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NATURWISSENSCHAFTLICHE
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The Discontinuous Galerkin method for coastal ocean modeling

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2016-12-09

Outline

Model

Application to the Lena delta

Model

3D shallow water equations

System of hydrostatic primitive equations with free surface

Free surface elevation
$$\partial_t \xi + \boldsymbol{\nabla}_{xy} \cdot \int_z^\xi \boldsymbol{u}_{xy} \mathrm{d}\tilde{z} = 0\,,$$

Continuity equation
$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0\,,$$

Momentum equations in conservative form

$$\partial_t \boldsymbol{u}_{xy} + \boldsymbol{\nabla} \cdot (\boldsymbol{u}_{xy} \otimes \boldsymbol{u}) - \partial_z(\nu_t \partial_z \boldsymbol{u}_{xy}) + g\boldsymbol{\nabla}_{xy}(\xi + \frac{1}{\rho_0}\int_z^\xi \rho \mathrm{d}\tilde{z}) - f_c \boldsymbol{e}_z \times \boldsymbol{u}_{xy} = \boldsymbol{f}_u\,,$$

Transport equation(s) for temperature / salinity $\partial_t r + \boldsymbol{\nabla} \cdot (\boldsymbol{u} r) - \partial_z(\nu_r \partial_z r) = f_r$ for $r \in \{s, \theta\}$,

Transport equation(s)[1] for turbulence variables $\partial_t m - \partial_z(\nu_m \partial_z m) = f_m\left(\|\partial_z \boldsymbol{u}_{xy}\|^2, \partial_z \rho\right)$ for $m \in \{k, \psi\}$.

ξ – free surface elevation, $\boldsymbol{\nabla}_{xy} = (\partial_x, \partial_y)$ $f_\cdot$ – body forces,
$\boldsymbol{u} = (u, v, w)$ – velocity, $\boldsymbol{u}_{xy} = (u, v)$, $\nu_\cdot$ – eddy viscosity coefficients,
θ – temperature, $\boldsymbol{e}_z = (0, 0, 1)$, $\rho = \rho(\theta, s)$ – density (equation of state),
s – salinity, g – gravity constant, f_c – Coriolis coefficient.

[1] Umlauf, Burchard (2003), doi:10.1357/002224003322005087

3D shallow water equations

System of hydrostatic primitive equations with free surface

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Continuity equation $$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0\,,$$

Barotropic model

Momentum equations in conservative form

$$\partial_t \boldsymbol{u}_{xy} + \boldsymbol{\nabla} \cdot (\boldsymbol{u}_{xy} \otimes \boldsymbol{u}) - \partial_z(\nu_t \partial_z \boldsymbol{u}_{xy}) + g \boldsymbol{\nabla}_{xy}(\xi + \cancel{\frac{1}{\rho_0}\int_z^{\xi}} \rho \mathrm{d}\tilde{z}) - f_c \boldsymbol{e}_z \times \boldsymbol{u}_{xy} = \boldsymbol{f}_u\,,$$

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3D shallow water equations

System of hydrostatic primitive equations with free surface

Baroclinic model

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Baroclinic model with turbulence

Momentum equations in conservative form

$$\partial_t \boldsymbol{u}_{xy} + \boldsymbol{\nabla} \cdot (\boldsymbol{u}_{xy} \otimes \boldsymbol{u}) - \partial_z(\nu_t \partial_z \boldsymbol{u}_{xy}) + g \boldsymbol{\nabla}_{xy}(\xi + \frac{1}{\rho_0} \int_z^\xi \rho \mathrm{d}\tilde{z}) - f_c \boldsymbol{e}_z \times \boldsymbol{u}_{xy} = \boldsymbol{f}_u\,,$$

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ξ – free surface elevation, $\boldsymbol{u} = (u, v, w)$ – velocity, θ – temperature, s – salinity,

$\boldsymbol{\nabla}_{xy} = (\partial_x, \partial_y)$, $\boldsymbol{u}_{xy} = (u, v)$, $\boldsymbol{e}_z = (0, 0, 1)$, g – gravity constant,

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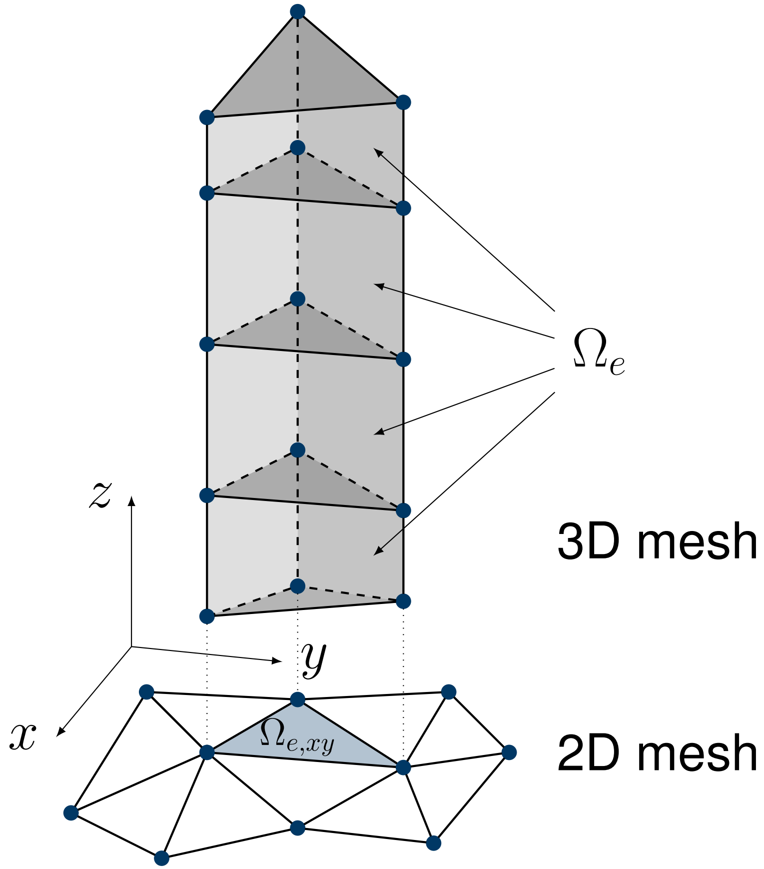
[1] Umlauf, Burchard (2003), doi:10.1357/002224003322005087

UTBEST3D[6]

- Physics:
  - **Baroclinic** or **barotropic** system
  - **Wetting/drying**
  - **Vertical eddy viscosity**: algebraic, 1st, and 2nd order closures
- Numerics[2]:
  - up to 2nd order **discontinuous Galerkin** approx. in space and up to 3rd in time
  - Non-conforming **prismatic meshes**
  - Local mesh and space refinement (**hp-adaptivity**)
  - Individual choice of discretization for each unknown
  - **Riemann solvers**: Lax-Friedrichs, Roe's, HLL, HLLC
  - **Slope limiters**: Vertex-based[3]
- Written in `C++`, **parallelized** with MPI and OpenMP
- Domain partitioning with METIS[4]
- **Scalable** and **portable**[5]: runs on x86, Power, ARM

[2] Dawson, Aizinger (2005), doi:10.1007/s10915-004-4139-3 — Aizinger, Dawson (2007), doi:10.1016/j.cma.2006.04.010
[3] Aizinger (2011), doi:10.1007/978-3-642-17770-5_16 — Kuzmin (2010), doi:10.1016/j.cam.2009.05.028
[4] Karypis, Kumar (1999), doi:10.1137/S1064827595287997
[5] Reuter, Aizinger, Köstler (2015), doi:10.1016/j.compfluid.2015.05.020
[6] `https://math.fau.de/utbest3d`

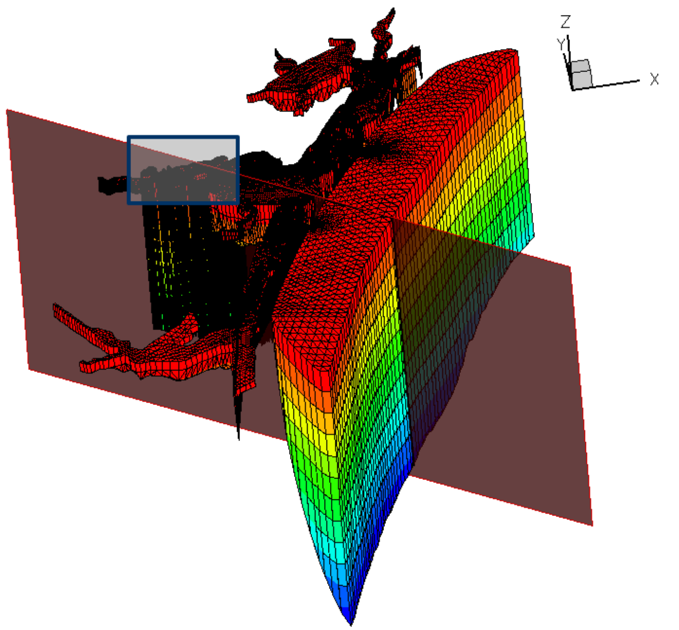
Mesh

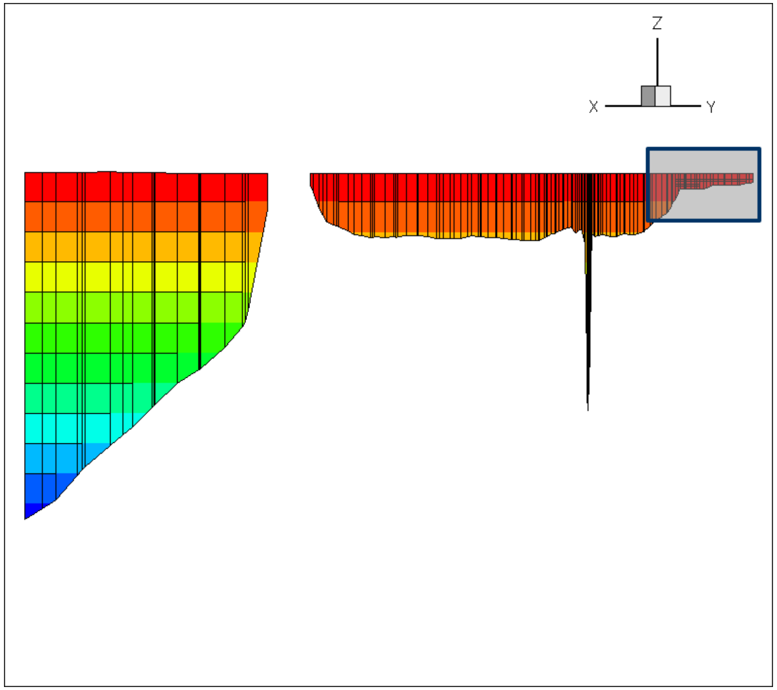


- Input data: 2D **triangulation** with **bathymetry** at nodes
- 3D mesh: columns of **prismatic elements** aligned with 2D elements
- **Non-parallel** faces on top and bottom
- Dynamic **merging/unmerging** of thin elements
- Different layer structure in subdomains possible

Dynamic merging and subdomains: Corpus Christi Bay circulation I

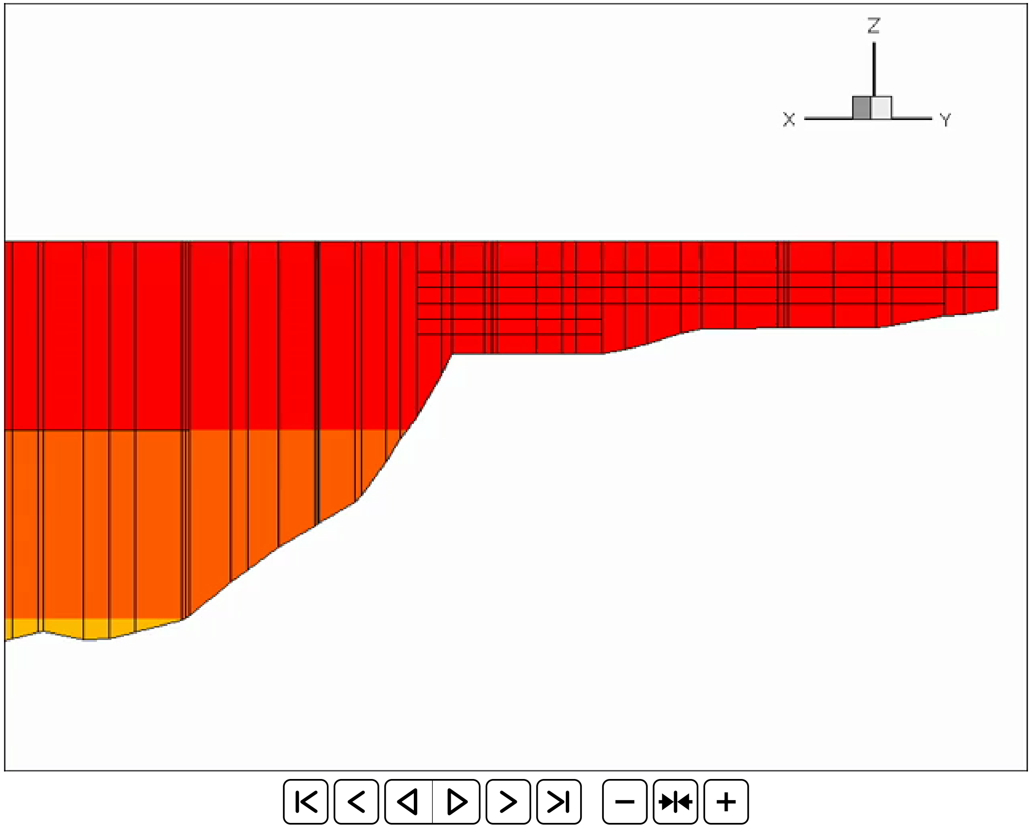
- Area of interest: **Nueces Bay**, max. depth 2m
- 12 vertical layers. Layer thickness in Nueces Bay: 0.16m, outside: 2m





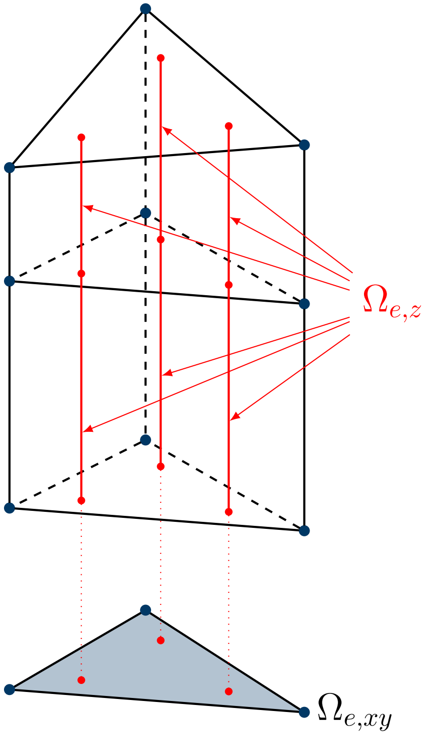
Dynamic merging and subdomains: Corpus Christi Bay circulation II

- Nueces bay mesh: 12 x 0.16m layers. Global mesh: 12 x 2m layers

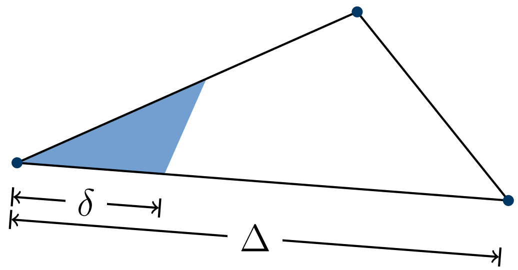


Segments for turbulence variables

- Vertical **segments** aligned with 2D quadrature points.
- Turbulence quantities are **1D variables**



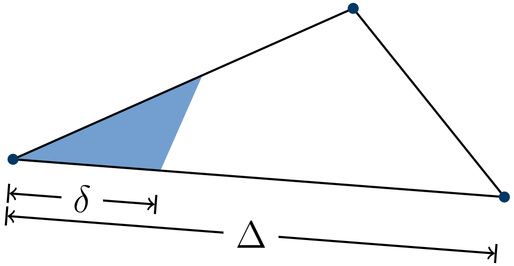
Wetting / Drying

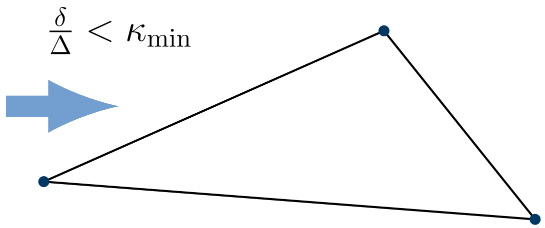


Traditionally: two approaches

1. Partially wet elements
   - Dry elements contain **some water**
     or
   - Wet elements with **dry areas**
2. Split cells: elements subdived into wet and dry parts

Wetting / Drying





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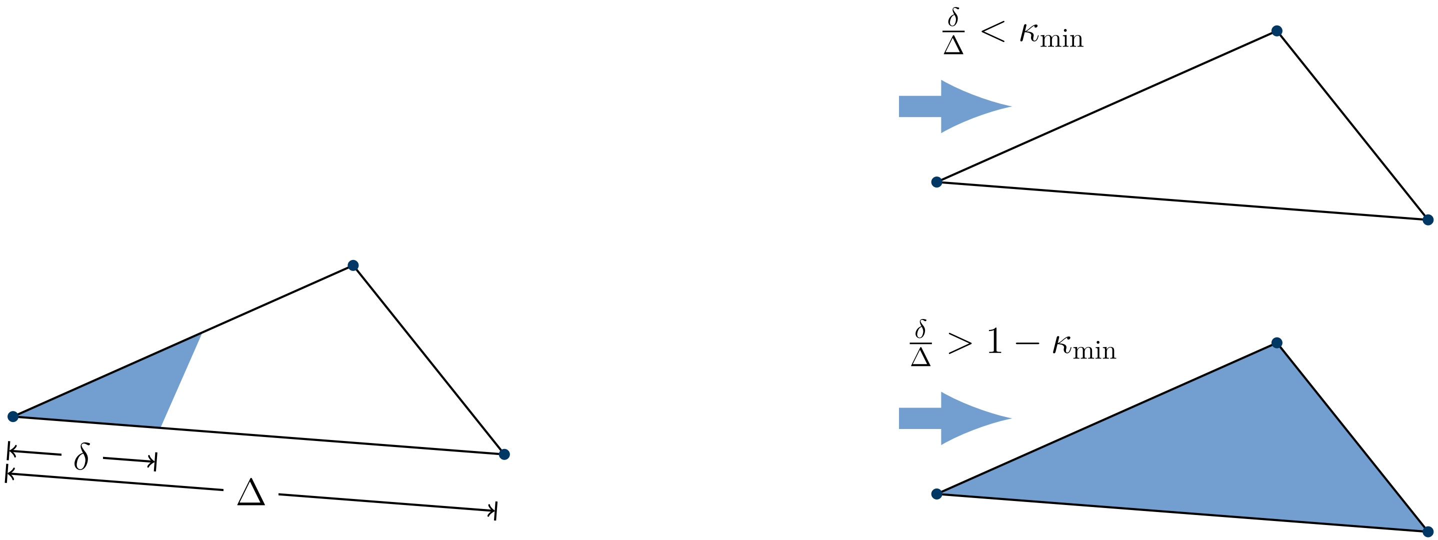

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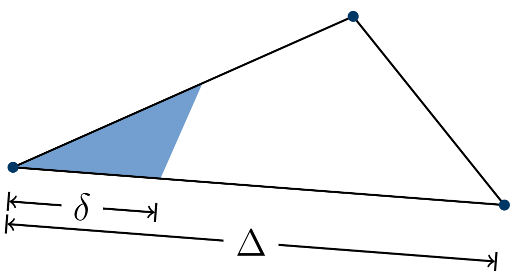

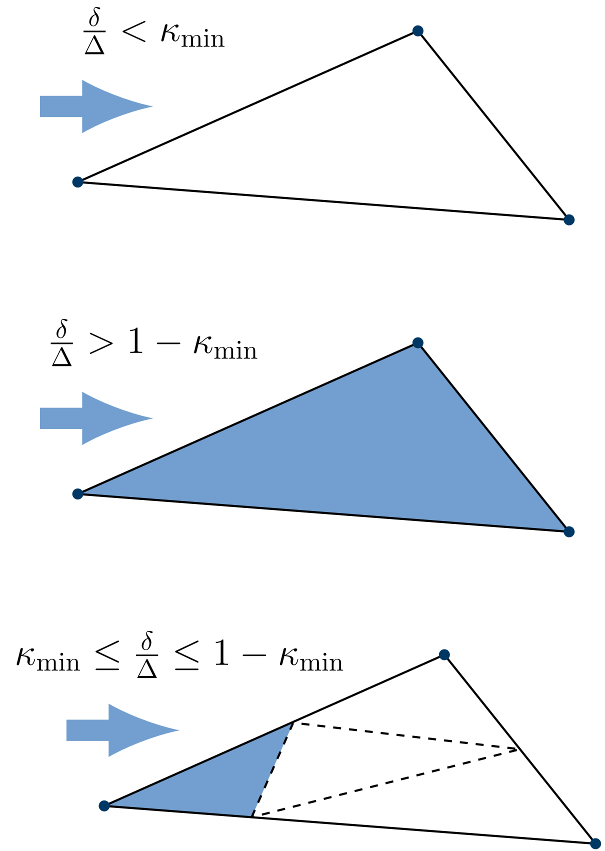

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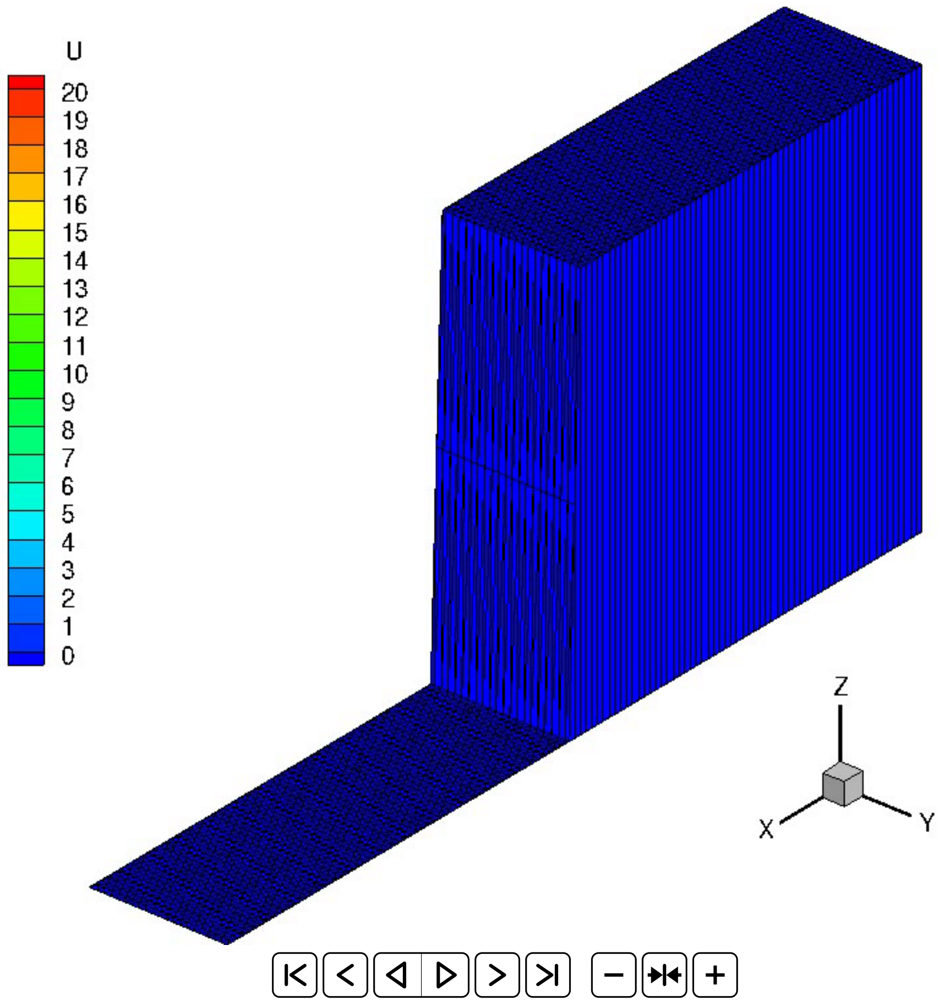
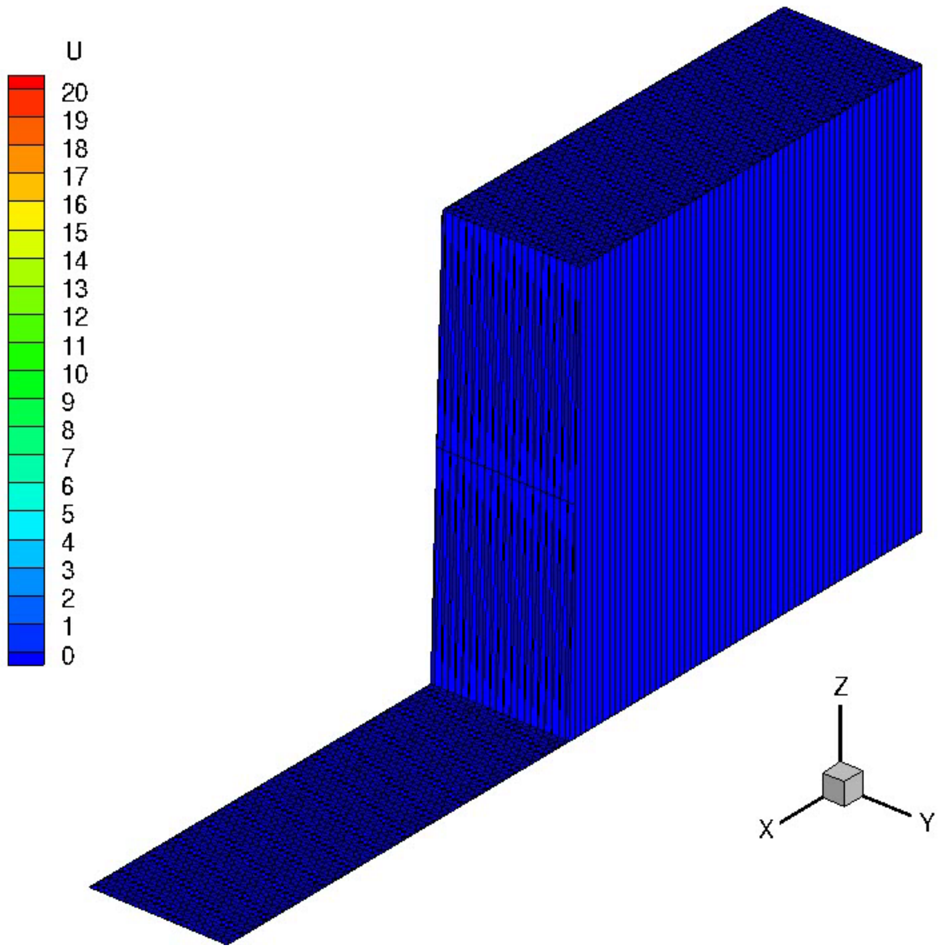
Dam Break

Piecewise constant approximation



Dam Break

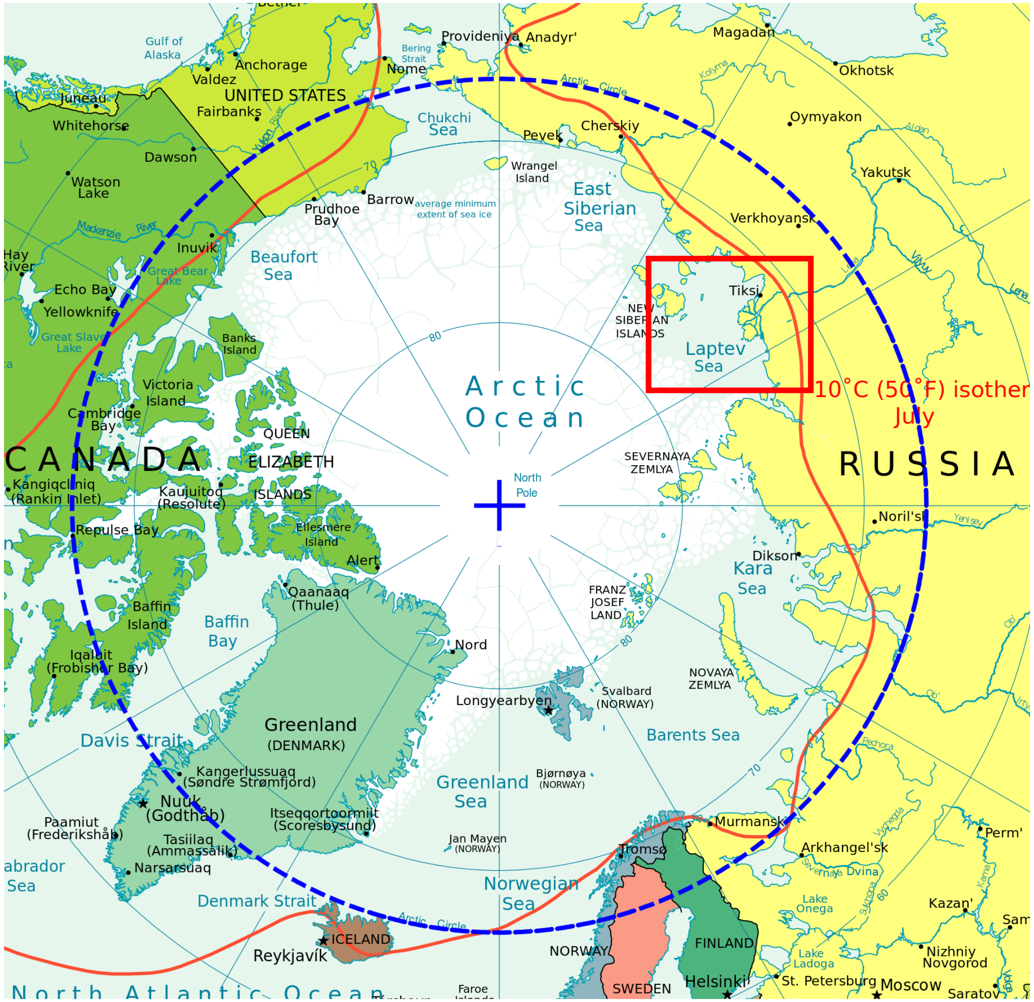
Piecewise linear approximation



Application to the Lena delta

Lena delta - problem setting (1)

- Joint work with **Vera Fofonova (Alfred Wegener Institute, Bremerhaven)** and **Tony Ying**
- Second largest river discharging in the Arctic Ocean
- Substantial impact in the context of climate change



Lena delta - problem setting (2)

- Highly complex delta geometry
- A number of unique physical processes and characteristics
- Sensitive environmental system
- New study object for 3D numerical modeling

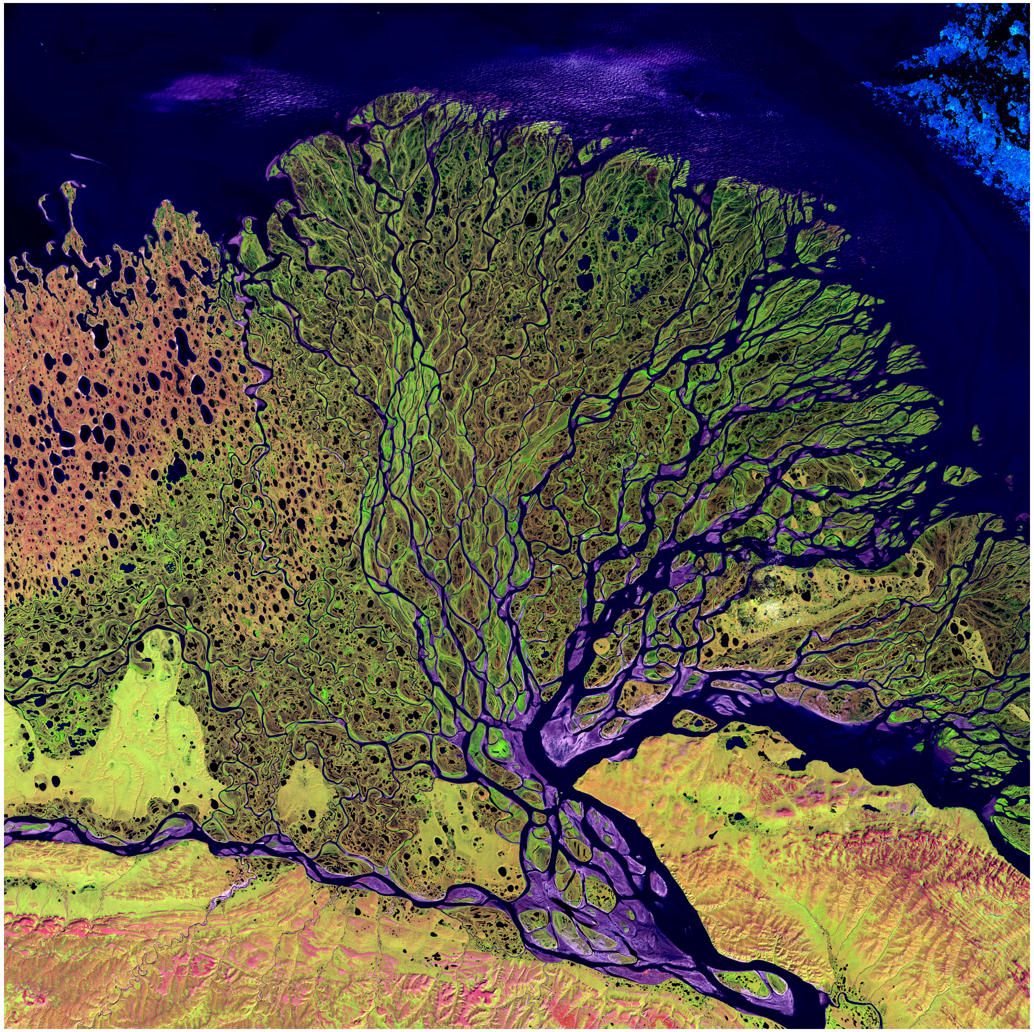
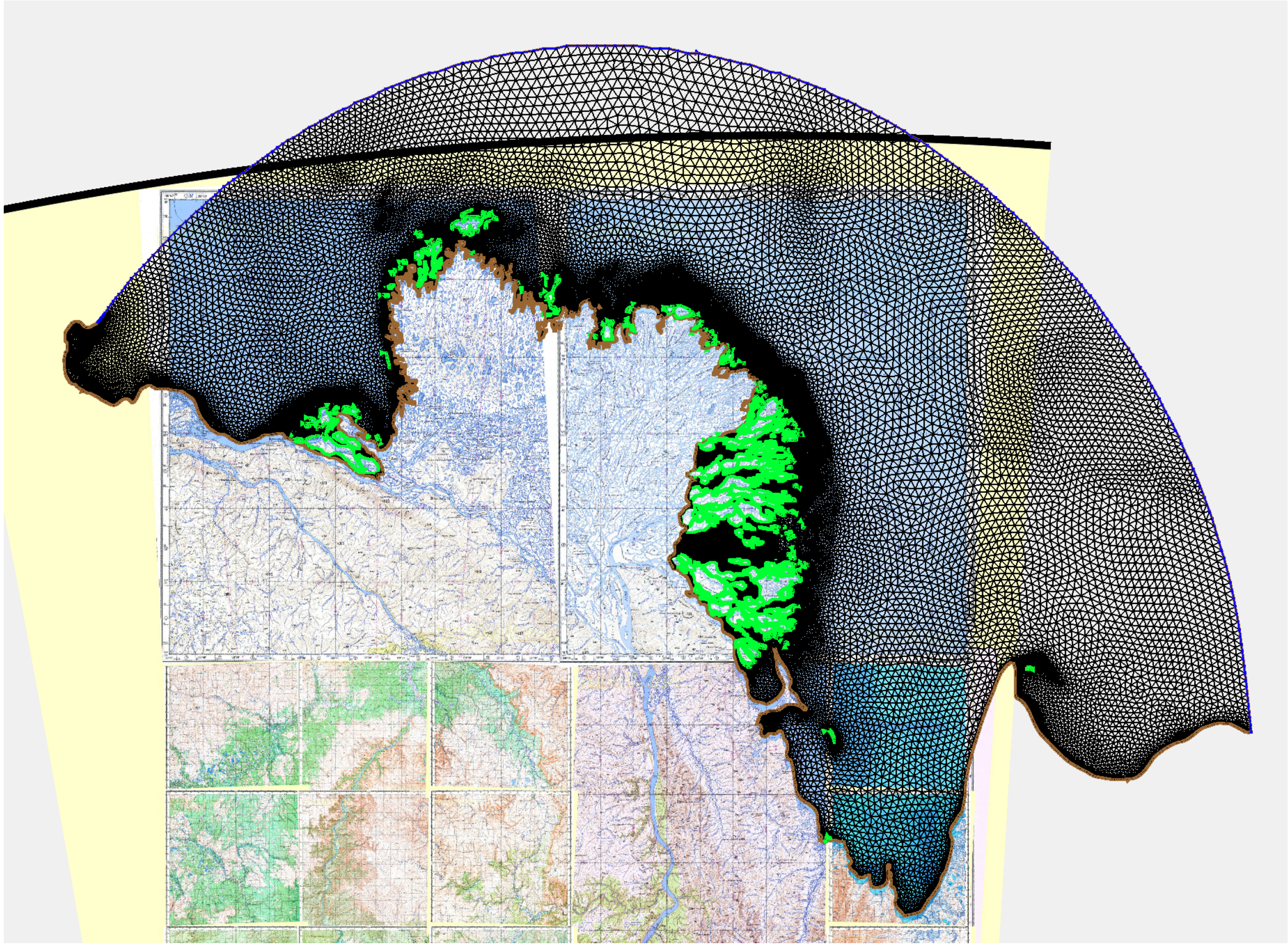


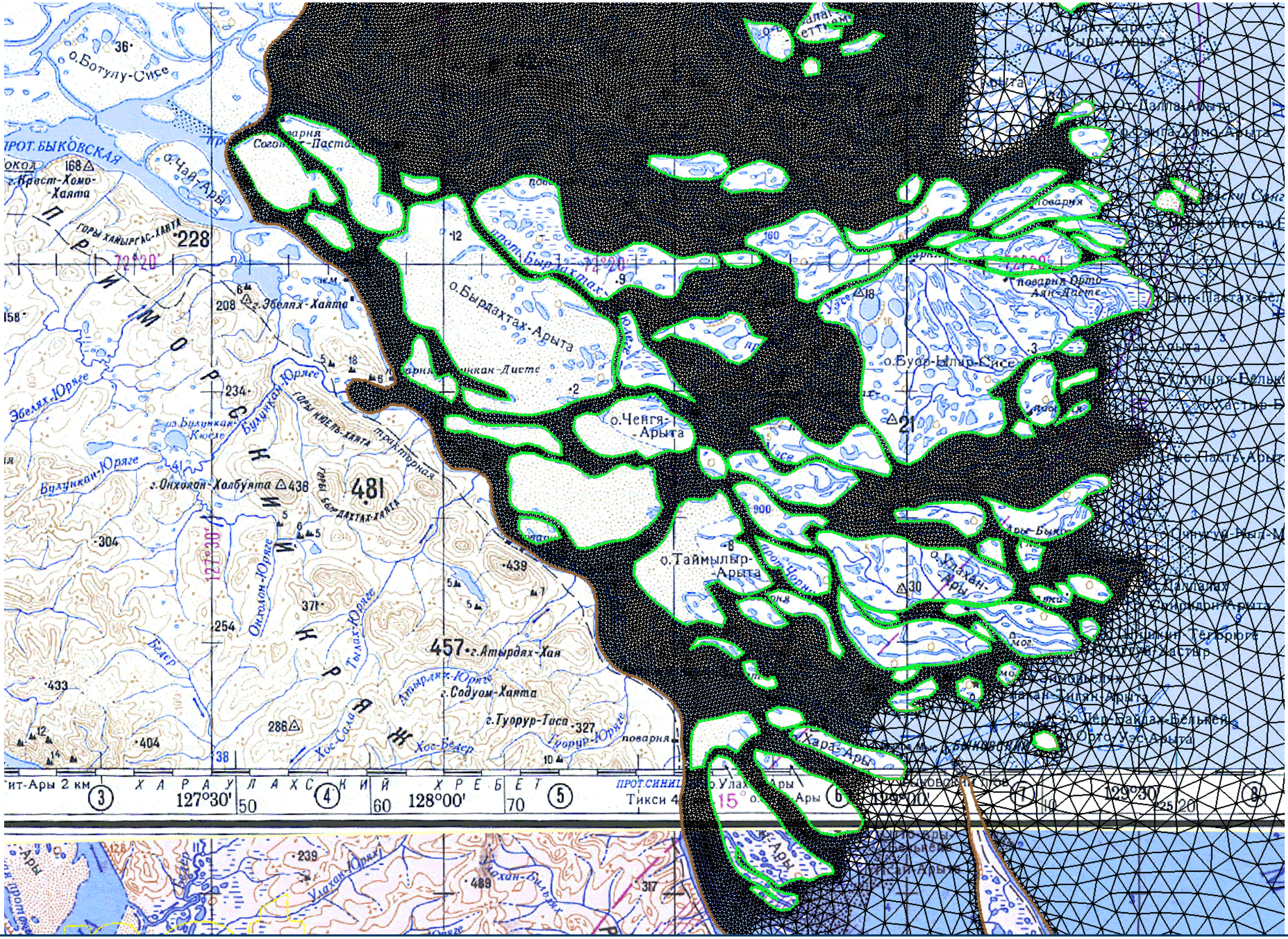
Image: NASA Landsat

Lena delta - data challenge (1)

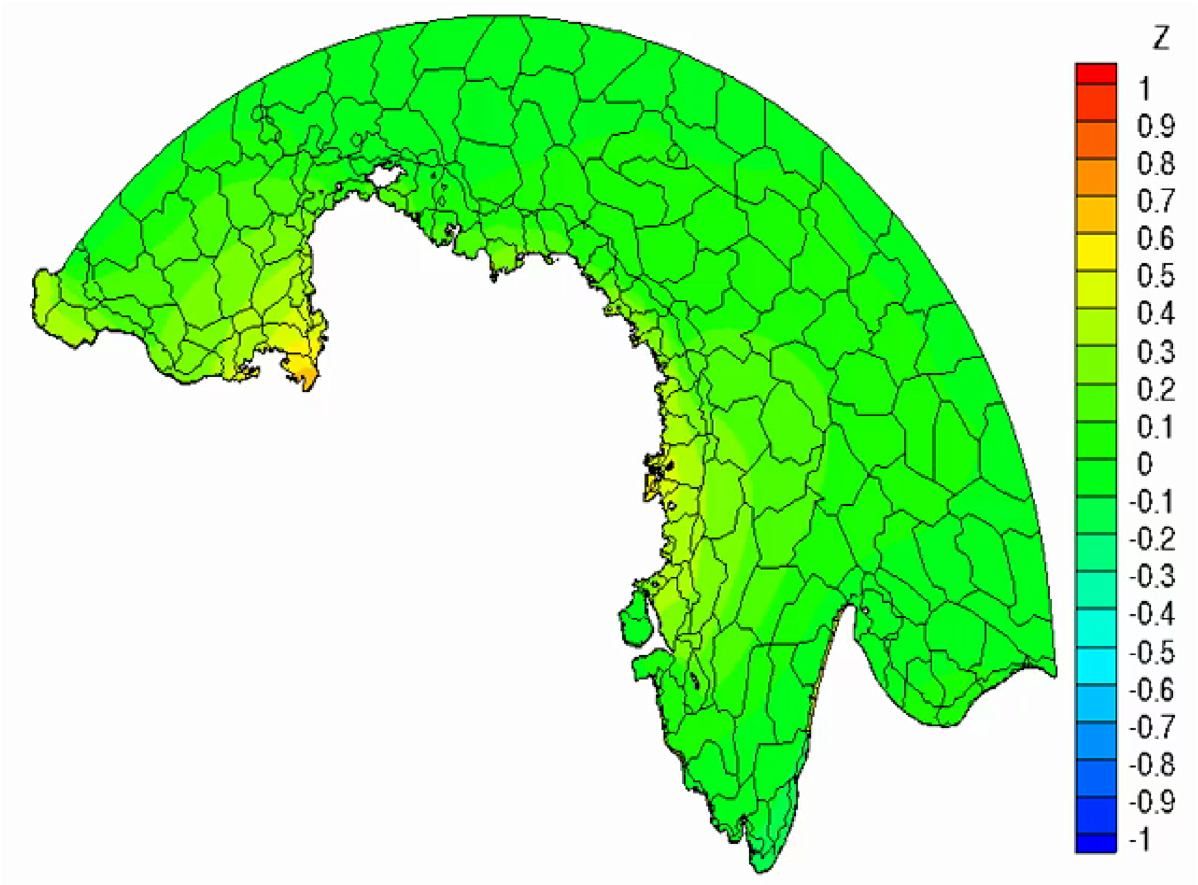
Main bathymetry data source for the delta: **soviet military topographic maps**.



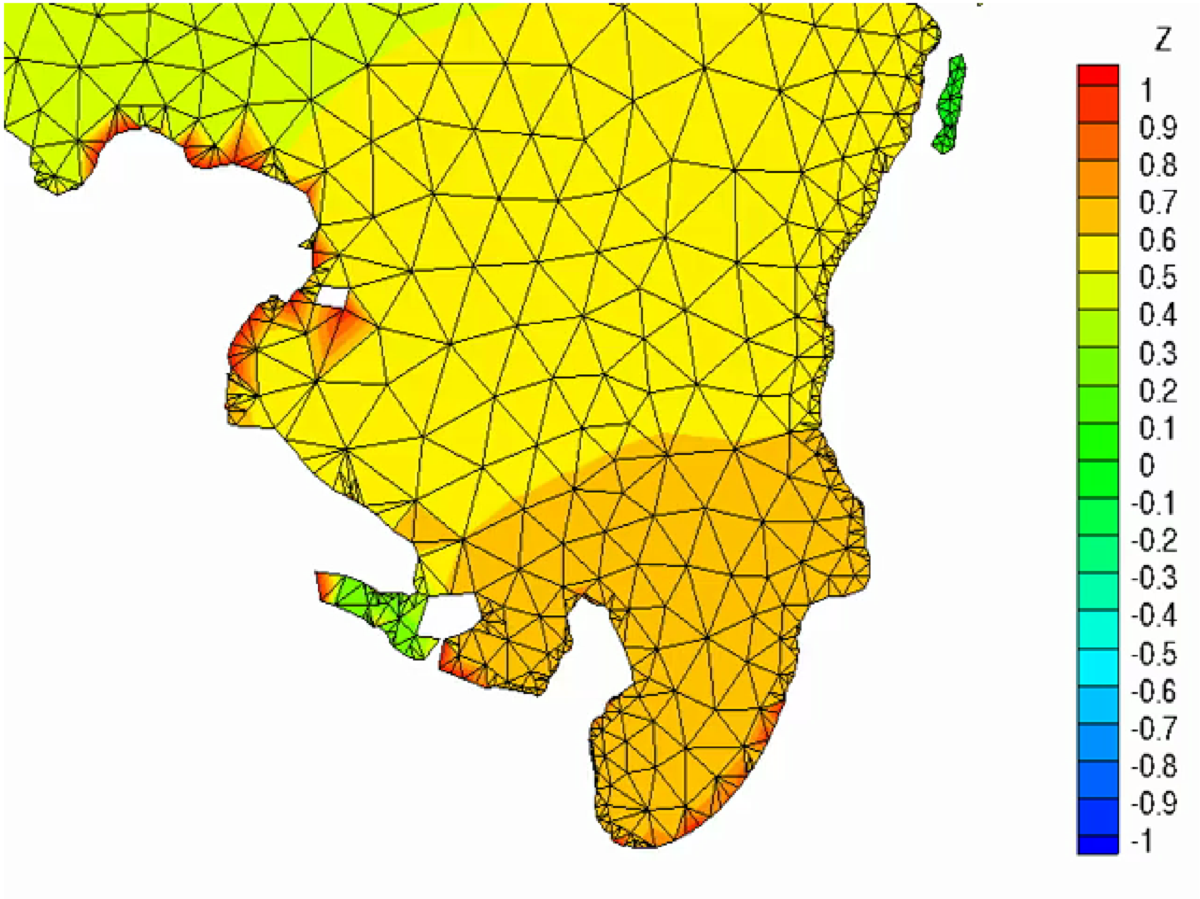
Lena delta - data challenge (2)



Lena delta - first results (1)



Lena delta - first results (2)



Take home messages

- **UTBEST3D**, a numerical solver for the primitive hydrostatic equations
- Hierarchical physical model
- **Lena delta** as challenging study object for numerical modeling

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- **UTBEST3D**, a numerical solver for the primitive hydrostatic equations
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Thank you for your attention!

https://math.fau.de/utbest3d
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Appendix

LDG-Discretization

Exemplary for transport equations

Mixed formulation[7]

$$\partial_t r + \nabla \cdot (\boldsymbol{u} r) - \partial_z(\nu_r q_r) = f_r$$
$$q_r = \partial_z r \,.$$

Weak formulation

$$\int_{\Omega_e} \partial_t r \, \varphi_r \, \mathrm{d}\boldsymbol{x} + \int_{\partial\Omega_e} \left(\widehat{(\boldsymbol{u} r)} \cdot \boldsymbol{n} + \widehat{\nu_r q_r n_z} \right) \varphi_r \, \mathrm{d}\boldsymbol{s} -$$
$$\int_{\Omega_e} \left((\boldsymbol{u} r) \cdot \nabla + \nu_r q_r \partial_z \right) \varphi_r \, \mathrm{d}\boldsymbol{x} = \int_{\Omega_e} f_r \, \varphi_r \, \mathrm{d}\boldsymbol{x}$$
$$\int_{\Omega_e} q_r \lambda_r \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega_e} \widehat{r n_z} \lambda_r \mathrm{d}\boldsymbol{s} - \int_{\Omega_e} r \partial_z \lambda_r \mathrm{d}\boldsymbol{x} \,,$$

with $\boldsymbol{n} = (n_x, n_y, n_z)$ exterior unit normal to $\partial\Omega_e$, and test functions $\varphi_r, \lambda_r \in H^1(\Omega_e)$.

Choose fluxes and approximate Riemann solvers[8]

Local basis representation

$$r_h(t, \boldsymbol{x}) = \sum_i R_i(t) \phi_{r,i}(\boldsymbol{x}) \,, \qquad q_{r,h}(t, \boldsymbol{x}) = \sum_i Q_{r,i}(t) \phi_{q_r,i}(\boldsymbol{x}) \,.$$

Test with $\phi_{r,i}, \phi_{q_r,i}$ and build linear system of equations

[7] Cockburn, Shu (1998), doi:10.1137/S0036142997316712
[8] Dawson, Aizinger (2005), doi:10.1007/s10915-004-4139-3

Discretization of the primitive continuity equation[9]

$$\partial_t h + \boldsymbol{\nabla}_{xy} \cdot \int_{z_b}^{\xi} \boldsymbol{u}_{xy} \mathrm{d}\tilde{z} = 0$$

2D form of weak formulation

$$\int_{\Omega_{e,xy}} \partial_t h \, \varphi_h \mathrm{d}x\mathrm{d}y + \int_{\partial\Omega_{e,xy}} \int_{z_b}^{\xi} \boldsymbol{u}_{xy} \mathrm{d}z \cdot \boldsymbol{n} \, \varphi_h \mathrm{d}\boldsymbol{s} - \int_{\Omega_{e,xy}} \int_{z_b}^{\xi} \boldsymbol{u}_{xy} \mathrm{d}z \cdot \boldsymbol{\nabla}_{xy} \varphi_h \mathrm{d}x\mathrm{d}y = 0$$

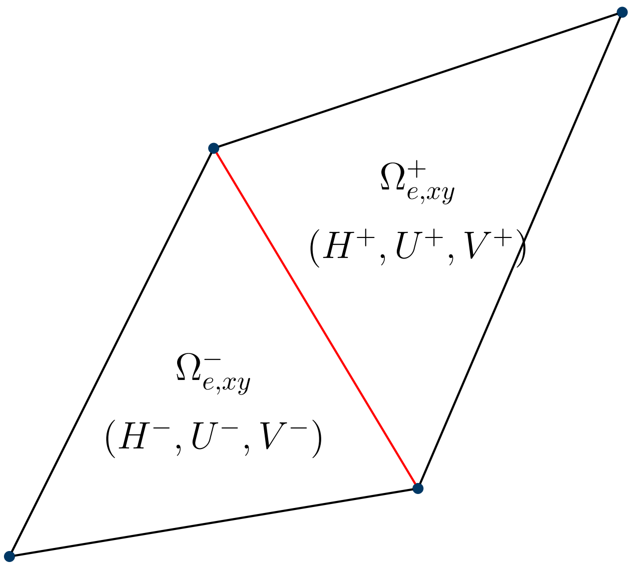
3D form of weak formulation

$$\int_{\Omega_{e,xy}} \partial_t h \, \varphi_h \mathrm{d}x\mathrm{d}y + \sum_{\boldsymbol{\Pi}_{xy}\Omega_e = \Omega_{e,xy}} \int_{\partial\Omega_{e,\text{lat}}} \boldsymbol{u}_{xy} \cdot \boldsymbol{n} \, \varphi_h \mathrm{d}\boldsymbol{s} - \sum_{\boldsymbol{\Pi}_{xy}\Omega_e = \Omega_{e,xy}} \int_{\Omega_e} \boldsymbol{u}_{xy} \cdot \boldsymbol{\nabla}_{xy} \varphi_h \mathrm{d}\boldsymbol{x} = 0$$

with $\Omega_{e,\text{lat}}$ lateral boundary of Ω_e.

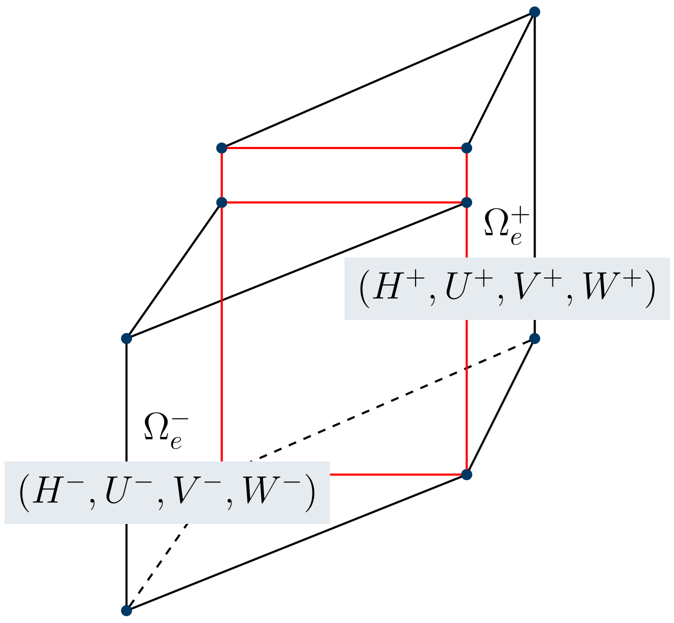
[9] Aizinger (2004), PhD Dissertation, UT Austin

Riemann problem in 3D



$\int_{\partial\Omega^-_{e,xy} \cap \partial\Omega^+_{e,xy}} \hat{k}(\cdot^-, \cdot^+)\, \varphi \mathrm{d}\boldsymbol{s}$

Riemann problem in 3D



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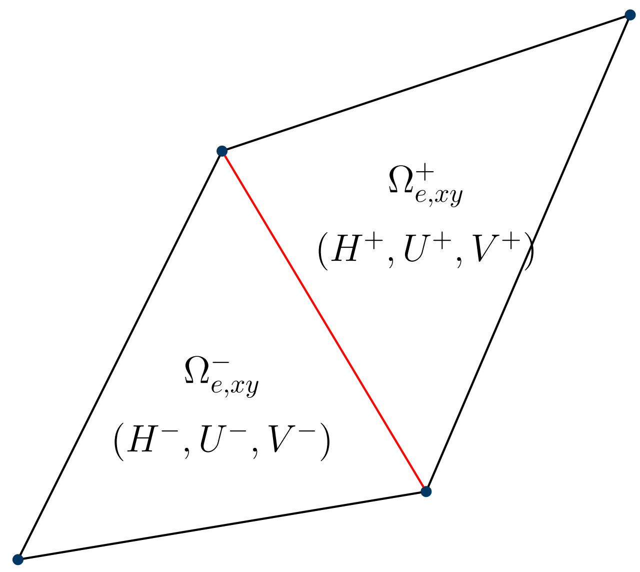
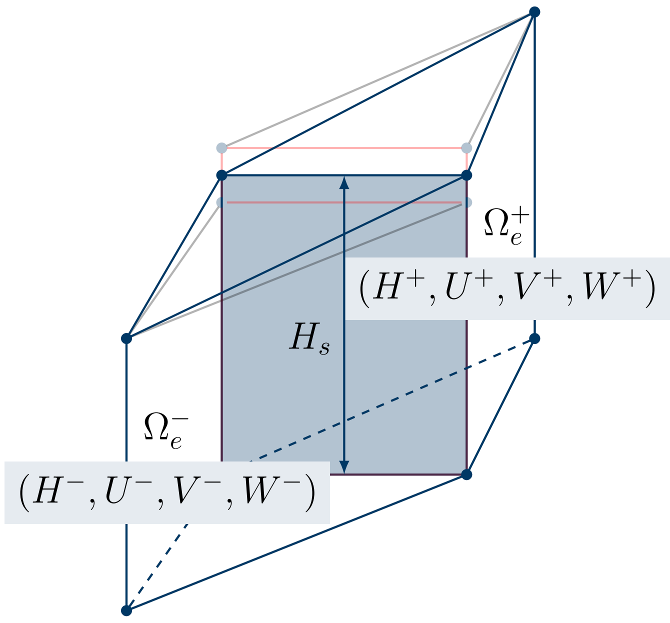
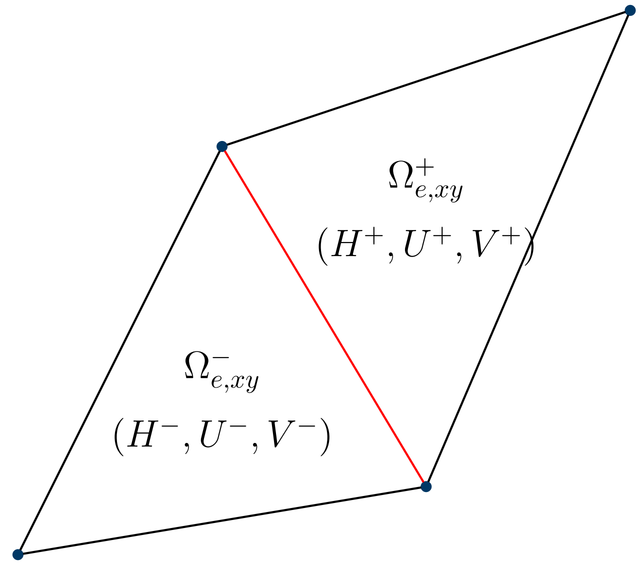


$\int_{\partial\Omega_{e,xy}^- \cap \partial\Omega_{e,xy}^+} \hat{k}(\cdot^-, \cdot^+)\, \varphi \mathrm{d}\boldsymbol{s}$

Riemann problem in 3D



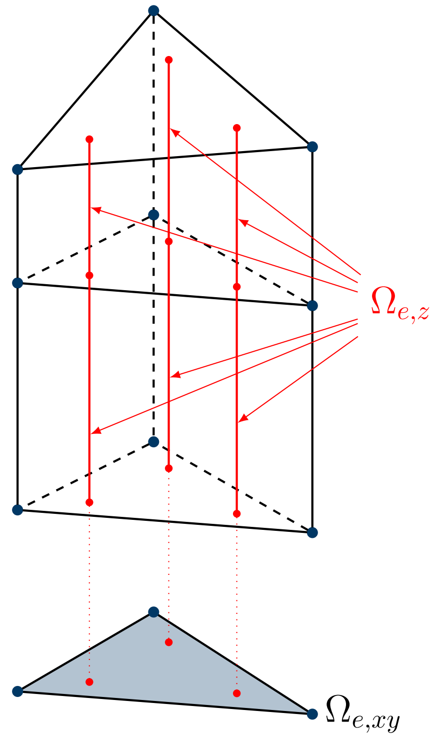


$\int_{\partial\Omega_e^- \cap \partial\Omega_e^+} \hat{k}(\cdot^-, \cdot^+)\, \varphi \mathrm{d}\boldsymbol{s}$

$\int_{\partial\Omega_{e,xy}^- \cap \partial\Omega_{e,xy}^+} \hat{k}(\cdot^-, \cdot^+)\, \varphi \mathrm{d}\boldsymbol{s}$

Discretization of turbulence transport equations

Vertical **segments** aligned with 2D quadrature points.



Mixed formulation

$$\begin{aligned}\partial_t m - \partial_z(\nu_m q_m) &= f_m \\ f_m &= \partial_z m\,.\end{aligned}$$

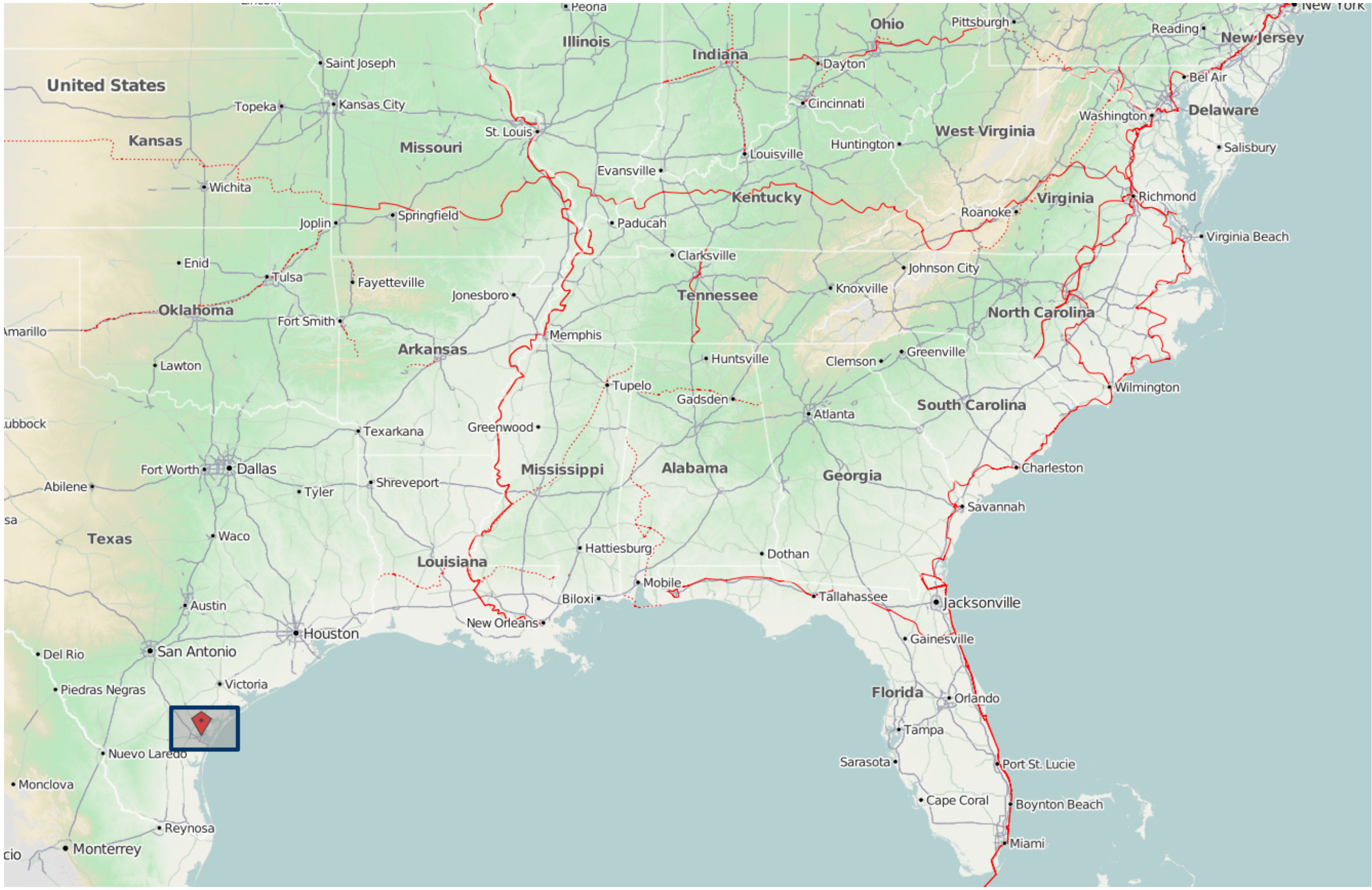
Weak formulation

$$\int_{\Omega_{e,z}} \partial_t m\, \varphi_m\, \mathrm{d}z + (\nu_m q_m n_z\, \varphi_m)\Big|_{\partial\Omega_{e,z}} - \int_{\Omega_{e,z}} \nu_m q_m \partial_z\, \varphi_m\, \mathrm{d}z = \int_{\Omega_{e,z}} f_m\, \varphi_m\, \mathrm{d}z$$

$$\int_{\Omega_{e,z}} q_m \lambda_m \mathrm{d}z = (m n_z \lambda_m)\Big|_{\partial\Omega_{e,z}} - \int_{\Omega_{e,z}} m \partial_z \lambda_m \mathrm{d}z\,,$$

with $\boldsymbol{n} = (0, 0, n_z)$ exterior unit normal to $\partial\Omega_{e,z}$, and test functions $\varphi_m, \lambda_m \in H^1(\Omega_{e,z})$.

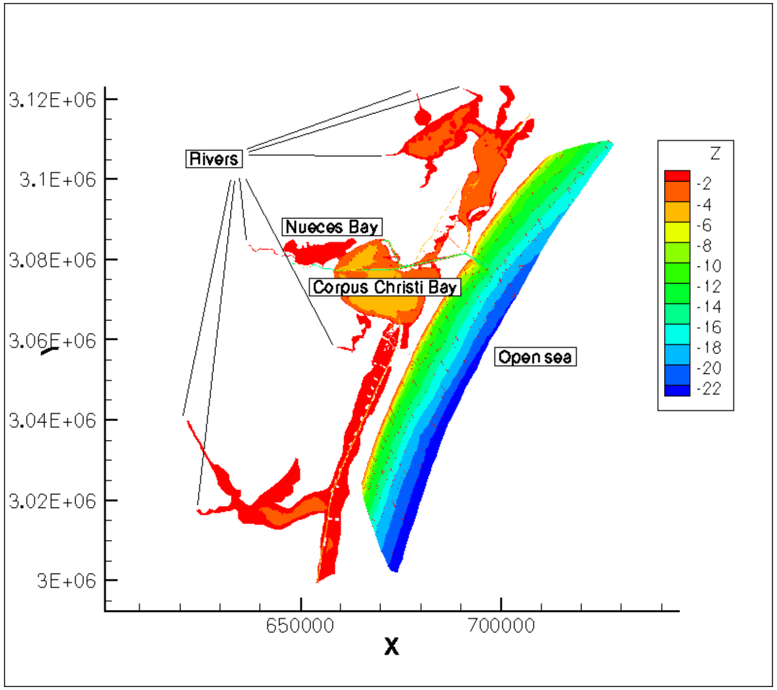
Corpus Christi Bay circulation I

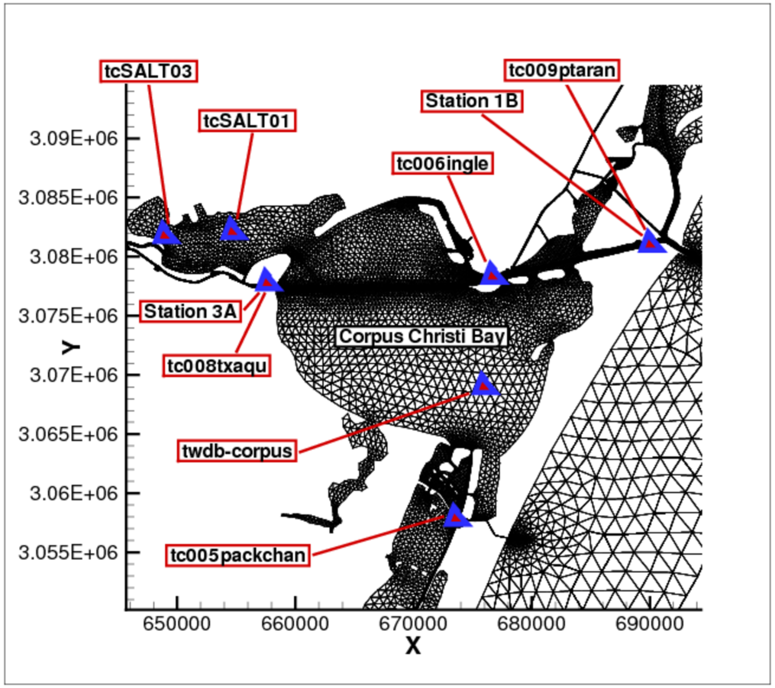


`http://openstreetmap.de/karte.html?zoom=6&lat=33.97063&lon=-90.05908&layers=0B00TT`

Corpus Christi Bay circulation[10] II

- Texas Water Development Board (TWDB) initiated **environmental study**.
- Specified river inflows, wind, precipitation, evaporation, open sea elevation and salinity for 2000-2001.
- Total of 18 **monitoring stations** for water elevation, velocity, salinity.

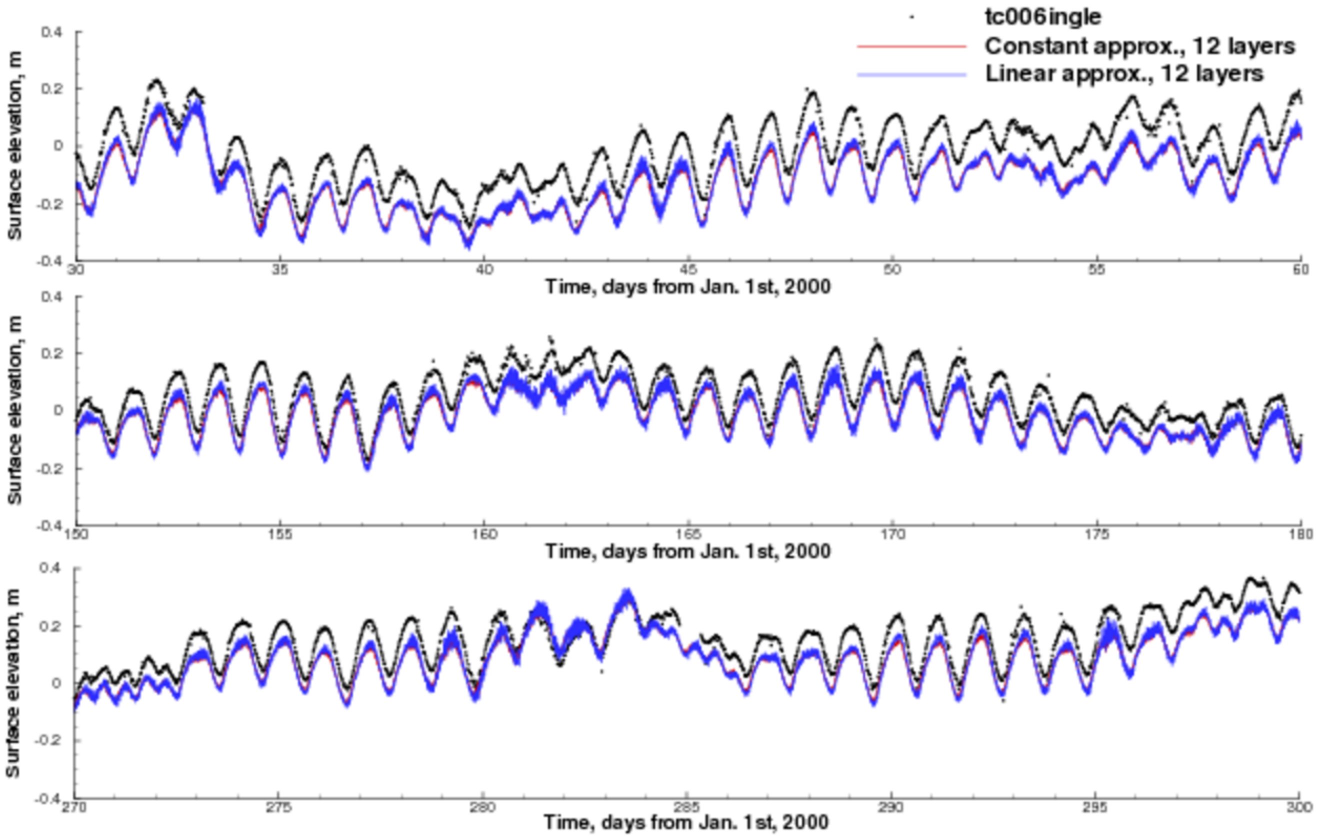




[10] Aizinger, Proft, Dawson, Pothina, Negusse (2013), doi:10.1007/s10236-012-0579-8

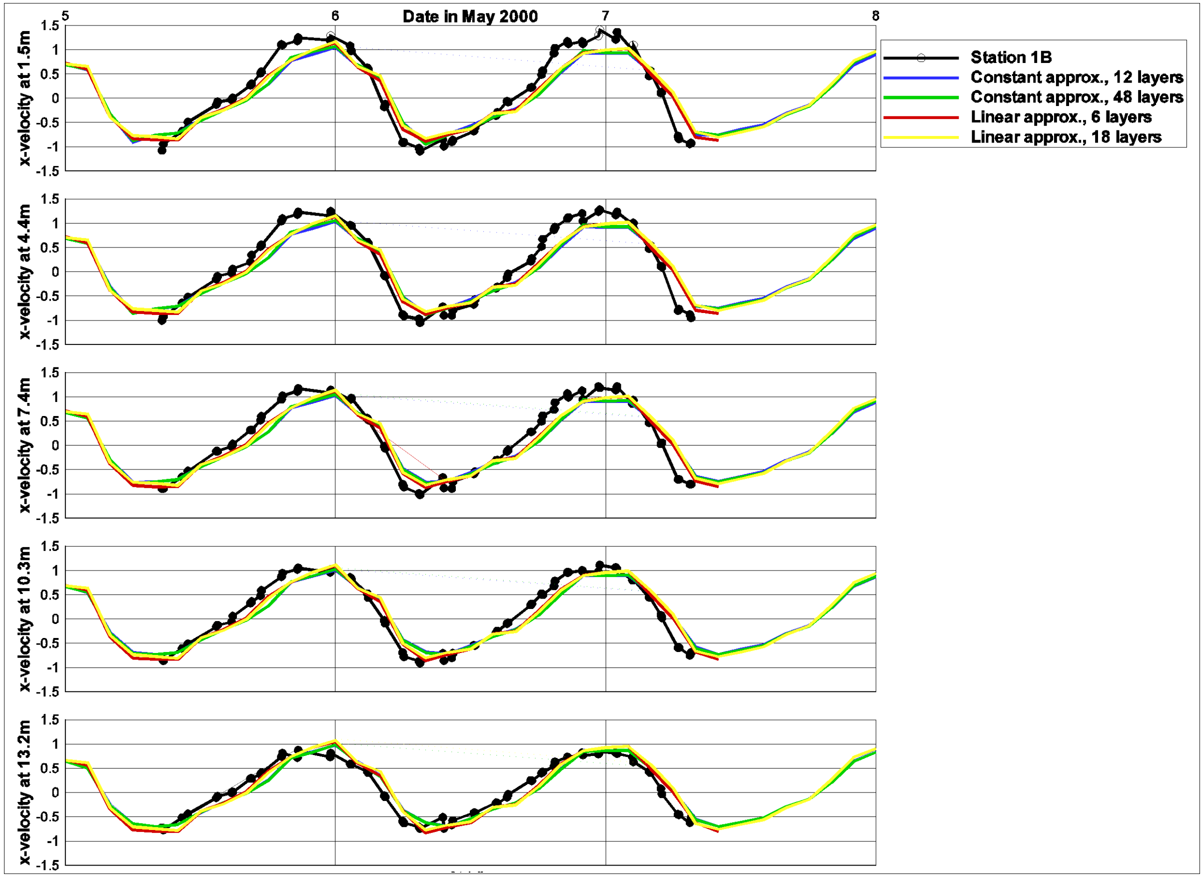
Corpus Christi Bay circulation V

- Elevation station tc006 Ingleside



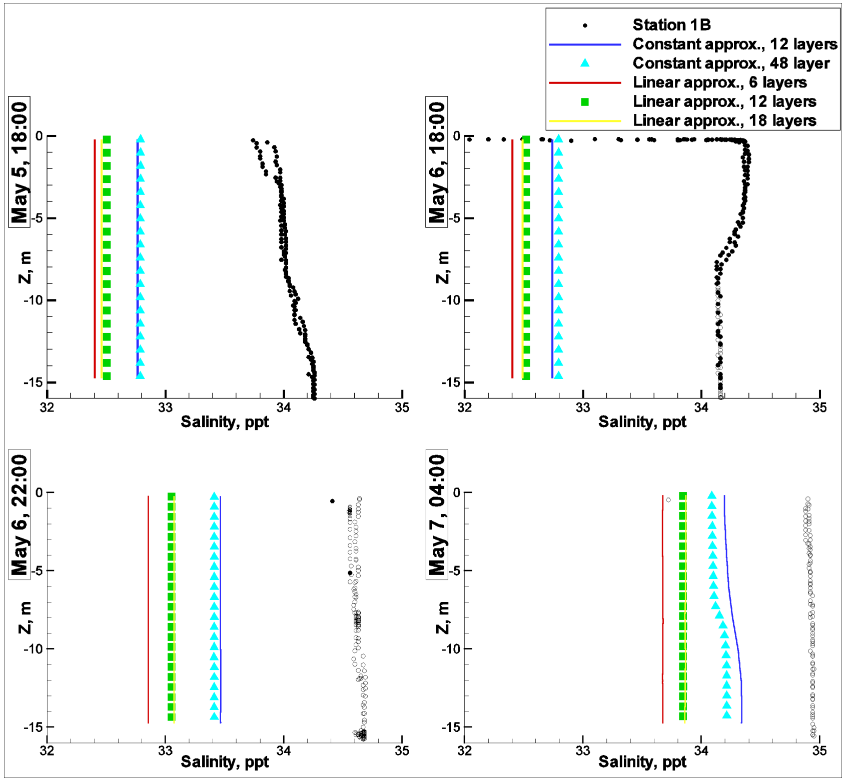
Corpus Christi Bay circulation VI

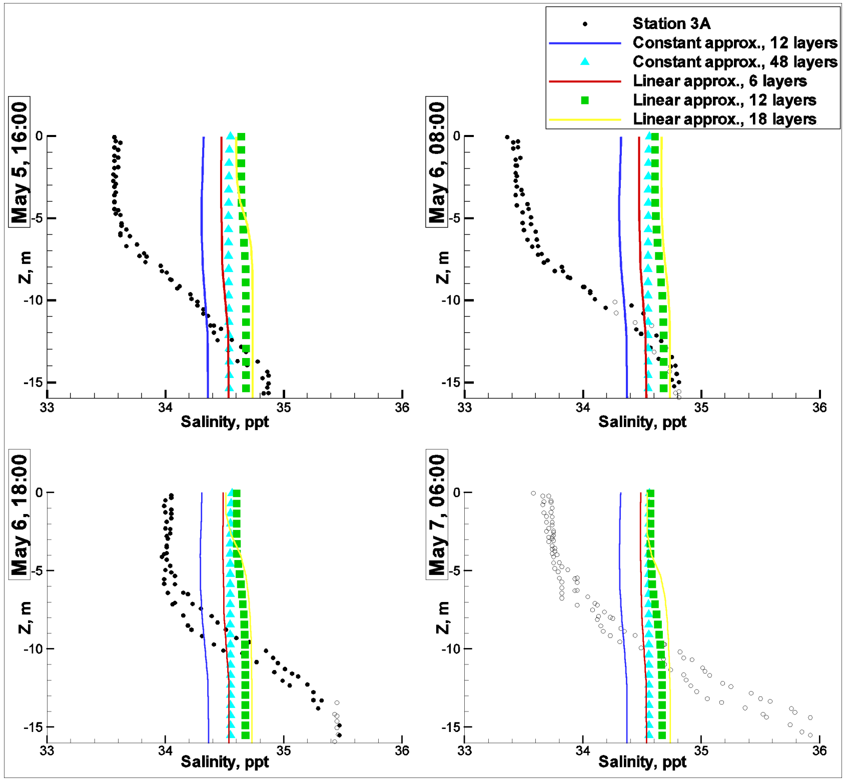
- Station 1B x-**velocity** depth profile



Corpus Christi Bay circulation VII

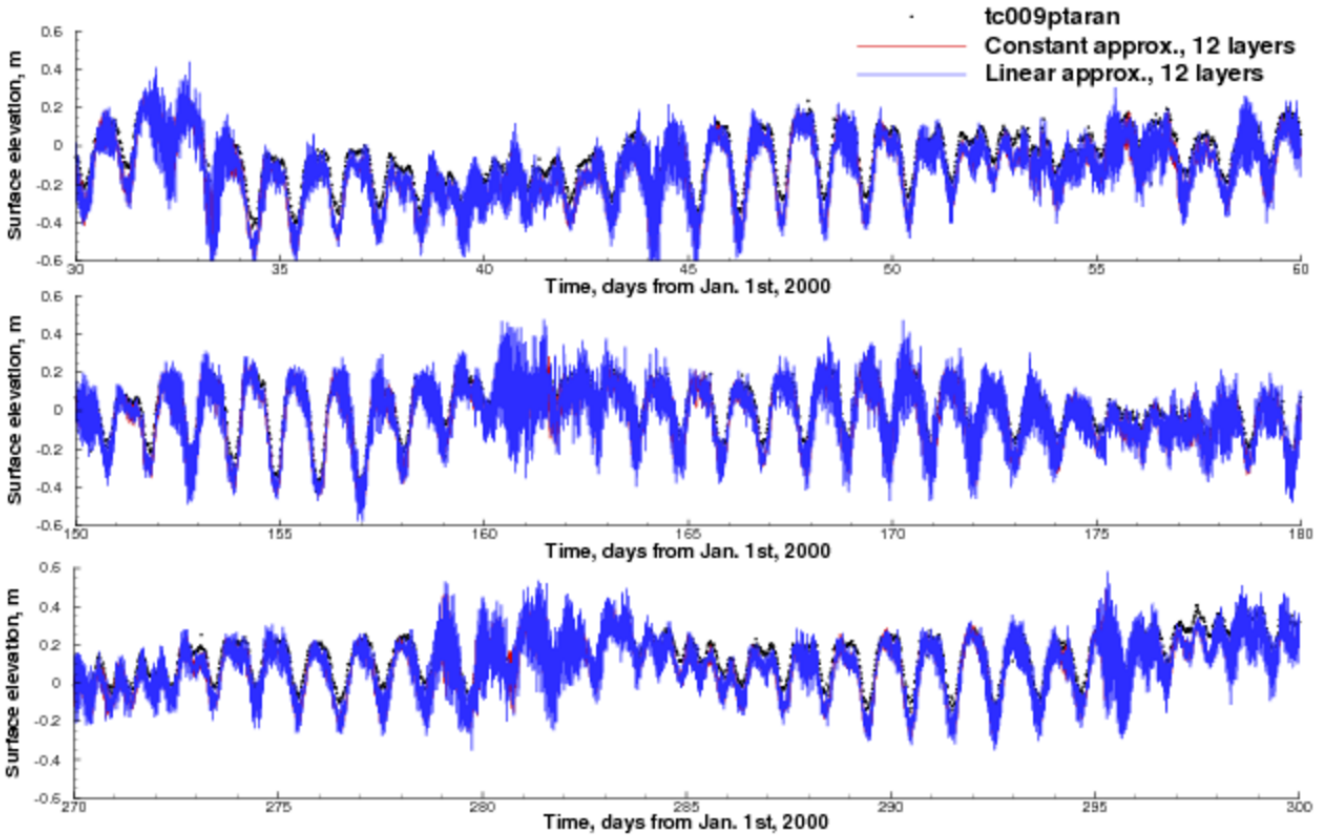
- **Salinity** depth profile: Station 1B vs. Station 3A





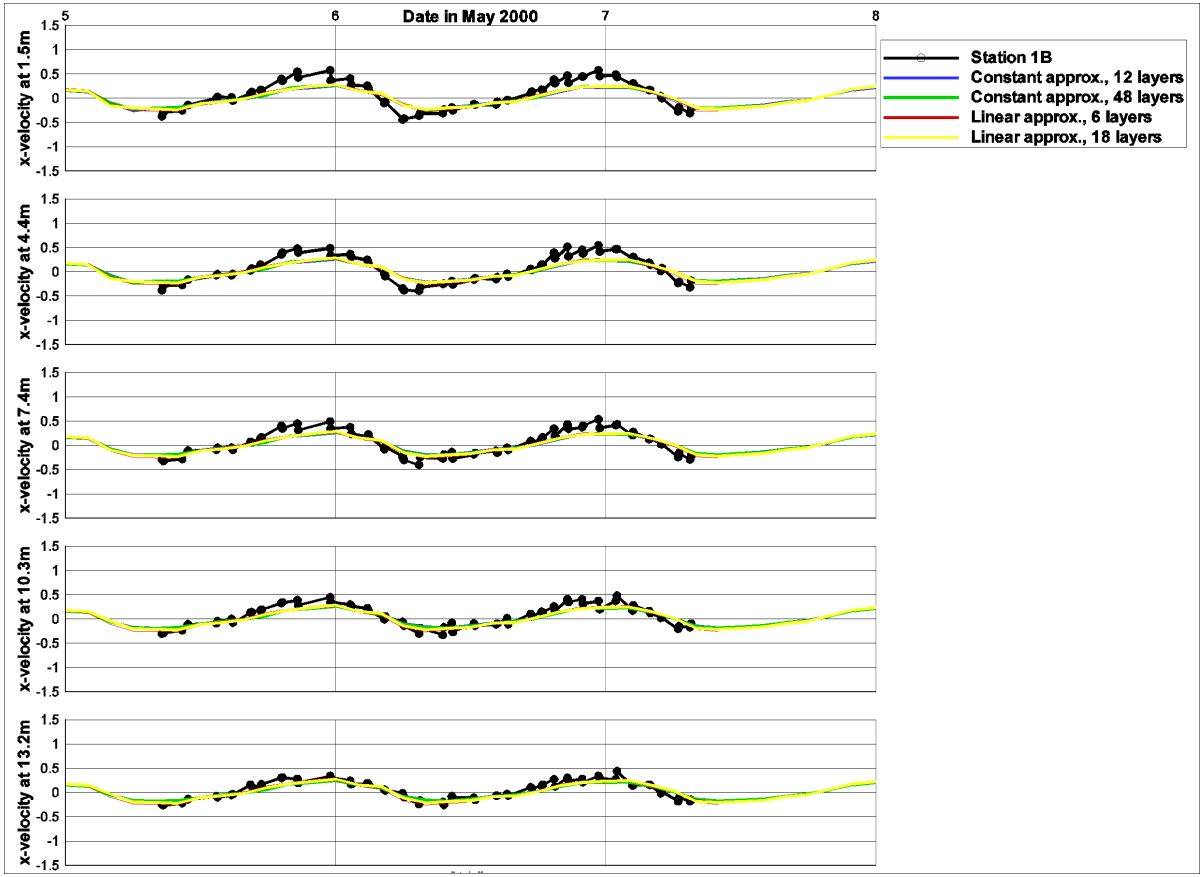
Corpus Christi Bay circulation

- Elevation station tc009 Port Aransas



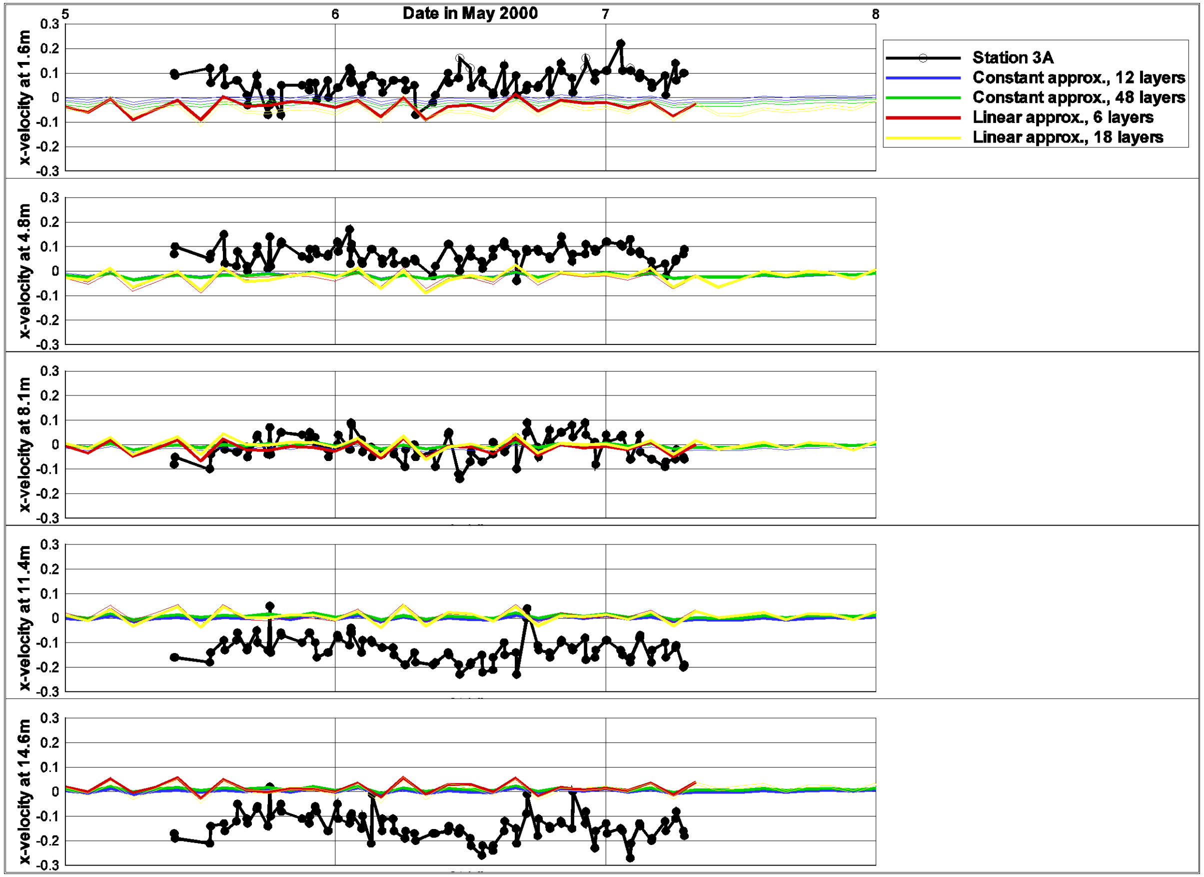
Corpus Christi Bay circulation

- Station 1B y**-velocity** depth profile



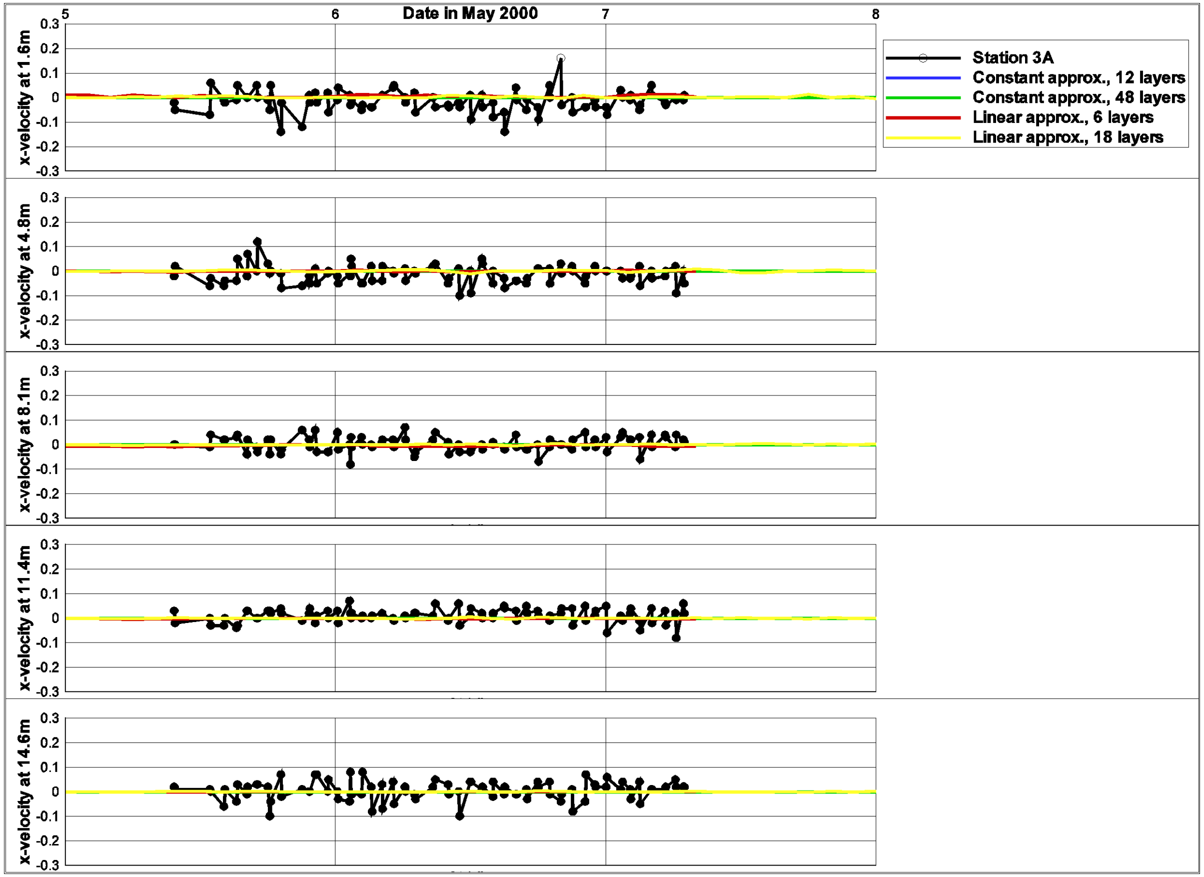
Corpus Christi Bay circulation

- Station 3A x-**velocity** depth profile



Corpus Christi Bay circulation

- Station 3A y-**velocity** depth profile



Corpus Christi Bay circulation

- **Salinity** station tcSALT01