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A Newtonian Fragment relating to Centripetal Forces.

By W. W. ROUSE BALL. Read May 12th, 1892.

1. The demonstrations given by Newton in his *Principia* are geometrical, though there is little doubt that in establishing the truth of some of his results he used fluxions.* To his contemporaries the language and methods of geometry were familiar, while to most of them the calculus was unknown; hence it was natural and reasonable that the proofs should be presented in a geometrical form. I presume that the fluxional analysis by which a result was obtained was generally thrown aside as soon as a synthetic geometrical proof had been found; and, as far as I know, the only proposition in that book of which Newton's fluxional demonstration has been published is his determination of the form of the solid of least resistance, of which the result alone was given in the *Principia*.†

2. Among the numerous sheets of rough work and calculation which

* *Commercium Epistolicum*, edition of 1722, p. 39; S. P. Rigaud, *Essay on the First Publication of the Principia*, Oxford, 1838, p. 24; Sir David Brewster, *Life of Newton*, second edition, Edinburgh, 1860, vol. i., p. 347.

† *Principia*, book ii. scholium to prop. 35 in the first edition, and to prop. 34 in the second and third editions.

are preserved in the Portsmouth Collection is a fragment* on the law of the centripetal force under which any orbit—and particularly a parabola of any order—can be described. The theorem to which the analysis leads is so inconvenient of application as to be practically useless, and, probably for that reason, it was not inserted in the *Principia*. Such interest as it possesses lies rather in its illustrating the way in which Newton arrived at the law given below for the description of any parabola under a central force.

The fragment consists merely of a double sheet of folio paper, containing calculations and rough drafts of notes subsequently incorporated in the second edition; but there are so many erasures and corrections that, were it not for Newton's proverbially clear handwriting, it would be difficult to read the manuscript.

3. The two inside pages contain (i.) a note on the velocity of sound from which the second paragraph added in the second edition to the scholium to book II. prop. 50 is taken, and (ii.) numerical calculations showing that the moon is retained in her orbit by the earth's attraction. The latter calculations refer explicitly to book III. prop. 4 as printed in the first edition, where the mean distance of the moon in syzygy is taken as 61 times the earth's radius; but he here says that more accurate calculations show that the mean distance of the moon is 59·9 (or 59·8) semi-diameters of the earth if the earth be at rest, and is $60\frac{1}{3}$ (or $60\frac{1}{4}$) semi-diameters if the earth and moon revolve about their common centre of gravity; in the second edition the distance is taken as being approximately equal to $60\frac{1}{2}$ radii of the earth: though in both editions he uses the number 60 as giving a sufficiently close approximation.

Of the two outside pages, the first begins with the draft of a scholium — presumably to book I. prop. 7 — which is entitled "Methodus investigandi vires quibus corpora in Orbibus propositis revolventur amplior reddi potest per Propositiones sequentes," and then follow three rules, of which the last is incomplete. Book I. prop. 7 is on the law of force under which a circle can be described about a centre of force S in its own plane. In the first edition, S is a point on the circumference, and there are no corollaries. In the second edition, S is any point in the plane; cor. 1 relates to the case when S is on the circumference; cor. 2 is concerned with the comparison of the forces to two different points about which the circle

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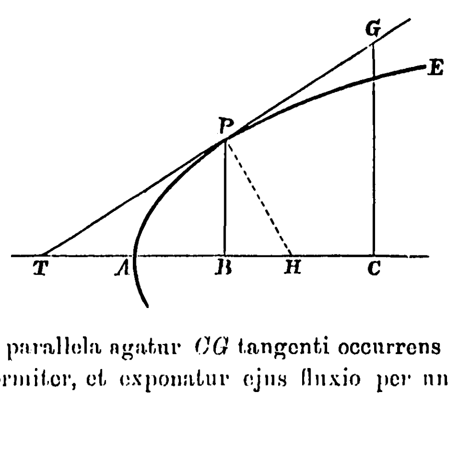
* *Portsmouth Collection*, I. VIII. 2.

might be described; and in cor. 3, the result of cor. 2 is extended to any orbit. The first rule, given here by Newton, is on similar orbits described about centres of force similarly situated. The second rule and the third (so far as it is written) seem to be the same, and are given in a form substantially identical with the third corollary, which was added to prop. 7 in the second edition. On the next page Newton refers to these rules, and says that therefore "per Prop. 10 et Cor. 3, Prop. 7" the law of force under which a central conic can be described about any point is known, because (by prop. 10) the law to the centre is known. This is the alternative proof for book I. props. 11 and 12, which was introduced in the second edition, and is printed in most modern text-books. The above rules enable us to find the law of force to any point under which a given orbit can be described, provided the law to one point is known.

4. Newton next returns to the problem (already treated in prop. 6) of the direct determination of the centripetal force to any given point under which any given orbit can be described. This is the proposition to which I referred above as illustrative of his use of fluxions. I quote the manuscript textually, merely adding punctuation. Perhaps it may be convenient to prefix the remark that the square of a quantity x is generally denoted by x^q (for *x quadratus*), and the cube is denoted by x^{cub}.

"Si Lex vis centripetae investiganda sit qua corpus P in orbe quocunque APE circa centrum quodcunque datum O movebitur: institui potest calculus per methodum sequentem.

"Sit AB Abscissa Curvae propositae, per centrum O transiens; sitque BP ejus Ordinata in dato quovis angulo Abscissae insistens. Ducatur TPG Orbem tangens in P, et Abscissae occurrens in T; et



Ordinatae BP parallela agatur CG tangenti occurrens in G. Fluat Abscissa uniformiter, et exponatur ejus fluxio per unitatem. Et si

Ordinata BP dicatur v, vis centripeta qua corpus P in Orbe APE circa centrum C movebitur erit ut

$$-\frac{CP \times \ddot{v}}{CG^{cub}}.\text{"}$$

5. Newton then applies this theorem to determine the centripetal force under which any parabola can be described. In this application, H is the foot of the normal drawn through P, and the axes are assumed to be rectangular. The manuscript continues as follows.

"Ut si aequatio ad Curvam sit $ax = v^n$ (ubi a quantitatem quamvis datum et x Abscissam denotat, et n index est dignitatis Ordinatae v). Methodus fluxionum dabit primo

$$a = n\dot{v}v^{n-1},$$

deinde,

$$0 = (nn-n)\,\dot{v}\dot{v}\,v^{n-2} + n\,\ddot{v}\,v^{n-1},$$

seu

$$\ddot{v} = -\frac{(n-1)\,\dot{v}\dot{v}}{v},$$

id est,

$$= -\frac{(n-1) \times PB}{TB^q}.$$

Et propterea vis centripeta erit ut

$$\frac{(n-1) \times PB \times CP}{TB^q \times CG^{cub}},$$

id est (ob datum $n-1$, et aequalia $TB \times CG$ et $PB \times TC$) ut

$$\frac{CP}{TB \times TC \times CG^q},$$

vel quod perinde est ut

$$\frac{CP}{PB \times TC^q \times CG}$$

$$= \frac{CP}{BH \times TC^{cub}}.\text{"}$$

6. Newton also applies the method to the curve whose equation is $ax^m + bx^n = cy^r$. He finds $\ddot{y}$, but the expression for the force thence derived is not simple or symmetrical, and he appears to have thrown the theorem aside as not being useful.

7. No demonstration of the theorem quoted above in art. 4 is given

in the manuscript; but, if the axes be rectangular, the result follows at once from book I. prop. 6. For, with the usual notation, taking (see figure, art. 4) $AB = x$, $BP = v$, and denoting OP by r, the perpendicular from O on TP by p, the angle T by ψ, and the radius of curvature at P by ρ, we have

$$F = \frac{h^2}{p^3}\frac{dp}{dr}$$

$$= \frac{h^2}{p^3}\frac{CP}{\rho}$$

$$= \frac{h^2 CP}{\rho CG^3 \cos^3\psi}.$$

Now
$$\frac{1}{\rho\cos^3\psi} = \frac{d\psi}{ds}\frac{ds}{dx}\sec^2\psi$$

$$= \frac{d}{dx}(\tan\psi)$$

$$= \frac{d}{dx}\left(\frac{dv}{dx}\right)$$

$$= \ddot{v}.$$

Hence
$$F \propto \frac{CP\ddot{v}}{CG^3}.$$

8. As first written, Newton stated that the axes must be rectangular, and it would seem that the subsequent alteration to oblique axes was an error. For, if the axes be inclined at an angle ω, an argument similar to that in the last article shows that

$$F = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{h^2}{p^3}\frac{r}{\rho} = \frac{h^2 CP}{\rho CG^3 \sin^3(\omega-\psi)},$$

and

$$\frac{1}{\rho\sin^3(\omega-\psi)} = \frac{d\psi}{ds}\left\{\frac{1}{\sin\omega}\frac{ds}{dx}\right\}\operatorname{cosec}^2(\omega-\psi) = \frac{1}{\sin\omega}\frac{d}{dx}\{\cot(\omega-\psi)\}.$$

Hence, in order that Newton's result may be true, we must have $\cot(\omega-\psi)$ equal to $\frac{dv}{dx}$, that is, to $\frac{\sin\psi}{\sin(\omega-\psi)}$; this is the case only if $\omega = \frac{1}{2}\pi$, that is, if the axes be rectangular.

9 The date of this fragment is not of much importance, but I am inclined to fix it as about the year 1694, when we know that Newton

was engaged in revising the first edition of the *Principia*. The reference to prop. 7 cor. 3 looks at first sight as if the manuscript were subsequent to the second edition; but I think that this reference relates merely to the manuscript of his proposed addition to that proposition, of which the three rules mentioned above are, I believe, a rough draft. On the other hand, the numerical calculation relating to book III. prop. 4 gives a number as occurring on page 406, line 25, and can refer only to the first edition; moreover many of the remarks alluded to above in art. 3 would be meaningless if written subsequent to the publication of the second edition in 1713. Altogether I feel no doubt that the manuscript was written before the issue of the second edition.

The Harmonic Functions for the Elliptic Cone. By E. W. Hobson.

Communicated in abstract January 14th, 1892. Received May 30th, 1892.

The harmonic functions for the circular cone were introduced by Mehler; an account of his theory is given in Heine's *Kugelfunctionen*. In the present communication, I give some indications of a theory of the more general harmonic functions which are required for the corresponding potential problems connected with the elliptic cone; I propose to call these harmonics *elliptic conal harmonics*.

It is first shown that the normal functions are of the form

$$\frac{1}{\sqrt{r}} \begin{matrix}\sin \\ \cos\end{matrix} (p \log r)\, A_p(\mu)\, B_p(\nu),$$

where r is the radius vector, and μ, ν are elliptic coordinates, the latter referring to the elliptic cones, and p is a constant; the functions $A_p(\mu)$, $B_p(\nu)$ satisfy differential equations which are the same as Lamé's, except that the degree n is no longer a positive integer, but a complex quantity $-\frac{1}{2}+pc$, so that the functions are really Lamé's functions of complex degree. I have next considered the forms of the solutions of the differential equations satisfied by