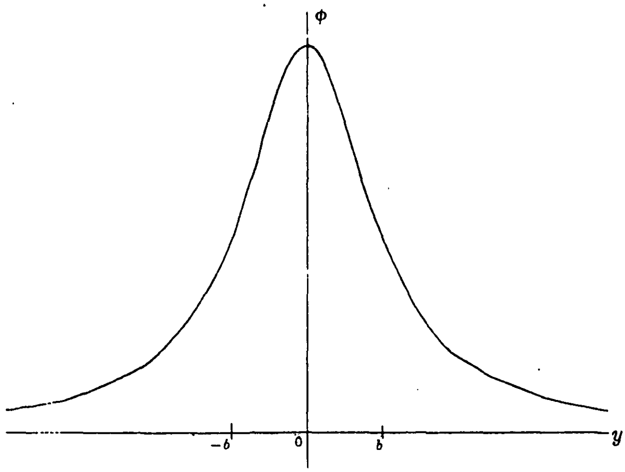

FIG. 2.

We have now $$\int_{-\infty}^{\infty} F(a)\,e^{-ika}\,da = \pi e^{-kb}, \tag{21}$$

and therefore
$$\begin{aligned} f(y) &= \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-kb+i(ky+\frac{1}{4}\pi)}\,k^{\frac{1}{2}}\,dk \\ &= \frac{e^{\frac{1}{4}i\pi}}{\sqrt{\pi}\,(b-iy)^{\frac{3}{2}}}\,\Gamma(\tfrac{3}{2}) \\ &= \frac{e^{\frac{1}{4}i\pi}}{2\,(b-iy)^{\frac{3}{2}}}, \end{aligned} \tag{22}$$

provided that in the denominator that value of the square root of $b-iy$ be taken which is real and positive when $y=0$. It is understood, as before, that the real part of the expression is alone to be preserved.

If we include the imaginary part, the formula (22) corresponds to

$$F(y) = \frac{1}{b+iy}. \tag{23}$$

This may immediately be verified. Since

$$\frac{d}{d\zeta}\,\frac{\zeta}{\sqrt{(b+iy-i\zeta^2)}} = \frac{b+iy}{(b+iy-i\zeta^2)^{\frac{3}{2}}}, \tag{24}$$

the formula (22) makes

$$\int f(y-\zeta^2)\,d\zeta = \frac{e^{-\frac{1}{4}i\pi}}{2\,(b+iy)}\,\frac{\zeta}{\sqrt{(b+iy-i\zeta^2)}}. \tag{25}$$

Taking this between the limits 0 and ∞, we find

$$\int_0^\infty f(y-\zeta^2)\,d\zeta = \frac{1}{2(b+iy)}, \tag{26}$$

in accordance with (13).

If we substitute from (23) and (25) in (11), making use of (8), we obtain

$$\phi = \frac{\frac{1}{2}}{b+i(ct+y)} + \frac{\frac{1}{2}}{b+i(ct-y)}$$
$$+\frac{\frac{1}{2}e^{-\frac{1}{4}i\pi}}{\sqrt{\{b+i(ct-r)\}}}\left\{\frac{\xi+\eta}{b+i(ct+y)}+\frac{\xi-\eta}{b+i(ct-y)}\right\}, \tag{27}$$

where the square root is to be interpreted as before.

In order to isolate the real part of this expression we write

$$ct+y = b\tan\alpha, \quad ct-y = b\tan\beta, \quad ct-r = b\tan\omega, \tag{28}$$

where the angles α, β, ω may range from $-\frac{1}{2}\pi$ through 0 to $+\frac{1}{2}\pi$. We find

$$\phi = \frac{1}{2b}\left\{\cos^2\alpha+\cos^2\beta+\frac{\xi+\eta}{\sqrt{b}}\cos^{\frac{1}{2}}\omega\cos\alpha\cos(\tfrac{1}{2}\omega+\alpha+\tfrac{1}{4}\pi)\right.$$
$$\left.+\frac{\xi-\eta}{\sqrt{b}}\cos^{\frac{1}{2}}\omega\cos\beta\cos(\tfrac{1}{2}\omega+\beta+\tfrac{1}{4}\pi)\right\}, \tag{29}$$

corresponding to a primary wave

$$\phi = \frac{b}{b^2+(ct+y)^2} = \frac{\cos^2\alpha}{b}. \tag{30}$$

The origin of t is the instant at which the centre of the primary wave would coincide with the plane $y=0$ if the screen were absent. It may also be noted that the values of $\xi+\eta$, $\xi-\eta$ in the four quadrants of the plane xy are respectively as follows :—

$$\left.\begin{aligned}\xi+\eta &= +\sqrt{(r+y)},\ +\sqrt{(r+y)},\ +\sqrt{(r+y)},\ -\sqrt{(r+y)},\\ \xi-\eta &= +\sqrt{(r-y)},\ -\sqrt{(r-y)},\ -\sqrt{(r-y)},\ -\sqrt{(r-y)},\end{aligned}\right\} \tag{31}$$

as appears at once from (7) and (8).

3. To interpret this solution we notice in the first place that when the ratio ct/b is large and negative we have, in the region for which $y > 0$,

$$\beta = -\tfrac{1}{2}\pi, \quad \omega = -\tfrac{1}{2}\pi,$$

approximately. The formula (29) therefore reduces to

$$\phi = \frac{\cos^2\alpha}{2b}\left\{1+\sqrt{\left(\frac{r+y}{b}\right)}\cos^{\frac{1}{2}}\omega\right\}, \tag{32}$$

i.e., since

$$(r+y)/b = \tan\alpha - \tan\omega, \tag{33}$$

practically to the form (30), the primary wave being as yet unaffected by the presence of the screen. In the region where $y < 0$ we have

$$\alpha = -\tfrac{1}{2}\pi, \quad \omega = -\tfrac{1}{2}\pi,$$

nearly, whence, having regard to (31),

$$\phi = \frac{\cos^2\beta}{2b}\left\{1-\sqrt{\left(\frac{r-y}{b}\right)}\cos^{\frac{1}{2}}\omega\right\}. \tag{34}$$

Since

$$(r-y)/b = \tan\beta - \tan\omega, \tag{35}$$

this is practically insensible. That the formula should indicate any disturbance at all on the far side of the plane $y = 0$, at this stage, is due to the fact that our assumed primary wave has no clearly defined front.

When, on the other hand, ct/b is large and positive, we have, in the region for which $y > 0$, $\alpha = \frac{1}{2}\pi$, and therefore $\cos\alpha = 0$, nearly. The value of $\cos\beta$ will also be small except in the neighbourhood of the plane $y = ct$. Excluding for the present the critical region near the plane $x = 0$, we will further suppose that $r - ct$ is large compared with b, so that $\omega = -\frac{1}{2}\pi$, nearly. The formula thus reduces to

$$\phi = \frac{\cos^2\beta}{2b}\left\{1 \pm \sqrt{\left(\frac{r-y}{b}\right)}\cos^{\frac{1}{2}}\omega\right\}, \tag{36}$$

where the upper or lower sign is to be taken according as $x \gtrless 0$. In virtue of (35) this is equivalent to

$$\phi = \frac{\cos^2\beta}{b}, \quad \text{or} \quad 0. \tag{37}$$

Hence in the first quadrant of the plane xy we have a regularly reflected wave

$$\phi = \frac{b}{b^2+(ct-y)^2}, \tag{38}$$

whilst in the second quadrant the disturbance is insensible. Again, in the region for which $y < 0$, we have $\beta = \frac{1}{2}\pi$, nearly, and the more important part of the disturbance is given by

$$\phi = \frac{\cos^2 \alpha}{2b} \left\{ 1 \mp \sqrt{\left(\frac{r+y}{b}\right)} \cos^{\frac{1}{2}} \omega \right\}, \tag{39}$$

where the rule of signs is as before. In virtue of (33) this reduces to

$$\phi = 0, \quad \text{or} \quad \frac{\cos^2 \alpha}{b}. \tag{40}$$

The disturbance is accordingly negligible in the region behind the screen, in the fourth quadrant, whilst in the third quadrant the primary wave proceeds undisturbed.

That the main part of the disturbance at any instant is to be sought in the neighbourhood of the planes $y = \pm ct$ is indicated already by the formula (27). The same equation suggests, in addition, an examination of the circumstances in the neighbourhood of the cylindrical surface $r = ct$. If we exclude the neighbourhood of the points where this touches the planes $y = \pm ct$, we have $\alpha = \frac{1}{2}\pi$, $\beta = \frac{1}{2}\pi$, nearly; and therefore

$$\cos \alpha = b/(r+y), \qquad \cos \beta = b/(r-y), \tag{41}$$

approximately. The formula (29) then gives

$$\phi = \frac{1}{2\sqrt{b}} \left\{ \frac{\pm 1}{\sqrt{(r+y)}} + \frac{\pm 1}{\sqrt{(r-y)}} \right\} \cos^{\frac{1}{2}} \omega \cos(\tfrac{1}{2}\omega + \tfrac{3}{4}\pi), \tag{42}$$

where the combinations of signs to be taken in the several quadrants are $++$, $+-$, $+-$, $--$ respectively. This may be interpreted as representing a cylindrical wave diverging from the edge of the screen; but the amplitude at a distance r from the edge, in any assigned direction, is of the order $\sqrt{(b/r)}$ as compared with that of the primary wave, and therefore usually negligible.

4. It remains to examine the conditions near the edges of the reflected and transmitted waves. Since the circumstances are exactly similar in these two regions it will be sufficient to take the latter case. At the medial plane of the transmitted wave we have $y = -ct$, and we will suppose as before that the ratio ct/b is large. We have then $\alpha = 0$, whilst β is nearly $= \frac{1}{2}\pi$. Also

$$\xi + \eta = \mp \sqrt{(r+y)} = \mp \sqrt{(r-ct)} = \mp \sqrt{(-b \tan \omega)}. \tag{43}$$

Thus (29) becomes

$$\phi = \frac{1}{2b}\left\{1 \mp \sqrt{(-\sin\omega)}\cos\left(\tfrac{1}{2}\omega + \tfrac{1}{4}\pi\right)\right\}, \tag{44}$$

where ω ranges from 0 (at the edge of the geometrical shadow) to $-\frac{1}{2}\pi$.

Again, over the plane in question, we have $r^2 - c^2t^2 = x^2$, and therefore, if x be small compared with ct,

$$-\tan\omega = x^2/2bct. \tag{45}$$

The following table gives a series of corresponding values of $x/\sqrt{(2bct)}$ and $b\phi$, calculated from (45) and (44); and the results are exhibited graphically in Fig. 3. It will be seen that the value of ϕ shades off gradually, without fluctuation, in the neighbourhood of the edge of the geometrical shadow. The transition is the more rapid the smaller the value of b, *i.e.*, the smaller the breadth of the primary wave, the horizontal scale being proportional to $\sqrt{b}$.

$\pm\frac{x}{\sqrt{(2bct)}}$	$b\phi$	$\pm\frac{x}{\sqrt{(2bct)}}$	$b\phi$
0	$\begin{cases}\cdot500\\\cdot500\end{cases}$	·852	$\begin{cases}\cdot158\\\cdot842\end{cases}$
·281	$\begin{cases}\cdot397\\\cdot603\end{cases}$	1·000	$\begin{cases}\cdot112\\\cdot888\end{cases}$
·398	$\begin{cases}\cdot350\\\cdot650\end{cases}$	1·173	$\begin{cases}\cdot072\\\cdot928\end{cases}$
·490	$\begin{cases}\cdot310\\\cdot690\end{cases}$	1·401	$\begin{cases}\cdot041\\\cdot959\end{cases}$
·570	$\begin{cases}\cdot275\\\cdot725\end{cases}$	1·754	$\begin{cases}\cdot018\\\cdot982\end{cases}$
·644	$\begin{cases}\cdot243\\\cdot757\end{cases}$	2·513	$\begin{cases}\cdot005\\\cdot995\end{cases}$
·714	$\begin{cases}\cdot213\\\cdot787\end{cases}$	∞	$\begin{cases}0\\1\cdot000\end{cases}$

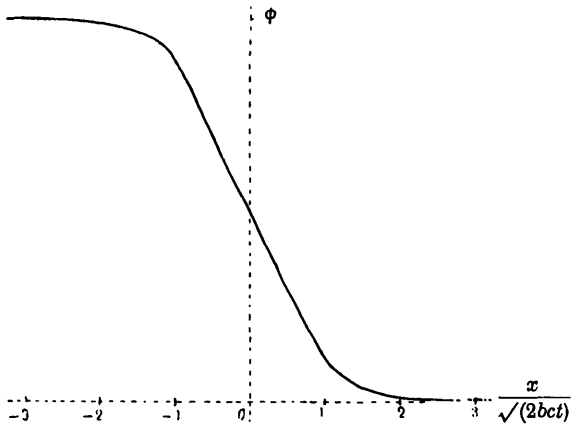

FIG. 3.

The figure gives an instantaneous view of the distribution of ϕ in the medial plane of the transmitted wave, but since the epochs of maximum amplitude are different for different points of any fixed plane $y =$ const., it is desirable to supplement it by an investigation of the whole history of the disturbance at various points, especially at such as are near the edge of the geometrical shadow.

This may be conducted as follows. Fixing our attention on any one point we may conveniently take the angle α as our independent variable, this being connected with t by the relation

$$ct = -y + b \tan \alpha. \tag{46}$$

The corresponding value of ω is given by

$$\tan \omega = \tan \alpha - (r+y)/b, \tag{47}$$

whilst β can be found, if necessary, from the relation

$$\tan \beta = \tan \alpha - 2y/b. \tag{48}$$

It appears, however, from (28) that when ct and $-y$ are both large compared with b, the angle β is very nearly equal to $\frac{1}{2}\pi$, so that the terms in (29) which contain $\cos \beta$ have little influence during the more important part of the disturbance. We have then, approximately,

$$\phi = \frac{1}{2b} \left\{ \cos^2 \alpha \pm \sqrt{\left(\frac{r+y}{b}\right)} \cos^{\frac{1}{2}} \omega \cos \alpha \cos \left(\tfrac{1}{2}\omega + \alpha + \tfrac{1}{4}\pi\right) \right\}. \tag{49}$$

This function can be tabulated without very much labour. I have chosen for computation the points for which

$$r+y = b, \quad \text{or} \quad x^2 = -2by. \tag{50}$$

At any given distance $(-y)$ from the plane of the screen there are two of these, one lying just outside, and the other just inside the geometrical shadow. The resulting curves, shewing the variation of ϕ with t, are given in Fig. 4. The scale corresponds to that of Fig. 2, which may now be taken to represent the history of the disturbance which would be produced at any point by the primary wave alone.

In the later stages of the disturbance, the terms neglected in (49) may of course become comparable with those which are retained.

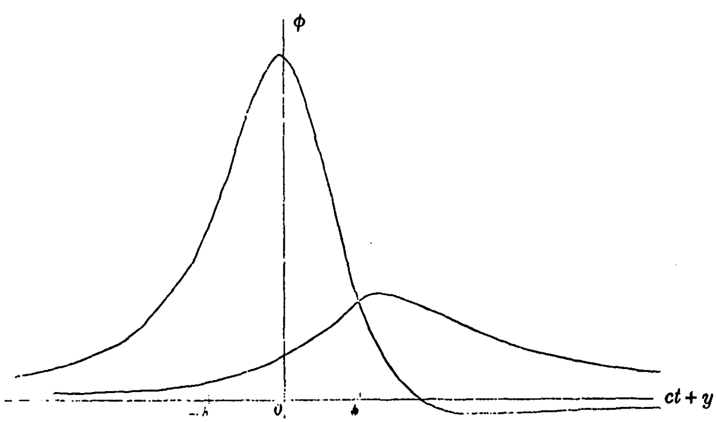

FIG. 4.

5. The second form of the diffraction problem, where the condition (3) which was to be satisfied at both faces of the screen is replaced by

$$\phi = 0, \tag{51}$$

need not be dwelt upon at length. The general solution corresponding to a primary wave of the type (1) is

$$\phi = \int_0^{\xi+\eta} f(ct+y-\zeta^2)\, d\zeta - \int_0^{\xi-\eta} f(ct-y-\zeta^2)\, d\zeta + \tfrac{1}{2}F(ct+y) - \tfrac{1}{2}F(ct-y), \tag{52}$$

where the function $f(y)$ is determined by (15).

In the case of
$$F(y) = \frac{1}{b+iy}, \tag{53}$$

we should find

$$\phi = \frac{\frac{1}{2}}{b+i(ct+y)} - \frac{\frac{1}{2}}{b+i(ct-y)} + \frac{\frac{1}{2}e^{-\frac{1}{4}i\pi}}{\sqrt{\{b+i(ct-r)\}}}\left\{\frac{\xi+\eta}{b+i(ct+y)} - \frac{\xi-\eta}{b+i(ct-y)}\right\}, \tag{54}$$

the real part of which is

$$\phi = \frac{1}{2b}\left\{\cos^2\alpha-\cos^2\beta+\frac{\xi+\eta}{\sqrt{b}}\cos^{\frac{1}{2}}\omega\cos\alpha\cos(\tfrac{1}{2}\omega+\alpha+\tfrac{1}{4}\pi) - \frac{\xi-\eta}{\sqrt{b}}\cos^{\frac{1}{2}}\omega\cos\beta\cos(\tfrac{1}{2}\omega+\beta+\tfrac{1}{4}\pi)\right\}, \tag{55}$$

in the notation of (28).

The interpretation would take exactly the same course as before, the chief difference being in the sign of the reflected wave. The main part of the disturbance on the far side of the plane $y = 0$ will be given as before by the formula (49); and the curves in Fig. 4 will still serve to represent the course of events near the edge of the shadow.

6. The preceding results admit of a great variety of applications. Their relation to the theory of electric waves has been indicated in § 1; but even if we confine ourselves to acoustics we find that there is still a considerable freedom of interpretation. Thus in the form of the problem treated in §§ 1–4 the function ϕ may be taken to represent either the velocity-potential or the condensation. In § 5, again, ϕ may be identified with the component (v) of the velocity normal to the plane xz, the positive half of this plane being occupied as before by a rigid screen.

If in the particular case to which Fig. 2 refers we identify ϕ with the *condensation* (usually denoted by s), we have a primary wave in which s has one sign only. It has been pointed out, however, by Stokes that a plane wave which is merely the ultimate form of a wave diverging from a source must necessarily contain both condensed and rarefied portions. A wave of this character, such as may be supposed to have originated in a local condensation, is obtained if in equation (30) we take ϕ to represent the *velocity-potential*, from which s is derived by the relation

$$c^2 s = \frac{\partial\phi}{\partial t}. \tag{56}$$

The consequent distribution of s is shewn in Fig. 5.

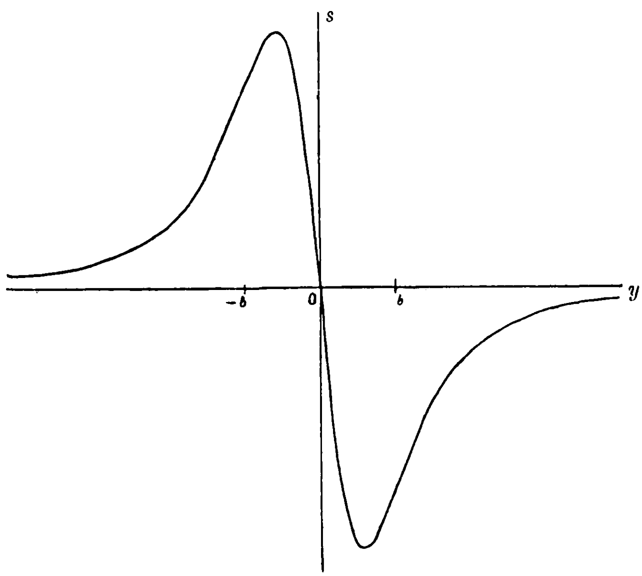

FIG. 5.

If we deal with the complex form (27), we deduce

$$cs = -\frac{\frac{1}{2}i}{\{b+i(ct+y)\}^2} - \frac{\frac{1}{2}i}{\{b+i(ct-y)\}^2}$$

$$-\frac{\frac{1}{4}ie^{-\frac{1}{4}i\pi}}{\{b+i(ct-r)\}^{\frac{3}{2}}}\left\{\frac{\xi+\eta}{b+i(ct+y)} + \frac{\xi-\eta}{b+i(ct-y)}\right\}$$

$$-\frac{\frac{1}{2}ie^{-\frac{1}{4}i\pi}}{\{b+i(ct-r)\}^{\frac{1}{2}}}\left\{\frac{\xi+\eta}{\{b+i(ct+y)\}^2} + \frac{\xi-\eta}{\{b+i(ct-y)\}^2}\right\}. \qquad (57)$$

Introducing the notations of (28), and rejecting the imaginary part, we obtain

$$cs = -\frac{1}{2b^2}\left\{2\sin\alpha\cos^3\alpha + 2\sin\beta\cos^3\beta\right.$$

$$+\frac{\xi+\eta}{2\sqrt{b}}\cos^{\frac{3}{2}}\omega\cos\alpha\sin\left(\tfrac{3}{2}\omega+\alpha+\tfrac{1}{4}\pi\right) + \frac{\xi-\eta}{2\sqrt{b}}\cos^{\frac{3}{2}}\omega\cos\beta\sin\left(\tfrac{3}{2}\omega+\beta+\tfrac{1}{4}\pi\right)$$

$$\left.+\frac{\xi+\eta}{2\sqrt{b}}\cos^{\frac{1}{2}}\omega\cos^2\alpha\sin\left(\tfrac{1}{2}\omega+2\alpha+\tfrac{1}{4}\pi\right) + \frac{\xi-\eta}{2\sqrt{b}}\cos^{\frac{1}{2}}\omega\cos\beta\sin\left(\tfrac{1}{2}\omega+2\beta+\tfrac{1}{4}\pi\right)\right\}, \qquad (58)$$

corresponding to $$cs = -\frac{2b(ct+y)}{\{b^2+(ct+y)^2\}^2} = -\frac{2\sin\alpha\cos^3\alpha}{b^2}, \tag{59}$$
in the primary wave.

It seems unnecessary to go through the interpretation in detail. The extreme values of s in the primary wave correspond to $\alpha = \pm\frac{1}{6}\pi$, and it is possible without much trouble to obtain an instantaneous picture of the distribution of s in the planes

$$y = -ct \pm \frac{1}{\sqrt{3}}\, b,$$

but the circumstance that the various particles are caught, so to speak, in different phases of their evolutions detracts, even more than in the case of Fig. 4, from the significance of the results. The main interest consists in tracing the history of the disturbance near the edge of the geometrical shadow, at a great distance from the screen. We have then, with sufficient accuracy,

$$cs = -\frac{1}{2b^2}\left\{2\sin\alpha\cos^3\alpha \mp \tfrac{1}{2}\sqrt{\left(\frac{r+y}{b}\right)}\cos^{\frac{3}{2}}\omega\cos\alpha\sin\left(\tfrac{3}{2}\omega+\alpha+\tfrac{1}{4}\pi\right)\right.$$
$$\left.\mp\sqrt{\left(\frac{r+y}{b}\right)}\cos^{\frac{1}{2}}\omega\cos^2\alpha\sin\left(\tfrac{1}{2}\omega+\alpha+\tfrac{1}{4}\pi\right)\right\}. \tag{60}$$

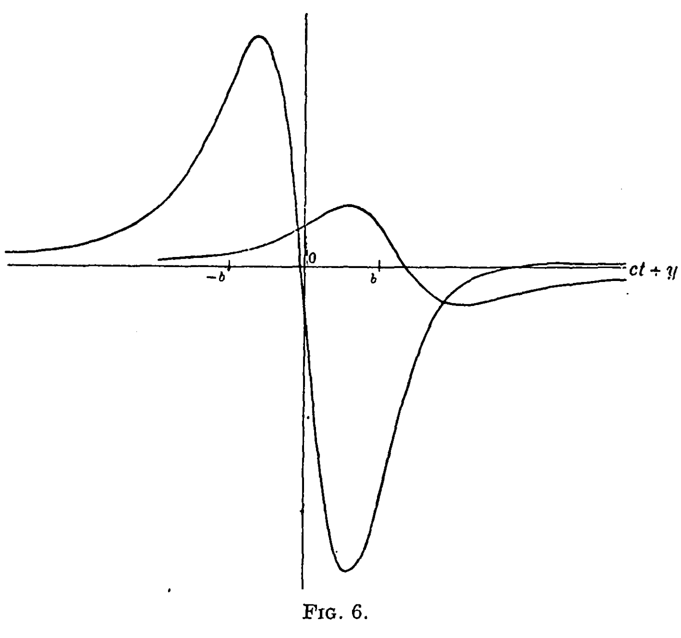

FIG. 6.

I have computed from this formula the variation of s with t at the points, just outside and just inside the geometrical shadow, for which $r+y=b$. The results are shewn in Fig. 6, which corresponds in scale with Fig. 5. The terms neglected in (60) would of course be comparable with those retained, in the later stages of the disturbance.

7. One curious point remains to be noticed. In the case of the formula (27) the time-integral

$$\int_{-\infty}^{\infty} \phi \, dt \tag{61}$$

has the same value, viz. π/c, in all parts of the field. This easily follows by the method of contour integration, for the integral of each term in the second line of the formula referred to, taken along the real axis of t and an infinite semicircle on the negative side of this axis, must vanish, since the singular points $t=(r+ib)/c$, $t=(\pm y+ib)/c$ lie outside the region thus bounded. The fact that the integral (61) has the same value at points well within the shadow as in the primary wave shews that the smaller amplitude at such points must be compensated by a longer duration. This is already indicated to some extent by the form of the second curve in Fig. 4.

It appears in the same way that in the case of (54) the integral (61) everywhere vanishes. It has already been remarked that in the form of the problem to which (54) relates ϕ may be taken to represent the y-component of the velocity in a wave incident on a rigid semi-infinite screen. It follows that on this view the total displacement

$$\int_{-\infty}^{\infty} v \, dt$$

of a particle in the direction normal to the screen is zero at *all* points, whereas, if the primary wave (30) had not been interrupted, the displacement would have had the uniform value π/c, on the scale of our formulæ. The conclusion may be unexpected, especially as regards particles which would at first sight appear to be far removed from the influence of the screen, but the obvious explanation is that a considerable positive displacement is followed sooner or later by a negative one of much smaller extent, but correspondingly greater duration. The first curve in Fig. 4, which is valid also as an approximation for a certain range of t in the present form of the question, indicates how this may come about. To follow out the matter completely, it would be necessary to take account of the terms in (55) which we have neglected, but which ultimately become comparable with those retained.

It will be surmised that the uniformity of the value of the integral (61) throughout the medium is a result not confined to the particular types of wave here considered, or even to the case of plane waves. A more general proof may be sought in the following manner. If we integrate both sides of the equation

$$\frac{\partial^2\phi}{\partial t^2} = c^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) \tag{62}$$

with respect to t between the limits $-\infty$ and $+\infty$, and write

$$\psi = \int_{-\infty}^{\infty} \phi\, dt, \tag{63}$$

we find

$$\nabla^2\psi = 0, \tag{64}$$

on the hypothesis that $\partial\phi/\partial t$ vanishes at both limits. The theorem

$$\iiint \left\{\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right\} dx\, dy\, dz = -\iint \psi \frac{\partial\psi}{\partial n}\, dS, \tag{65}$$

applied to an infinite region with fixed internal boundaries, then shews that ψ is constant, provided we can assert that the part of the surface integral which relates to an infinite enclosing boundary vanishes. Its precise value may then be found by a consideration of the circumstances at some particular point. Thus in our present case of a plane screen, where $v = 0$, the value must be zero if ϕ denotes the velocity-component v.