NOTE ON THE n-DIMENSIONAL CYCLES OF AN ALGEBRAIC n-DIMENSIONAL VARIETY.

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1. It is proposed here to prove anew and for a surface with *arbitrary singularities*, a proposition given by É. Picard [1]) for the case where the singularities are ordinary, proposition which may be stated thus: — *The two dimensional cycles of a surface may be generated by linear cycles of the general curve H of a linear system $|H|$, ∞^2 at least, when this curve describes a closed cycle, in a linear pencil of the system.*

In the method followed here, we make use of the properties of the double integrals of the second kind, while with Picard, the order if anything is reversed. We obtain a transcendental definition of the invariant I of Zeuthen-Segre, which together with all results obtained is easily extended to a variety of any number of dimensions with arbitrary singularities. This transcendental definition generalizes the one of Picard, and we may add that the extensions obtained would be very laborious, if possible, with the topological methods which he followed.

2. Let $F_m(zyz) = 0$ be the equation of an algebraic surface of order m, with *arbitrary singularities*, $|H|$ the system of its plane sections, H_y an arbitrary curve of the pencil cut out by the planes parallel to the (xz) plane, $J_y = \int \frac{P(xyz)\,dx}{F'_z}$ an abelian integral belonging to H_y, where P is a polynomial in (xyz), adjoint to H_y (subadjoint to F). If $b_1, b_2, \ldots, b_\nu$ are the critical values of y, that is those for which some of the periods of J_y cease to be uniform in y, then [2]) corresponding to b_i, there exist s_i independent periods $\Omega_{i1}, \Omega_{i2}, \ldots, \Omega_{is_i}$ algebraic near b_i, and such that the difference of the values which any period may take in the neighborhood of b_i, is a sum of integral multiples of these. If P has been properly chosen, that is, if necessary multiplied by a convenient polynomial in y, the $(\Omega_i)'s$ are finite near b_i. This being

[1]) É. Picard et G. Simart, *Théorie des fonctions algébriques de deux variables indépendantes*, vol. II (Paris, Gauthier-Villars, 1906), p. 335.

[2]) S. Lefschetz, *Equation of* Picard-Fuchs *for an algebraic surface with arbitrary singularities* [Bulletin of the American Mathematical Society, vol. XXI (1914-1915), pp. 227-232]. See also Picard-Simart, loc. cit. [1]), vol. II, p. 331.

assumed, it follows at once that if

$$\Pi = \sum_{1}^{\nu} {}_i \sum_{1}^{s_i} {}_{h_i} \lambda_{ih_i} \int_b^{b_i} \Omega_{ih_i}(y)\,dy$$

and

$$\sum\sum \lambda_{ih_i} \Omega_{ih_i} = 0$$

then Π is a period of

$$J = \int\int \frac{P(xyz)\,dx\,dy}{F'_z}$$

relative to a superficial cycle of F. Suppose J to be of the second kind, then it has no residues, and the area of this cycle is finite, or may be made so. This cycle which is of the type considered in § 1, will be called *an effective cycle relatively to* H_y. We wish to prove therefore that *all cycles of F are effective cycles with respect to H_y*. The following notations will be used

$1 + R_2 =$ superficial connectivity of F

$R'_2 =$ number of independent effective cycles

$1 + r =$ linear connectivity

$\rho_0 =$ number of double integrals of the second kind not of the form $\int\int \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) dx\,dy$

$\rho =$ number of Picard

$N = \sum_{1}^{\nu} {}_i s_i$.

As in the simple case, it may be shown that [3])

$$R'_2 = N - (4p + m - 1) + 2r$$

$$\rho_0 = R'_2 - (\rho - 1).$$

We wish to prove that $R'_2 = R_2$.

3. Let $|D|$ be a linear system ∞^2 at least, the general curve of which is irreducible. There is a rational function $A(xyz)$, such that $A = h$ on D, h being a constant for all points of D and when h varies this curve describes a certain pencil $|D_h|$.

The definition of the effective cycles with respect to D_h is immediate, let R''_2 be their number. Let $\psi(xyz)$ be the denominator of $A(xyz)$, and D_∞, the curve D_h for $h = \infty$. We may consider the integrals of the type

$$J = \int\int \frac{P(xyz)\,dx\,dy}{[\psi(xyz)]^\alpha F'_z}$$

which become infinite only in the neighborhood of D_∞, so that $\frac{P}{\psi^\alpha}$ is of order $m-4$ in (xyz). If Γ is an effective cycle of F with respect to D_h, generated by the cycle

[3]) Picard-Simart, Loc. cit. [1]), vol. II, p. 373, 407.

γ of this last curve when h describes the closed path δ in its complex plane, we have

$$\int\int_{\Gamma} = \int_{\delta} dh \int_{\gamma} \frac{P\,dx}{\psi^{\alpha}[F'_2 R'_y - F'_y R'_2]}.$$

Following a method already referred to [4]), it may be shown that:

a) There are $\infty^{R''_2}$ double integrals of the second kind such that none has all its effective periods relative to D_h equal to zero.

b) Among these there are $\infty^{\rho-1}$ of the type $\int dh \int \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial h}\right) dx$ that is of the type $\int\int \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) dx\,dy$ since the two are equivalent [5]). It follows that there are

$$\rho_0 = R''_2 - (\rho - 1)$$

independent integrals of the second kind not of this form.

Hence

$$R''_2 = R'_2 = \rho_0 + \rho - 1$$

that is: — *The number of effective cycles relative to an arbitrary linear pencil contained in a system ∞^2 at least with irreducible general curve, is independent of the system considered.*

4. Suppose now that $R'_2 < R_2$. Then there exists at least one superficial cycle Γ of F, not effective relatively to H_y. I say that if the integer k is sufficiently great, there is a pencil contained in the linear system $|kH|$, having for effective cycles all those of H_y, and also Γ. These being $R'_2 + 1$ in number, we will obtain a contradiction with the proposition just proved, from which will follow that Γ is not independent from the R'_2 effective cycles of H_y, and therefore that $R'_2 = R_2$.

Let Δ be any one of the $R'_2 + 1$ cycles considered. Consider on this real surface in the four dimensional image [6]) (F) of F, a system of curves, such that only one goes through any point of it. Let γ be one of them. If the system $|kH|$ is ∞^s, we could find a curve kH such that its real image (kH) in (F) go through s arbitrary points on γ. By taking k sufficiently great, we can get a system $|kH|$, such that there is a surface (kH) having as many points in common with any assigned curve γ as we please. Then by small deformations of small cycles of γ, it will be changed into a cycle in (kH). The same deformation being carried out in a continuous manner for each γ, will change Δ into a cycle Δ', equivalent to it on (F). Let M_s be the manifold of the curves kH. The cycle Δ' is generated by a certain linear cycle γ' of (kH), when its representative point in (M_s) describes a certain closed path δ.

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[4]) Exposed in a paper in course of publication, entitled « *Multiple integrals on an algebraic variety* », § 28. The propositions stated are easily obtained by making the notion of effective period the central point in the theory of the integrals of the type considered.

[5]) PICARD-SIMART, loc. cit. [1]), vol. II, p. 161.

[6]) In what follows we will designate by (M) the $2d$ dimensional real image of the d dimensional algebraic manifold M.

It is clear that k, may be so chosen, that each of the $R'_2 + 1$ cycles of (F), such as Δ, be transformed into others equivalent to them, and generated by certain linear cycles of (kH), when its representative point in (M_s) describes certain closed paths, say $\delta_1, \delta_2, \ldots, \delta_{R'_2+1}$. The variety M_s is an s-flat, therefore it contains an infinity of rational curves, let σ be one of them, (σ) its real image in (M_s), l the dimension of the system of rational irreducible curves of the same order, to which it belongs. By properly chosing σ, l may be made arbitrarily large. Hence if we assign any number of points on the $(\delta)'s$, we may so chose σ, that (σ) will pass trough them. A small deformation performed upon small arcs of these paths, will change them into equivalent paths in (σ), and correspondingly the two-dimensional cycles (Δ'), will have become cycles (Δ''), $R'_2 + 1$ in number and equivalent to them on (F). These cycles are generated by linear cycles of (kH), when its representative point in (M_s) describes certain closed paths in (σ), that is *when kH remains in a linear pencil of curves.* To this pencil will correspond $R'_2 + 1$ effective cycles of F. The announced contradiction being established, *our theorem is proved.*

REMARKS I. The deformations of the superficial cycles being as small as we chose to make them, are always possible, since in the neighborhood of points common to any cycle Δ_i and to the singular variety of (E), the transformed cycle Δ''_i will differ by as little as we care to, from Δ_i.

II. The consideration of infinity, if troublesome, may be avoided by replacing (F), by its transformed by reciprocal vectors, with respect to any point not on it.

5. Since $R'_2 = R_2$ we have

$$\rho_0 = R_2 - (\rho - 1).$$

Hence $I = R_2 - 2r - 1$ is like R_2 and ρ, a relative invariant of F. If $\overline{F}$ is the transformed of F having only ordinary singularities, $\overline{I}$ corresponding value of I is the invariant of ZEUTHEN-SEGRE. If I' is the value of this invariant for F, $\overline{\rho}$ the number of PICARD for $\overline{F}$, we have $I' - \rho = \overline{I} - \overline{\rho} = I - \rho \therefore I = I'$ and we have obtained a transcendental definition of I, which holds for a surface with any singularities. Explicitly:

$$I = N - 4p - m = \sum_1^{\nu} s_i - 4p - m$$

where p is the genus of the plane sections. This transcendental definition can easily be worded for any system on F, even if the general curve is reducible and composed with curves of an irrational pencil, case considered by CASTELNUOVO-ENRIQUES [7]).

This definition is easily generalized to an n-dimensional variety V_n.

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[7]) G. CASTELNUOVO ed F. ENRIQUES, *Sopra alcune questioni fondamentali nella teoria delle superficie algebriche* [Annali di matematica pura ed applicata, serie III, t. VI (1901), pp. 165-225], § 6.

Let its equation be $F(x_1, \ldots, x_n, t) = 0$, and let J and J_n be the two integrals

$$J = \int \cdots \int \frac{P(x_1, x_2, \ldots, x_n, t)}{F'_t} dx_1, \ldots, dx_n$$

$$J_n = \int \cdots \int \frac{P(x_1, \ldots, x_n, t)}{F'_t} dx_1, \ldots, dx_{n-1}$$

the last being relative to the sections by the hyperplanes $x_n = C$. The periods of J_n are algebraic functions of x_n if n is odd, and there is always a certain number N of critical periods of J_n, corresponding to critical values of x_n. If $R_i + 1$, $r_i + 1$ are the i^{th} dimensional connectivity of V_n and its hyperplane sections respectively, the number

$$I = N - 2r_{n-1} - r_{n-2}$$

is a relative invariant of V_n with respect to birational transformations. It is *the invariant of* ZEUTHEN-SEGRE *generalized*. If ρ_0 is the number of proper integrals of the second kind of multiplicity n relative to V_n we have [8])

$$\rho_0 = I + 2R_{n-1} + R_{n-2} - \lambda.$$

The number λ is an invariant which, like ρ, depends upon the systems of curves in V_n and their relation to its $(n-1)$-fold integrals. The proofs do not differ from those given.

6. The geometric definition [9]) of I is easily extended to the case of a V_n with general singularities. We proceed to show this briefly for the case $n = 3$.

Let then $|S|$ and $|S_1|$ be two linear pencils of surfaces in V_3, belonging each to a more general system ∞^2 at least with irreducible characteristic curve. We suppose accordingly that the only base manifolds of the pencils are irreducible curves C, C_1 of genus p, p_1, respectively. Let also n, n_1 be the number of points common to two surfaces of $|S|$ and one of $|S_1|$, or to two surfaces of $|S_1|$ and one of $|S|$.

If D is the curve locus of the contacts of the surfaces of one pencil with those of the other, and P its genus, M, M_1 the number of the points of intersection of D with S, S_1, then S determines on D a linear series g'_{M_1}. This series has $2M_1 + 2P - 2$ double points, among which are formed the following

a) The X points where a surface of $|S|$ touches C.

b) The N points of V_3 which are nodes for some surface of $|S|$.

c) The δ points for which the intersection of the surfaces of the pencils going through them has a tacnode. They are equivalent to two contacts infinitely close.

It follows

$$2M + 2P - 2 = X + N + \delta.$$

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[8]) loc. cit. [4]), § 27.

[9]) Compare with C. SEGRE, *Intorno ad un carattere delle superficie e delle varietà superiori algebriche* [Atti della R. Accademia delle Scienze di Torino, t. XXXI (1895-1896), pp. 485-501], more particularly, pp. 497-501. The proof here is much the same as his. Having no access to his paper, the writer became aware of this only when the above was ready for printing.

Similarly

$$2M_1 + 2P - 2 = X_1 + N_1 + \delta$$

$$\therefore N + X - 2M = N_1 + X_1 - 2M_1.$$

But

$$X = 2n_1 + 2p_1 - 2$$

$$X_1 = 2n + 2p - 2.$$

Furthermore if I, I_1 are the invariants of ZEUTHEN-SEGRE for S and S_1 we have

$$n_1 + I + 4\pi = M$$

$$n + I_1 + 4\pi = M_1$$

where π is the genus of the intersection of S and S_1.

If we substitute above we obtain

$$N - 2I - 2p = N_1 - 2I_1 - 2p_1.$$

Finally if $R_2 + 1$ and $R_2^{(1)} + 1$ are the superficial connectivities of S and S_1 we have (§ 5)

$$R_2 = I + 2r + 1$$

$$R_2^{(1)} = I_1 + 2r + 1$$

where $r + 1$ is the common linear connectivity of S, S_1 and V_3. Hence

$$N - 2R_2 - 2p = N_1 - 2R_2^{(1)} - 2p_1.$$

This expression coincides with that of § 5, and we see that it is an invariant of V_3, as was to be proved.

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