out of a lower row, that $a_{2,r}$ may be taken to be zero: and that the concomitant vanishes unless β_2 is even.

Hence, in order to give a non-zero concomitant, the number of letters in each row of PN must be even. Also we may suppose that there are no sets $a_{r,s}$, $r \neq s$.

There is then only one independent form, viz.,

$$f\left(\frac{\beta_1}{2}, \frac{\beta_2}{2}, \ldots, \frac{\beta_k}{2}; 0, 0, \ldots, 0\right).$$

34. If there is a product of concomitants of the quadratic which is of the same degree, and of the same order in each kind of variable, as the above, then the above is reducible.

Hence the only irreducible forms for a single q-ary quadratic are given by

$$f(1, 1, \ldots, 1; 0, 0, \ldots, 0),$$

where the number of rows of the PN from which the expression is obtained is $\not> q$.

Thus the irreducible concomitants of a q-ary quadratic are q in number; they may be written symbolically

$$a_{1x}^2, (ab)_{2x}^2, \ldots, (a_1 a_2 \ldots a_{q-1})_{q-1x}^2, (a_1 a_2 \ldots a_q)^2.$$

On the Series $1+\left(\frac{p}{1}\right)^3+\left\{\frac{p(p+1)}{1.2}\right\}^3+\ldots$ *By* F. Morley.

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A curious formula for the sum of the cubes of the coefficients in the Maclaurin series for

$$(1-x)^{-p}$$

for the special case of p a negative integer was proposed by me long since, and was proved by Dixon (*Messenger of Mathematics*,

* [The title of the paper was changed in the revision: *cf. Proc.*, Vol. XXXIV., p. 49.—Sec.]

Vol. xx.) and by Richmond (*ib.*, Vol. xxi.), and again, as an illustration of generating functions, by MacMahon (*Quarterly Journal*, Vol. xxxiii.).

I wish now to sum the series for p a real positive number; where, by ordinary tests of convergence, we take

$$p < \tfrac{2}{3}.$$

1. The special hypergeometric series of the second kind

$$f(p, t) = 1 + \left(\frac{p}{1}\right)^3 t + \left\{\frac{p(p+1)}{1.2}\right\}^3 t^2 + \dots \qquad (1)$$

is the sum of terms independent of x and y in

$$(1-x)^{-p}(1-y)^{-p}(1-t/xy)^{-p}, \qquad (2)$$

where x, y, t are real and positive,

$$x < 1, \quad y < 1, \quad t < xy,$$

and by $(1-x)^{-p}$ we mean

$$1 + \frac{p}{1}x + \frac{p(p+1)}{1.2}x^2 + \dots,$$

that is

$$\frac{1}{\Gamma p}\sum_0^\infty \frac{\Gamma(p+n)}{\Gamma(1+n)} x^n.$$

Now (2) is

$$(1-x-y+xy)^{-p}(1-t/xy)^{-p}$$
$$= (1+xy)^{-p}(1-t/xy)^{-p}\sum \frac{\Gamma(p+m)}{\Gamma p\,\Gamma(1+m)}\left(\frac{x+y}{1+xy}\right)^m.$$

For m odd there is no term independent of x and y, but for m even, say $= 2n$, we have such terms arising from

$$\frac{1}{\Gamma p}\sum (1+xy)^{-p-2n}(1-t/xy)^{-p}\frac{\Gamma(p+2n)}{\Gamma(1+2n)}\,\frac{\Gamma(1+2n)}{\Gamma^2(1+n)}(xy)^n.$$

That is, if $z = xy$, so that z is positive and < 1, $f(p, t) =$ sum of terms independent of z in

$$\frac{1}{\Gamma p}\sum \frac{\Gamma(p+2n)}{\Gamma^2(1+n)} z^n (1+z)^{-p-2n}\left(1-\frac{t}{z}\right)^{-p}$$
$$= \frac{1}{\Gamma p}\sum \frac{\Gamma(p+2n)}{\Gamma^2(1+n)}\,\frac{t^n}{\Gamma p\,\Gamma(p+2n)}$$
$$\times\left[\frac{\Gamma(p+n)\,\Gamma(p+2n)}{\Gamma(1+n)\,\Gamma 1} - \frac{\Gamma(p+n+1)\,\Gamma(p+2n+1)}{\Gamma(2+n)\,\Gamma 2}t + \dots\right],$$

or $f(p, t) = \frac{1}{\Gamma^2 p} \Sigma t^n \frac{\Gamma(p+n)\,\Gamma(p+2n)}{\Gamma^3(1+n)} F(p+n, p+2n, 1+n; -t),$ (3)

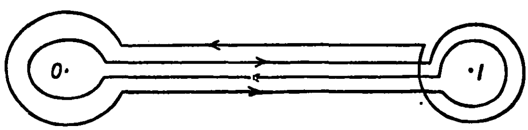
with the usual notation $F(\alpha, \beta, \gamma; x)$ for a hypergeometric series of the first kind.

The series F is such that

$$\gamma = 1 - \alpha + \beta;$$

on this depends further progress.

2. We wish to make $t = -1$, but the series F is then useless. We replace then the series by an integral, taken over the path in figure.



The hypergeometric function

$$\int_{(0,1)} x^{\beta-1}(1-x)^{\gamma-\beta-1}(1+tx)^{-\alpha}\,dx$$

taken over this path becomes, when we write

$$\gamma = 1 - \alpha + \beta,$$

$$\int_{(0,1)} x^{\beta-1}(1-x)^{-\alpha}(1+tx)^{-\alpha}\,dx,$$

and when $t = 1$ is $\int_{(0,1)} x^{\beta-1}(1-x^2)^{-\alpha}\,dx,$

or, if $x^2 = y$ and the consequent y-path be denoted by $(0^2, 1)$, is

$$\tfrac{1}{2}\int_{(0^2,1)} y^{\frac{1}{2}\beta-1}(1-y)^{-\alpha}\,dy.$$

When x passes from near 0 to near 1 so does y; when x circles 1 y circles 1; but when x circles 0 y circles 0 twice.

Hence, expressing Euler's first integral $B(\beta, \beta')$ on the one hand by

$$-(1-\exp 2\pi i\beta)(1-\exp 2\pi i\beta')\,B = \int_{(0,1)} x^{\beta-1}(1-x)^{\beta'-1}\,dx,$$

on the other hand by $\frac{\Gamma\beta\,\Gamma\beta'}{\Gamma(\beta+\beta')}$

(see Klein, *Hypergeometric Function*, p. 142; Schlesinger, *Handbuch* *der Differentialgleichungen*, Vol. III., p. 452), we have from the first form

$$\int_{(0^2,\,1)} y^{\frac{1}{2}\beta-1}(1-y)^{-a}\,dy$$

$$= -B\left(\frac{\beta}{2},\ 1-a\right)(1-\exp 2\pi i\beta)[1-\exp 2\pi i\,(1-a)]$$

and $\int_{(0,\,1)} x^{\beta-1}(1-x)^{-a}\,dx$

$$= -B\,(\beta,\ 1-a)(1-\exp 2\pi i\beta)[1-\exp 2\pi i\,(1-a)].$$

Hence $\lim\limits_{t=1}\int_{(0,\,1)} x^{\beta-1}(1-x)^{-a}(1+tx)^{-a}\,dx$

$$= \tfrac{1}{2}\,\frac{B\left(\frac{\beta}{2},\ 1-a\right)}{B\,(\beta,\ 1-a)}\int_{(0,1)} x^{\beta-1}(1-x)^{-a}\,dx;$$

and therefore

$$\lim_{t=1} F(a,\ \beta,\ 1-a+\beta;\ -1) = \tfrac{1}{2}\,\frac{B\left(\frac{\beta}{2},\ 1-a\right)}{B\,(\beta,\ 1-a)} = \tfrac{1}{2}\,\frac{\Gamma\frac{\beta}{2}\Gamma\,(1-a+\beta)}{\Gamma\beta\Gamma\left(1-a+\frac{\beta}{2}\right)}. \qquad (4)$$

Hence (3) becomes, writing $a = p+n$, $\beta = p+2n$,

$$f(p,\ 1) = \frac{1}{2\Gamma^2 p}\,\Sigma\,\frac{\Gamma\,(p+n)\,\Gamma\,(p+2n)\,\Gamma\left(\frac{p}{2}+n\right)\Gamma\,(1+n)}{\Gamma^3(1+n)\,\Gamma(p+2n)\,\Gamma\left(1-\frac{p}{2}\right)}$$

$$= \frac{1}{2\Gamma^2 p\,\Gamma\left(1-\frac{p}{2}\right)}\,\Sigma\,\frac{\Gamma\,(p+n)\,\Gamma\left(\frac{p}{2}+n\right)}{\Gamma^2\,(1+n)}.$$

But (Forsyth, *Differential Equations*, p. 197)

$$\Sigma\,\frac{\Gamma\,(p+n)\,\Gamma\,(q+n)}{\Gamma\,(1+n)\,\Gamma\,(r+n)} = \frac{\Gamma p\,\Gamma q\,\Gamma\,(r-p-q)}{\Gamma\,(r-p)\,\Gamma\,(r-q)},$$

when $r > p+q$.

Hence $f(p,\ 1) = \dfrac{1}{2\Gamma^2 p\,\Gamma\left(1-\frac{p}{2}\right)}\,\dfrac{\Gamma p\,\Gamma\left(\frac{p}{2}\right)\Gamma\left(1-\frac{3p}{2}\right)}{\Gamma\,(1-p)\,\Gamma\left(1-\frac{p}{2}\right)}.$

$$= \frac{\Gamma\left(1+\frac{p}{2}\right)\Gamma\left(1-\frac{3p}{2}\right)}{\Gamma(1+p)\,\Gamma(1-p)\,\Gamma^3\left(1-\frac{p}{2}\right)}$$

$$= \frac{p\frac{\pi}{2}}{\sin\frac{p\pi}{2}}\,\frac{\sin p\pi}{p\pi}\,\frac{\Gamma\left(1-\frac{3p}{2}\right)}{\Gamma^3\left(1-\frac{p}{2}\right)};$$

so that, finally,

$$1+\left(\frac{p}{1}\right)^3+\left\{\frac{p(p+1)}{1.2}\right\}^3+\ldots = \cos\frac{p\pi}{2}\,\frac{\Gamma\left(1-\frac{3p}{2}\right)}{\Gamma^3\left(1-\frac{p}{2}\right)}. \qquad (5)$$

This for p a negative integer $-m$ gives at once the old result that

$$1-\binom{m}{1}^3+\binom{m}{2}^3-\ldots,$$

which is evidently 0 for m odd, is

$$(-)^n\frac{(3n)!}{(n!)^3} \text{ for } m=2n.$$

For any number p real or complex, such that $|p| < \frac{2}{3}$, formula (5) is true.

3. The simplest of some series whose sums follow from (5) are worth noting. We have

$$\log\Gamma(1+x) = -\gamma x+\tfrac{1}{2}s_2x^2-\tfrac{1}{3}s_3x^3+\ldots,$$

where $|x| < 1$, γ is Euler's constant, and

$$s_m = \sum_{n=1}^{\infty}\frac{1}{n^m}.$$

Also $$\log\cos\frac{\pi}{2} = \sum_1^{\infty}\log\left\{1-\frac{p^2}{(2m-1)^2}\right\},$$

Hence, from (5), taking logarithms and expanding in powers of p,

$$\frac{s_2}{2}(3^2-3)\left(\frac{p}{2}\right)^2+\frac{s_3}{3}(3^3-3)\left(\frac{p}{2}\right)^3+\frac{s_4}{4}(3^4-3)\left(\frac{p}{2}\right)^4+\ldots$$

$$-s_2\left(\frac{1}{2^2}\right)p^2-\frac{s_4}{2}\left(1-\frac{1}{2^4}\right)p^4-\ldots$$

$$= s_3p^3+3p^4\left[\frac{1}{2^3}+(1+\tfrac{1}{2})\frac{1}{3^3}+(1+\tfrac{1}{2}+\tfrac{1}{3})\frac{1}{4^3}+\ldots\right].$$

Hence $$\frac{1}{2^3}+(1+\tfrac{1}{2})\frac{1}{3^3}+(1+\tfrac{1}{2}+\tfrac{1}{3})\frac{1}{4^3}+\dots$$

$$=\tfrac{1}{3}\frac{24}{2^5}s_4=\frac{s_4}{4}=\frac{\pi^4}{360}. \qquad (6)$$

Again, our series is the sum of coefficients of all powers of xyz in

$$\{(1-x)(1-y)(1-z)\}^{-p},$$

or in $$\exp p\,(\sigma_1+\tfrac{1}{2}\sigma_2+\tfrac{1}{3}\sigma_3+\dots),$$

where $$\sigma_n=x^n+y^n+z^n.$$

The part contributed by the coefficient of p^4 is the sum of coefficients of $(xyz)^n$ in

$$\frac{1}{4!}\left(\sigma_1+\frac{\sigma_2}{2}+\frac{\sigma_3}{3}+\dots\right)^4,$$

$$=\frac{6}{2!}\left\{\frac{1}{2.2^2}+\frac{1}{1.2.3^2}+\frac{1}{1.3.4^2}+\frac{1}{1.4.5^2}+\frac{1}{2.3.5^2}+\dots\right\}$$

$$\times\left\{1+\frac{1}{2^4}+\frac{1}{3^4}+\dots\right\},$$

$$=3s_4\left[\sum_3^\infty\frac{1}{P_r r^2}+\tfrac{1}{8}\right],$$

where P_r is the product of any two relative primes whose sum is r.

But the coefficient of p^4 in the logarithm, and hence also in the series (5), was $\frac{3s_4}{4}$. Hence

$$\sum_3^\infty\frac{1}{P_r r^2}=\tfrac{1}{8}.$$