Mr. Tucker read a paper On a Group of Circles connected with the Nine-Points Circle, considered as the Locus of the Intersections of Orthogonal Simson Lines, and parts of a paper by Mr. R. A. Roberts, entitled Notes on the Plane Unicursal Quartic.

Communications were also made by the Treasurer, Mr. G. Heppel, and the President.

The following presents were received :—

"Royal Society—Proceedings," Vol. xxxvii., No. 233.

"Nautical Almanac," for 1888.

"Educational Times," for December.

"A Synopsis of Elementary Results in Pure and Applied Mathematics, containing Propositions, Formulæ, and Methods of Analysis, with abridged Demonstrations," by G. S. Carr, M.A., Vol. i., Sections x., xi., and xii., 8vo; London, 1884.

"Beiblätter zu den Annalen der Physik und Chemie," Band 8, St. 11; Leipzig, 1884.

"Bulletin de la Société Mathématique de France," T. xii., No. 4; Paris, 1884.

"Bulletin des Sciences Mathématiques et Astronomiques," S. 2, T. viii.; Paris, Dec., 1884.

"Berichte über die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig,—Mathematisch-physische Classe," 1883, 8vo; Leipzig, 1884.

"Transactions of the Connecticut Academy of Arts and Sciences," Vol. vi., Part 1; Newhaven, 1884.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. v., No. 5; Coimbra.

"Acta Mathematica," 5 : 1; Stockholm, 1884.

"Crelle," "Journal für Mathematik," Bd. xcvii., H. 4; Berlin, 1884.

"Ueber die Frage des Weber'schen Gesetzes und Periodicitaetsgesetzes im Gebiete des Zeitsinnes," von G. Th. Fechner, 4to; Leipzig, 1884.

From M. Maurice d'Ocagne :—

"Sur la Droite Moyenne d'un Système de Droites quelconques situées dans un plan" (*Bulletin de la Soc. Math. de France*, Tom. xii., 1884).

"Sur les Transformations centrales des Courbes planes."

"Sur quelques Propriétés générales des Surfaces algébriques de degré quelconque" (*Comptes Rendus*, Nov., 1884).

Notes on the Plane Unicursal Quartic. By R. A. Roberts, M.A.

[*Read December 11th*, 1884.]

I collect in these notes a few miscellaneous properties of the plane unicursal quartic, most of which, I believe, have not been noticed before. I do not attempt to give any systematic account of this curve, but merely investigate such questions in connection with it as appear most likely to lead to results of some interest.

1. To find the parameters of the nodes of the general unicursal quartic.

The sextic giving these parameters is arrived at in Salmon's *Higher Plane Curves*, Art. 291 (*a*), but the result comes out in a rather complicated form. The following method gives the equation in the form of a single determinant, and has the advantage of being immediately applicable to unicursal curves of any degree.

Suppose the quartic to be determined by the equations

$$\left.\begin{aligned} \rho x &= f_1 = a_1\vartheta^4+b_1\vartheta^3+c_1\vartheta^2+d_1\vartheta+e_1 \\ \rho y &= f_2 = a_2\vartheta^4+b_2\vartheta^3+c_2\vartheta^2+d_2\vartheta+e_2 \\ \rho z &= f_3 = a_3\vartheta^4+b_3\vartheta^3+c_3\vartheta^2+d_3\vartheta+e_3 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(1),$$

the quantities f_1, f_2, f_3 being written without binomial coefficients, then, if L, M are two lines passing through a node, we must have

$$\left.\begin{aligned} L &= u\,(\vartheta-a)\,(\vartheta-a') \\ M &= v\,(\vartheta-a)\,(\vartheta-a') \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(2),$$

where u, v are quadratics in ϑ, and a, a' are the two parameters corresponding to the node. Hence substituting linear expressions in x, y, z for L, M respectively in $Lv-Mu=0$, we get a result which may be written

$$(a\vartheta^2+\beta\vartheta+\gamma)\,x+(a'\vartheta^2+\beta'\vartheta+\gamma')\,y+(a''\vartheta^2+\beta''\vartheta+\gamma'')\,z=0 \ldots (3),$$

and then

$$\left.\begin{aligned} ax+a'y+a''z &= 0 \\ \beta x+\beta' y+\beta'' z &= 0 \\ \gamma x+\gamma' y+\gamma'' z &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(4)$$

represent three lines passing through the node.

We now substitute f_1, f_2, f_3 for x, y, z in (3), and equate to zero the seven coefficients of ϑ^6, ϑ^5, &c., when we get

$$\left.\begin{aligned} aa_1+a'a_2+a''a_3 &= 0 \\ ab_1+a'b_2+a''b_3+\beta a_1+\beta' a_2+\beta'' a_3 &= 0 \\ ac_1+a'c_2+a''c_3+\beta b_1+\beta' b_2+\beta'' b_3+\gamma a_1+\gamma' a_2+\gamma'' a_3 &= 0 \\ ad_1+a'd_2+a''d_3+\beta c_1+\beta' c_2+\beta'' c_3+\gamma b_1+\gamma' b_2+\gamma'' b_3 &= 0 \\ ae_1+a'e_2+a''e_3+\beta d_1+\beta' d_2+\beta'' d_3+\gamma c_1+\gamma' c_2+\gamma'' c_3 &= 0 \\ \beta e_1+\beta' e_2+\beta'' e_3+\gamma d_1+\gamma' d_2+\gamma'' d_3 &= 0 \\ \gamma e_1+\gamma' e_2+\gamma'' e_3 &= 0 \end{aligned}\right\} \ldots (5).$$

But the parameters of the node must satisfy the equations obtained by substituting f_1, f_2, f_3 for x, y, z in the equations (4). Hence, taking the first and last of these equations, and eliminating a, a', &c. by means of (5), we get a determinant which we may write

$$\left|\begin{array}{lll} u_1, u_2, u_3, & & \\ & & v_1, v_2, v_3 \\ a_1, a_2, a_3, & & \\ b_1, b_2, b_3, & a_1, a_2, a_3, & \\ c_1, c_2, c_3, & b_1, b_2, b_3, & a_1, a_2, a_3 \\ d_1, d_2, d_3, & c_1, c_2, c_3, & b_1, b_2, b_3 \\ e_1, e_2, e_3, & d_1, d_2, d_3, & c_1, c_2, c_3 \\ & e_1, e_2, e_3, & d_1, d_2, d_3 \\ & & e_1, e_2, e_3 \end{array}\right| = 0 \quad \ldots\ldots\ldots\ldots\ldots(6),$$

where
$$u_1 = b_1\vartheta^3+c_1\vartheta^2+d_1\vartheta+e_1,$$
$$v_1 = a_1\vartheta^3+b_1\vartheta^2+c_1\vartheta+d_1,$$
$$u_2 = \&c.,$$

and the factors corresponding to $\vartheta = 0$ and ∞, respectively, have been divided out.

In exactly the same way we obtain a determinant with $3(n-1)$ rows to determine the parameters of the nodes of the general unicursal curve of the n^{th} degree.

2. By eliminating a, a', &c., between (5), and each pair of the equations (4), we obtain, in the form of determinants, three conics passing through the nodes of the quartic, and, by forming the Jacobian of these conics, we have the equation of the sides of the triangle formed by the nodes.

3. We can find the tangential equation of the nodes of the curve as follows. Let λ, μ, ν be the coordinates of a line passing through a node, then, from (4), we can evidently take

$$\left.\begin{array}{lll} \alpha = l\gamma+m\lambda, & \alpha' = l\gamma'+m\mu, & \alpha'' = l\gamma''+m\nu \\ \beta = l'\gamma+m'\lambda, & \beta' = l'\gamma'+m'\mu, & \beta'' = l'\gamma''+m'\nu \end{array}\right\} \ldots\ldots\ldots(7),$$

substituting which values of a, a', &c. in (5), we get seven equations from which we can eliminate γ, γ', γ'', l, m, l', m', when the required result comes out in the form of a determinant $\Delta^2 = 0$, where $\Delta = 0$ gives the three nodes each once.

4. By forming the discriminant of $\lambda f_1+\mu f_2+\nu f_3$, we see that the

tangential equation of the curve is of the form $S^3 = T^2$. We can also easily show that this is the case if the equation of the curve referred to the nodal triangle be given.

Writing $U \equiv x^2y^2+y^2z^2+z^2x^2+2xyz\,(ax+by+cz) = 0$ ....(8),

it is evident, by inversion, that the line $\lambda x+\mu y+\nu z = 0$ will touch the curve if the two conics

$$V \equiv x^2+y^2+z^2+2ayz+2bzx+2cxy,$$

$$V' \equiv \lambda yz+\mu zx+\nu xy,$$

touch one another. Forming then the discriminant of $V+2kV'$, we get

$$2\lambda\mu\nu k^3-(\lambda^2+\mu^2+\nu^2-2a\mu\nu-2b\nu\lambda-2c\lambda\mu)\,k^2$$
$$+2\,\{(bc-a)\,\lambda+(ca-b)\,\mu+(ab-c)\,\nu\}\,k$$
$$+1+2abc-a^2-b^2-c^2 = 0 \quad \text{.........................(9)},$$

and the discriminant of this equation with regard to k will give the tangential equation of U.

We thus find

$$(8\rho^3+27\Delta^2\lambda\mu\nu+9\Delta\rho\Sigma)^2-(4\rho^2+3\Delta\Sigma)^3 = 0 \quad \text{...........(10)},$$

where we have put

$$\lambda^2+\mu^2+\nu^2-2a\mu\nu-2b\nu\lambda-2c\lambda\mu = \Sigma,$$

$$(a-bc)\,\lambda+(b-ca)\,\mu+(c-ab)\,\nu = \rho,$$

$$1+2abc-a^2-b^2-c^2 = \Delta,$$

and λ, μ, ν are tangential coordinates, and a, b, c parameters.

5. Now, from (10), it is evident that $4\rho^2+3\Delta\Sigma = 0$ represents a conic touching the six inflexional tangents, and it is easy to see that Σ is the conic which touches the six tangents at the nodes. Thus we see that these two conics have double contact with each other, the point represented by ρ being the pole of the chord of contact.

6. Since the tangential equation of the curve can be written in the form $S^3-T^2 = 0$, it follows, from a result given in a paper "On Tangents to a Cubic forming a Pencil in Involution," published in the *Proceedings*, Vol. XIII., p. 25, that there will be a locus of the ninth order from any point of which the tangents to the curve will form a pencil in involution. The complete locus is of the forty-fifth degree, but it must be divisible by this special curve of the ninth degree.

7. To find the locus of the points from which the tangents drawn to the curve have their points of contact on a conic.

We know that the points of contact of the tangents from a point P lie on a cubic passing through the nodes, viz., the polar cubic of P. Now, if six points of intersection of a quartic and a cubic lie on a conic, the remaining six points of intersection must lie on another conic. Hence, since the nodes count doubly as intersections of the two curves, they must be the points of contact of a conic having triple contact with the cubic. But, by a property of the cubic, when this is the case, the points where the sides meet the curve again must lie on a line.

Now, the polar cubic of x', y', z' with regard to U is

$$x'x(y^2+z^2)+y'y(z^2+x^2)+z'z(x^2+y^2)+(ax'+by'+cz')\,xyz$$
$$+(ax+by+cz)(x'yz+y'zx+z'xy)=0 \quad \ldots\ldots\ldots\ldots(11).$$

Hence, for the points where x, y, z meet the curve again, we have

$$x=0,\ y(z'+bx')+z(y'+cx')=0;\quad y=0,\ x(z'+ay')+z(x'+cy')=0;$$
$$z=0,\ x(y'+az')+y(x'+bz')=0.$$

Hence, expressing that these points lie on a line, we get

$$(a-bc)\,x\,(y^2-z^2)+(b-ca)\,y\,(z^2-x^2)+(c-ab)\,z\,(x^2-y^2)=0 \ldots(12).$$

This, then, is the equation of a cubic such that the six tangents from any point thereof to the quartic have their points of contact on a conic.

8. I proceed to consider what this locus becomes for a few special forms of the equation (8). If in the foregoing equation (8) we have $c=1$, the node xy becomes a cusp, and the cubic (12) becomes divisible by the cuspidal tangent, the remaining factor being the conic $(a-b)(xy-z^2)+(1-ab)\,z\,(x-y)=0$, which is the locus of points from which the tangents have their five points of contact on a conic passing through the cusp.

Let $a=b=c=0$, then the cubic (12) vanishes identically, and we see that the points of contact of the tangents drawn from any point

to the curve $$x^2y^2+y^2z^2+z^2x^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

lie on a conic. We can find the equation of this conic as follows. The points of contact of the tangents from x', y', z' evidently lie on

the curve $$\frac{x'}{x^3}+\frac{y'}{y^3}+\frac{z'}{z^3}=0,$$

and, combining this with (13), we get

$$\left.\begin{array}{c}\frac{\vartheta}{x'^2} = \left(\frac{x}{x'}\right)^3\left(\frac{y}{y'}-\frac{z}{z'}\right), \quad \frac{\vartheta}{y'^2} = \left(\frac{y}{y'}\right)^3\left(\frac{z}{z'}-\frac{x}{x'}\right) \\ \frac{\vartheta}{z'^2} = \left(\frac{z}{z'}\right)^3\left(\frac{x}{x'}-\frac{y}{y'}\right)\end{array}\right\} \quad \ldots\ldots(14),$$

where ϑ is indeterminate. We have then

$$\left.\begin{array}{c}\vartheta\left(\frac{1}{x'^2}+\frac{1}{y'^2}+\frac{1}{z'^2}\right) = \left(\frac{x}{x'}+\frac{y}{y'}+\frac{z}{z'}\right)\Delta \\ \vartheta\left(\frac{x}{x'^3}+\frac{y}{y'^3}+\frac{z}{z'^3}\right) = \left(\frac{x^2}{x'^2}+\frac{y^2}{y'^2}+\frac{z^2}{z'^2}+\frac{yz}{y'z'}+\frac{zx}{z'x'}+\frac{xy}{x'y'}\right)\Delta\end{array}\right\} \ldots(15),$$

where $$\Delta \equiv \left(\frac{x}{x'}-\frac{y}{y'}\right)\left(\frac{y}{y'}-\frac{z}{z'}\right)\left(\frac{z}{z'}-\frac{x}{x'}\right).$$

Hence, eliminating ϑ between the two equations (15), we get the conic

$$(y'^2+z'^2)\,x^2+(z'^2+x'^2)\,y^2+(x'^2+y'^2)\,z^2+y'z'yz+z'x'zx+x'y'xy = 0.$$

9. I give here the locus of points whence the tangents to the curve have their points of contact on a conic for three special forms not included under equation (8). For the nodo-tacnodal quartic

$$x^4+y^2z^2+x^2y^2+2xy\,(ayz+bzx+cx^2) = 0$$

(see Salmon's *Higher Plane Curves*, Art. 289), the locus is the conic

$$2b\,(1-a^2)\,x^2+2\,(1-a^2)\,zx+(c-ab)\,yz+(a+bc-2ab^2)\,xy = 0.$$

For the oscnodal quartic

$$(yz+x^2)^2+2cxy\,(yz+x^2)+y^2\,(x^2+y^2+2hxy+2fyz) = 0,$$

we find similarly the line

$$(1-c^2-2f)\,x+(h-cf)\,y = 0.$$

For the quartic with a triple point, the locus is found to consist of three right lines passing through that point.

10. To find the relations connecting the parameters of four points on the curve which lie on a line.

If the line $lx+my+nz = 0$ meet the curve, we have, from (1),

$$lf_1+mf_2+nf_3 \propto \phi\,(\vartheta) = 0,$$

where $$\phi\,(\vartheta) = (\vartheta-\vartheta_1)\,(\vartheta-\vartheta_2)\,(\vartheta-\vartheta_3)\,(\vartheta-\vartheta_4).$$

Now, let a, a' be the parameters of a node, then we have $\phi\,(a)$ and

$\phi(a')$ respectively proportional to $lf_1+mf_2+nf_3$ and $lf'_1+mf'_2+nf'_3$. But $\frac{f_1}{f'_1} = \frac{f_2}{f'_2} = \frac{f_3}{f'_3}$ for a node obviously.

Hence we get

$$\phi(a)-k_1\phi(a') = 0,\quad \phi(\beta)-k_2\phi(\beta') = 0,\quad \phi(\gamma)-k_3\phi(\gamma') = 0 \ldots (16),$$

where β, β', γ, γ' are the parameters of the two other nodes, and k_1, k_2, k_3 are constants. These three relations are evidently not independent, but are only equivalent to two. There is, in fact, an identical linear relation of the form

$$l\{\phi(a)-k_1\phi(a')\}+m\{\phi(\beta)-k_2\phi(\beta')\}+n\{\phi(\gamma)-k_3\phi(\gamma')\} = 0 \quad \ldots\ldots\ldots\ldots (17).$$

By referring the curve to the triangle formed by the nodes, we find

$$k_1 = \frac{(a-\beta)(a-\beta')(a-\gamma)(a-\gamma')}{(a'-\beta)(a'-\beta')(a'-\gamma)(a'-\gamma')},\quad k_2 = \&c.$$

11. In the same way, if we seek the intersection of the curve with a conic, we get

$$\phi(a)-k_1^2\phi(a') = 0,\quad \phi(\beta)-k_2^2\phi(\beta') = 0,\quad \phi(\gamma)-k_3^2\phi(\gamma') = 0 \ldots (18),$$

where

$$\phi(t) = (t-\vartheta_1)(t-\vartheta_2)(t-\vartheta_3)(t-\vartheta_4)(t-\vartheta_5)(t-\vartheta_6)(t-\vartheta_7)(t-\vartheta_8).$$

In this case the three relations (18) are evidently independent. Hence, if a conic have quartic contact with the curve, the four points of contact must satisfy the equations

$$\phi(a)\pm k_1\phi(a') = 0,\quad \phi(\beta)\pm k_2\phi(\beta') = 0,\quad \phi(\gamma)\pm k_3\phi(\gamma') = 0 \ldots (19),$$

where now $\qquad \phi(t) = (t-\vartheta_1)(t-\vartheta_2)(t-\vartheta_3)(t-\vartheta_4)$.

We cannot take two negative signs in (19), as then, from (16) and (17), the points would necessarily lie on a line; but we may take two positive and the other negative, and thus we get three distinct systems of conics having quartic contact with the curve. By taking three positive signs in (19), we get another system of conics, which are evidently perfectly symmetrical with regard to the three nodes.

12. Now, if z be the line joining two of the nodes, it is evident that the curve can be written in the form $S^2-z^2S' = 0$, where S and S' are conics, and then

$$\vartheta^2z^2+2\vartheta S+S' = 0$$

will represent a conic having quartic contact with the curve. We thus have three different systems of such conics corresponding to each pair of nodes, which are evidently the same as those we have come upon above.

13. Again, suppose the quartic to be written in the form

$$U \equiv ay^2z^2 + bz^2x^2 + cx^2y^2 + 2xyz\,(fx + gy + hz) = 0 \quad \text{.........(20)},$$

then it is evident that the envelope of the system of conics

$$Al^2x^2 + Bm^2y^2 + Cn^2z^2 + 2Fmnyz + 2Gnlzx + 2Hlmxy = 0 \quad \text{......(21)},$$

subject to the condition $\quad l + m + n = 0 \quad$ ....(22),

is the curve U, where A, B, &c. are the differentials with regard to a, b, &c., of $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2$. The conic (21) evidently belongs to the symmetrical system obtained by taking all the signs positive in (19).

Since the conic (21) is transformed into a fixed conic by substituting x, y, z for lx, my, nz, it follows that any covariant of the triangle of reference and the conic (21) will be transformed into fixed curves by the same substitution. Hence, from the condition (22), it appears that the polars of the nodes with regard to (21) pass through fixed points. Also the poles of the sides with regard to (21) lie on conics circumscribing the triangle.

14. If we suppose two of the nodes to become the circular points at infinity I, J, the curve will become a bicircular quartic with a node, and the conic (21) is then the result of transforming a fixed conic by the substitution of u, v for $\frac{lu}{n}$, $\frac{mv}{n}$, respectively, where u, v are the lines joining the node to the points I, J. Now, this transformation is equivalent to turning the curve about the origin and altering the radii vectores in a constant ratio, the dilatation and the angle through which the figure is turned being connected by a certain relation determined by (22). Hence, in this case, it appears that the axes, asymptotes, tangents at the vertices and directrices of (21) will all turn about fixed points, and the centre, foci, vertices, &c. will move on fixed circles passing through the node. It is evident also that the eccentricity of (21) is given.

15. By taking the envelope of the tangential equation of (21), viz.,

$$\frac{a\lambda^2}{l^2} + \frac{b\mu^2}{m^2} + \frac{c\nu^2}{n^2} + \frac{2f\mu\nu}{mn} + \frac{2g\nu\lambda}{nl} + \frac{2h\lambda\mu}{lm} = 0,$$

subject to the condition (22), we get the tangential equation of the quartic, and the invariants of the biquadratic in $\frac{l}{m}$ will give the envelopes of lines divided harmonically and equi-anharmonically by the curve.

16. If a tangent to a conic U is homographic with a tangent to a conic V, their intersection will lie on a trinodal quartic having quartic contact with U and V. For a tangent to U, and a homographic tangent to V, can be written in the forms

$$\vartheta^2P+2\vartheta R+Q=0,\quad \vartheta^2P'+2\vartheta R'+Q'=0,$$

and the result of eliminating ϑ between these equations is

$$\left.\begin{array}{r} 4(RQ'-R'Q)(PR'-P'R)-(PQ'-P'Q)^2=0 \\ \text{or}\quad (PQ'+P'Q-2RR')^2-4(PQ-R^2)(P'Q'-R'^2)=0 \end{array}\right\}\ldots\ldots(23),$$

which evidently represent a quartic having quartic contact with $PQ-R^2$ and $P'Q'-R'^2$, of which the three points determined by the

equations $$\frac{P}{P'}=\frac{Q}{Q'}=\frac{R}{R'}\ldots\ldots\ldots\ldots(24)$$

are nodes. It is easy to show that the conics U, V belong to the system (21); for the variable conics of the same system as U, V are

evidently $$(P+kP')(Q+kQ')-(R+kR')^2=0\ \ldots\ldots\ldots\ldots(25).$$

But, if we take the nodes as triangle of reference, we must write, from (24),

$$\left.\begin{array}{ll} P=ax+\beta y+\gamma z, & P'=lax+m\beta y+n\gamma z \\ Q=a'x+\beta'y+\gamma'z, & Q'=la'x+m\beta'y+n\gamma'z \\ R=a''x+\beta''y+\gamma''z, & R'=la''x+m\beta''y+n\gamma''z \end{array}\right\}\ldots(26);$$

from which it appears that the conic (25) will be transformed into a fixed conic by substituting $\frac{x}{1+kl}$, $\frac{y}{1+km}$, $\frac{z}{1+kn}$ for x, y, z respectively.

17. In the same way as in § 10, we can show that, if a curve of the n^{th} degree meet the quartic, we must have

$$\phi(a)-k_1^n\phi(a')=0,\quad \phi(\beta)-k_2^n\phi(\beta')=0,\quad \phi(\gamma)-k_3^n\phi(\gamma')=0,$$

where $$\phi(\vartheta)=(\vartheta-\vartheta_1)(\vartheta-\vartheta_2)\ldots\ldots(\vartheta-\vartheta_{4n}).$$

It may be observed that this method is also applicable to unicursal curves of any degree, and will give all the relations connecting the parameters of the points of intersection with a curve whose degree is assigned.

18. If we attempted to find whether it were possible to inscribe an infinite number of closed polygons in a unicursal quartic which should also be circumscribed about the curve, we should be led to expect the well-known relation of the second degree connecting the sum and products of the parameters of the points where a tangent meets the curve again. The curve, then, would have to be of the fourth class. Such curves are: (1) the nodo-bicuspidal quartic, (2) the quartic with a triple point at which the tangents coincide, (3) the quartic with a cusp and a ramphoid cusp; but the two latter curves have no indeterminate constant, which must exist and be determined afterwards by a condition depending on the number of sides of the polygon. Let us consider, then, the nodo-bicuspidal quartic which may be written $$(xy+yz+zx)^2-m^2z^2xy = 0 \quad\ldots\ldots\ldots\ldots(27).$$

This equation is satisfied by assuming

$$x = 1+\vartheta^2-m\vartheta, \quad y = \vartheta^2(1+\vartheta^2-m\vartheta), \quad z = -\vartheta^2 \quad\ldots\ldots(28),$$

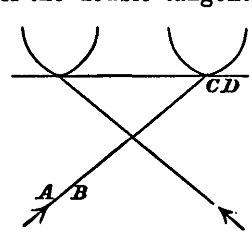
and then it is easy to see that the parameters of the points where any line meets the curve are connected by the relations

$$\Sigma_4^1\,\vartheta = m, \quad \Sigma_4^1\,\frac{1}{\vartheta} = m \quad\ldots\ldots\ldots\ldots(29).$$

Hence, putting $\vartheta_4 = \vartheta_3$, and eliminating ϑ_3, we get

$$(\vartheta_1+\vartheta_2)^2-m(\vartheta_1+\vartheta_2)(1+\vartheta_1\vartheta_2)+(m^2-4)\,\vartheta_1\vartheta_2 = 0 \ldots\ldots(30),$$

which is of the form mentioned above. For the triangle the value of m which we find is irrelevant, and therefore no such triangles can be described for any curve of the form (27). For the quadrilateral, we get $m^2 = -2$, and for this curve it can be shown then that the lines joining the points of contact of opposite sides intersect at the node.* The case $m^2 = -2$ is a very interesting result; it means that the tangent at each cusp passes through a point of contact of the double tangent.



The line in question taken twice is a degenerate form of quadrilateral, viz.,

AB quâ line joining two coincident points at cusp,

BC quâ tangent at cusp,

CD quâ double tangent,

and *DA quâ* tangent at cusp—are each of them a tangent to the curve.

* I am indebted to Professor Cayley for the following remarks on this case.

For x, y, z, writing $1+x$, $1-x$, $\frac{1}{2}y$, and for m^2 writing $-4k^2$, we have the Cartesian equation,

$$(x^2+y-1)^2-k^2y^2(x^2-1)=0,$$

where the curve is as in figure, and of course $k^2=\frac{1}{2}$ is the value for the case of the quadrilateral.

To find the values of m corresponding to polygons of a greater number of sides, we equate to zero certain determinants formed out of the coefficients of the expansion of the function

$$\{\lambda^3+(3-2k)\lambda^2+(k^2-3k+3)\lambda+1-k\}^{\frac{1}{2}},$$

in powers of λ, where $k=\frac{1}{4}m^2$, in the same way as has been done by Professor Cayley for polygons inscribed in one conic and circumscribed about another. (*Philosophical Magazine*, Vol. VI., p. 99.)

19. I now proceed to discuss some properties of the unicursal quartic with a triple point. If we suppose the quartic to be given by the equations (1), I find the condition that the curve should have a triple point as follows. If the curve have a triple point, it is evident that we must have

$$P+\vartheta Q=0 \quad\text{.............................(31)},$$

where P and Q are some pair of lines passing through the triple point. Hence, putting

$$P=\alpha x+\beta y+\gamma z,\quad Q=\alpha' x+\beta' y+\gamma' z,$$

we must have, identically, from (31),

$$(\alpha+\vartheta\alpha')f_1+(\beta+\vartheta\beta')f_2+(\gamma+\vartheta\gamma')f_3=0\text{............(32)}.$$

Equating then the coefficients of ϑ^5, ϑ^4, &c., in (32), to zero, and eliminating α, β, γ, α', β', γ' linearly, we get

$$\begin{vmatrix} a_1, & a_2, & a_3, & & & \\ b_1, & b_2, & b_3, & a_1, & a_2, & a_3 \\ c_1, & c_2, & c_3, & b_1, & b_2, & b_3 \\ d_1, & d_2, & d_3, & c_1, & c_2, & c_3 \\ e_1, & e_2, & e_3, & d_1, & d_2, & d_3 \\ & & & e_1, & e_2, & e_3 \end{vmatrix}=0\text{.................(33)}.$$

Also, solving for α, β, γ from five of these equations, and substituting in $\alpha f_1+\beta f_2+\gamma f_3=0$, we get the cubic which determines the three parameters of the triple point, as the coefficient of ϑ^4 vanishes.

20. To find the relations connecting the parameters $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$ of four collinear points on the curve.

If we take the lines x, y passing through the triple point, we may represent the curve thus

$$\rho x = \Theta_3, \quad \rho y = \vartheta\Theta_3, \quad \rho z = \Theta_4 \quad \text{..................(34)},$$

where Θ_3, Θ_4 are polynomials in ϑ of the third and fourth degrees, respectively. For the points, then, where the line $\lambda x + \mu y + \nu z = 0$ meets the curve, we have

$$(\lambda + \mu\vartheta)\,\Theta_3 + \nu\Theta_4 = 0.$$

Hence, if α, β, γ are the roots of Θ_3, we readily find

$$\frac{\phi(\alpha)}{l} = \frac{\phi(\beta)}{m} = \frac{\phi(\gamma)}{n} \quad \text{.......................(35)},$$

where $$\phi(t) = (t-\vartheta_1)(t-\vartheta_2)(t-\vartheta_3)(t-\vartheta_4),$$

and l, m, n are the values of Θ_4 corresponding to α, β, γ respectively.

21. From (35) we can easily find the relation connecting the parameters of the points where a tangent meets the curve again. Putting $\vartheta_4 = \vartheta_3$, and eliminating ϑ_3, we get

$$\frac{(\beta-\gamma)\sqrt{l}}{\sqrt{\{(\alpha-\vartheta_1)(\alpha-\vartheta_2)\}}} + \frac{(\gamma-\alpha)\sqrt{m}}{\sqrt{\{(\beta-\vartheta_1)(\beta-\vartheta_2)\}}} + \frac{(\alpha-\beta)\sqrt{n}}{\sqrt{\{(\gamma-\vartheta_1)(\gamma-\vartheta_2)\}}} = 0 \quad \text{............(36)}.$$

From this relation it is easy to show that it is impossible to inscribe a triangle in the quartic which shall be also circumscribed about the curve. For, if we put

$$P_\alpha = \frac{(\beta-\gamma)\sqrt{l}}{\sqrt{\{(\alpha-\vartheta_1)(\alpha-\vartheta_2)(\alpha-\vartheta_3)\}}}, \quad P_\beta = \text{\&c.},$$

we get, for a triangle, from (36),

$$\left.\begin{aligned} P_\alpha\sqrt{(\alpha-\vartheta_1)} + P_\beta\sqrt{(\beta-\vartheta_1)} + P_\gamma\sqrt{(\gamma-\vartheta_1)} &= 0 \\ P_\alpha\sqrt{(\alpha-\vartheta_2)} + P_\beta\sqrt{(\beta-\vartheta_2)} + P_\gamma\sqrt{(\gamma-\vartheta_2)} &= 0 \\ P_\alpha\sqrt{(\alpha-\vartheta_3)} + P_\beta\sqrt{(\beta-\vartheta_3)} + P_\gamma\sqrt{(\gamma-\vartheta_3)} &= 0 \end{aligned}\right\} \quad \text{......(37)},$$

which equations are manifestly inconsistent, as one of them, being cleared of radicals, gives a quadratic for ϑ.

22. To show that, if a triangle be inscribed in the curve, the lines

joining the triple point to the points where the sides meet the curve again form a pencil in involution.

Let a, b, c be the parameters of the vertices, and u_1, u_2, u_3 the three quadratics which determine the parameters of the points where the sides meet the curve again, then x, y, z being the sides of the triangle, we may write

$$x = (\vartheta - b)(\vartheta - c)\, u_1,\ y = (\vartheta - c)(\vartheta - a)\, u_2,\ z = (\vartheta - a)(\vartheta - b)\, u_3 \text{...(38)}.$$

Now we have seen that, when the curve has a triple point, we must have

$$(\alpha\vartheta + \alpha')(\vartheta - b)(\vartheta - c)\, u_1 + (\beta\vartheta + \beta')(\vartheta - c)(\vartheta - a)\, u_2$$
$$+ (\gamma\vartheta + \gamma')(\vartheta - a)(\vartheta - b)\, u_3 = 0 \text{ ....(39)},$$

identically. But, putting $\vartheta = a, b, c$ successively in (39), we get

$$\frac{\alpha'}{\alpha} = -a,\quad \frac{\beta'}{\beta} = -b,\quad \frac{\gamma'}{\gamma} = -c,$$

and then each term becomes divisible by $(\vartheta - a)(\vartheta - b)(\vartheta - c)$. We see thus that u_1, u_2, u_3 must be connected by a linear relation, and, therefore, the corresponding parameters form a system in involution.

We can show, conversely, that if the parameters of the points where the sides of a triangle inscribed in a unicursal quartic meet the curve again form a system in involution, then the curve must have a triple point. For, if we have $lu_1 + mu_2 + nu_3 = 0$, identically in (38), we get

$$lx(\vartheta - a) + my(\vartheta - b) + nz(\vartheta - c) = 0,$$

which shows that the curve has a triple point whose coordinates are

$$\frac{b-c}{l},\quad \frac{c-a}{m},\quad \frac{a-b}{n}.$$

Hence, if we suppose two sides of an inscribed triangle to touch the curve, the lines joining the triple point to the points of contact and the points where the third side meets the curve again are harmonically connected. Hence also, as we have seen before, it is impossible to describe a triangle which shall be simultaneously inscribed in and circumscribed about the curve.

Again, we can deduce the following theorem: If a quartic with a triple point be described through the vertices of a fixed triangle, and through fixed pairs of points on the sides, the locus of the triple point is a cubic curve, of which the fixed points on the sides are pairs of corresponding points.

23. If the pairs of points on the sides are at the extremities of the

diagonals of a complete quadrilateral, we know that the lines joining these points to an arbitrary point form a pencil in involution. Hence it would appear that a quartic can be described through the six points of intersection of the sides of a quadrilateral, and the vertices of the triangle formed by the diagonals, so as to have an arbitrary point for a triple point. We can verify this result independently as follows: Let x, y, z be the diagonals, and $x \pm y \pm z = 0$ the sides of the quadrilateral, then the equation of the quartic must be of the form

$$\phi \equiv ayz\,(y^2-z^2)+bzx\,(z^2-x^2)+cxy\,(x^2-y^2)+3xyz\,(lx+my+nz) = 0 \quad \text{.........}(40).$$

Now, if a curve have a triple point, the six second differentials must vanish for this point; hence, if x, y, z is the triple point, we have,

from (40),
$$cxy-bzx+lyz = 0,$$

$$ayz-cxy+mzx = 0, \quad bzx-ayz+nxy = 0,$$

$$\left.\begin{aligned} a\,(y^2-z^2)+lx^2+2mxy+2nzx &= 0 \\ b\,(z^2-x^2)+my^2+2lxy+2nyz &= 0 \\ c\,(x^2-y^2)+nz^2+2lzx+2myz &= 0 \end{aligned}\right\} \quad \text{...............}(41).$$

All these equations are satisfied by the values

$$\left.\begin{aligned} a = x^3, \; b = y^3, \; c = z^3 \\ l = x\,(y^2-z^2), \; m = y\,(z^2-x^2), \; n = z\,(x^2-y^2) \end{aligned}\right\} \quad \text{.........}(42),$$

showing that the triple point can be assumed arbitrarily.

24. For a quartic with a triple point, the invariants A and B (Salmon's *Higher Plane Curves*, Arts. 293, 294) vanish, and we can verify that this is the case, when $a, b,$ &c. have the values (42). Calculating A and B for the quartic (40), we get

$$\left.\begin{aligned} A &= -12\,(lmn+bcl+cam+abn) \\ B &= (lmn+bcl+cam+abn)^2 \end{aligned}\right\} \quad \text{...............}(43),$$

but these vanish when we put in for $a, b,$ &c., from (42). It may be observed that it is not possible to write a general quartic in the form (40); for, from (43), we see that the curve must satisfy the invariant relation $A^2 = 144B$.

25. To show that, if a triangle be inscribed in a quartic with a triple point so as to have its sides divided harmonically by the curve, the points where the sides meet the curve again must be at the extremities of the diagonals of a quadrilateral.

If the sides x, y, z of an inscribed triangle are divided harmonically by a quartic U, we have

$$U \equiv yz\,(b_1 y^2 + c_1 z^2) + zx\,(a_2 x^2 + c_2 z^2) + xy\,(a_3 x^2 + b_3 y^2)$$
$$+ 3xyz\,(lx + my + nz) = 0.$$

Calculating then the invariants A and B, we find

$$A = 12p + 4q, \quad B = (p-q)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(44),$$

where $p = lb_3c_2 + mc_1a_3 + nb_1a_2 - lmn, \quad q = a_2b_3c_1 + a_3b_1c_2.$

Now, we have seen that, when the curve has a triple point, A and B vanish; hence, from (44), we get $p = q = 0$, but $p = 0$ is the condition that the points on the sides should be at the extremities of the diagonals of a quadrilateral. I now proceed to show that, if such a triangle exist, the quartic must satisfy a special condition, and that there are then an infinite number of these triangles inscribed in the curve. If we take ∞, 0 as the parameters of the double points of the involution determined by the points where the sides meet the curve again, we may write the equations (38) as follows:

$$\left.\begin{aligned} \rho lx = (\vartheta^2 - k_1^2)(\vartheta - b)(\vartheta - c), \quad \rho my = (\vartheta^2 - k_2^2)(\vartheta - c)(\vartheta - a) \\ \rho nz = (\vartheta^2 - k_3^2)(\vartheta - a)(\vartheta - b) \end{aligned}\right\} \ldots(45).$$

But, if x divide the curve harmonically, we must have $k_1^2 = bc$, and similarly for y and z, $k_2^2 = ca$, $k_3^2 = ab$. Now, if we determine l, m, n, so that, when we put $x = 0$, $y = 0$, $z = 0$, we get $y^2 - z^2 = 0$, $z^2 - x^2 = 0$, $x^2 - y^2 = 0$, respectively, we find

$$l = \frac{(a-b)(a-c)}{a^{\frac{3}{2}}}, \quad m = \frac{(b-a)(b-c)}{b^{\frac{3}{2}}}, \quad n = \frac{(c-a)(c-b)}{c^{\frac{3}{2}}}.$$

Eliminating, then, ϑ between the equations (45), we obtain

$$(ax + \beta y + \gamma z)^3 \left(\frac{x}{a^3} + \frac{y}{\beta^3} + \frac{z}{\gamma^3}\right) = \left(\frac{x}{a} + \frac{y}{\beta} + \frac{z}{\gamma}\right)^3 (a^3x + \beta^3y + \gamma^3z) \ldots(46),$$

where we have put $a = \alpha^2$, $b = \beta^2$, $c = \gamma^2$. Now this equation (46) evidently does not represent the general quartic with a triple point, but one in which two inflexional tangents meet on the curve. If we seek the parameters of the triple point in this case, we find

$$\vartheta^3 - abc = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(47),$$

which shows that the double lines of the involution are absolutely fixed, and coincide with the Hessian lines of the tangents at the triple point.

It may be observed that, when the curve is written in the form (40), the tangents at the vertices are $cy-bz=0$, $az-cx=0$, $bx-ay=0$. Thus we see that these lines always pass through a point.

26. I give an independent proof that the problem, to inscribe a triangle in the quartic with a triple point, so that the sides may be divided harmonically by the curve, is either indeterminate or impossible.

Let the curve referred to the triangle formed by the triple point and the points of contact of a double tangent be written

$$x^2y^2-z(y-ax)(y-bx)(y-cx)=0,$$

then we may take $x=f$, $y=\vartheta f$, $z=\vartheta^2$, where $f=(\vartheta-a)(\vartheta-b)(\vartheta-c)$, and then, from (35), if any right line meet the curve, we have

$$\frac{\phi(a)}{a^2}=\frac{\phi(b)}{b^2}=\frac{\phi(c)}{c^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(48).$$

Now, if $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$ are harmonic, we have

$$2(\vartheta_1\vartheta_2+\vartheta_3\vartheta_4)=(\vartheta_1+\vartheta_2)(\vartheta_3+\vartheta_4) \quad\ldots\ldots\ldots\ldots\ldots(49).$$

Eliminating then ϑ_3 and ϑ_4 between (48) and (49), we get

$$\frac{a^2}{(a-b)(a-c)}\,\frac{\{2(\vartheta_1\vartheta_2+bc)-(b+c)(\vartheta_1+\vartheta_2)\}}{(a-\vartheta_1)(a-\vartheta_2)}$$
$$+\frac{b^2}{(b-a)(b-c)}\,\frac{\{2(\vartheta_1\vartheta_2+ca)-(c+a)(\vartheta_1+\vartheta_2)\}}{(b-\vartheta_1)(b-\vartheta_2)}$$
$$+\frac{c^2}{(c-a)(c-b)}\,\frac{\{2(\vartheta_1\vartheta_2+ab)-(a+b)(\vartheta_1+\vartheta_2)\}}{(c-\vartheta_1)(c-\vartheta_2)}=0\ldots(50).$$

Let us suppose now a conic defined by the equations

$$x=(b-c)^2(\vartheta-a)^2,\quad y=(c-a)^2(\vartheta-b)^2,\quad z=(a-b)^2(\vartheta-c)^2,$$

i.e.,
$$\sqrt{x}+\sqrt{y}+\sqrt{z}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(51),$$

then, if x, y, z are the coordinates of the intersection of the tangents at the points ϑ_1, ϑ_2, the equation (50) will become

$$a^2(b-c)^4yz(y+z-x)+b^2(c-a)^4zx(z+x-y)$$
$$+c^2(a-b)^4xy(x+y-z)=0 \quad\ldots\ldots\ldots(52).$$

Hence the problem becomes, to circumscribe triangles about the conic (51) which shall be inscribed in the cubic (52). Now I have shown, in § 12 of a paper published in the *Proceedings*, Vol. xv., p. 4, that when it is possible to circumscribe an infinite number of triangles about a conic so as to be inscribed in a cubic, then the points where the sides meet the cubic again must lie on a line, and this line must touch the conic. But this line for the cubic (52) is $x+y+z=0$, which cannot possibly touch the conic (51). If, however, we have

$$a^2(b-c)^4+b^2(c-a)^4+c^2(a-b)^4=0 \quad\text{.................(53)},$$

the cubic (52) becomes divisible by $x+y+z$, the remaining factor being the conic

$$a^2(b-c)^4yz+b^2(c-a)^4zx+c^2(a-b)^4xy=0;$$

and, since this conic circumscribes an infinite number of triangles circumscribed about (51), the problem then becomes indeterminate. We can verify that (53) is the condition that two inflexional tangents should meet on the curve. Let ϑ_1, ϑ_2 be the parameters of the inflexions, and ϕ the parameter of the intersection of the corresponding tangents; then, from (48), we have

$$\frac{a^2}{(a-\vartheta_1)^3(a-\phi)}=\frac{b^2}{(b-\vartheta_1)^3(b-\phi)}=\frac{c^2}{(c-\vartheta_1)^3(c-\phi)},$$

$$\frac{a^2}{(a-\vartheta_2)^3(a-\phi)}=\frac{b^2}{(b-\vartheta_2)^3(b-\phi)}=\frac{c^2}{(c-\vartheta_2)^3(c-\phi)},$$

whence, eliminating ϑ_1, ϑ_2, ϕ, we get (53).

27. From the theorem in § 21, it is easy to deduce that it is possible to inscribe an infinite number of triangles in the curve so that the pairs of points where the sides meet the curve again may be conjugate with regard to a fixed pair of lines passing through the triple point. When the fixed lines are the Hessian lines of the triple tangents, we see, from (47), that the triangles are those considered in the preceding paragraph.

[It has been pointed out to me by Professor Cayley that Clebsch (*Crelle*, t. 64, p. 64) has arrived at seven distinct systems of conics which have quartic contact with the plane unicursal quartic, whereas I have arrived at but four systems. Three of these systems appear to have been obtained by considering the cases of two negative and one positive sign in equations (19) of the text. But, from the identical relation (17), we see that these cases cannot exist. A few other statements of Clebsch, in the paper referred to above, seem to require similar modifications.]

January 8th, 1885.

J. W. L. GLAISHER, Esq., F.R.S., President, in the Chair.

Mr. F. R. Barrell, B.A., late Scholar of Pembroke College, Cambridge; Mr. S. O. Roberts, B.A., Scholar of St. John's College, Cambridge; and Mr. M. N. Dutt, B.A., Calcutta University, Professor of Mathematics, St. Stephen's College, Delhi,—were elected members. The Rev. T. C. Simmons was admitted into the Society.

The following communications were made :—

The Differential Equations of Cylindrical and Annular Vortices, Prof. M. J. M. Hill.

On Criticoids, Rev. R. Harley.

Multiplication of Symmetric Functions, Captain MacMahon.

Note on Symmetrical Determinants, Mr. Buchheim.

The President (Mr. Walker taking the chair) communicated Results in Elliptic Functions.

Mr. Tucker communicated a paper (supplementary) on Limits of Multiple Integrals, by Mr. MacColl, and read a Note by Prof. Cayley on the Binomial Equation $x^p-1=0$, Quinquisection (Second Note).

The following presents were received :—

"Annali di Matematica," Serie II., Tomo XII., Fasc. 4°; Dec., 1884.

"Educational Times," for January, 1885.

"Proceedings of the Academy of Natural Sciences of Philadelphia," Part II., May to October, 1884, 8vo; Philadelphia, 1884.

"Nieuw Archief voor Wiskunde," Deel XI.; Amsterdam, 1884.

"Beiblätter zu den Annalen der Physik und Chemie," B. VIII., St. 12.

"Atti della R. Accademia dei Lincei—Transunti," Vol. VIII., F. 16 ed ultimo; Rome, 1884.

The Binomial Equation $x^p-1=0$; *Quinquisection, Second Note.* *By* Prof. CAYLEY.

[*Read January 8th*, 1885.]

In the paper, "The Binomial Equation $x^p-1=0$; Quinquisection," *Proc. Lond. Math. Soc.*, t. 12 (1880), pp. 15–16, I considered for an exponent $p=5n+1$, the five periods X, Y, Z, W, T connected by the equations

$$\begin{array}{r|ccccc} & X, & Y, & Z, & W, & T \\ \hline X^2 = & a, & b, & c, & d, & e \\ XY = & f, & g, & h, & i, & j \\ XZ = & k, & l, & m, & n, & o \end{array}$$