

26. The congruence (3, 3) of the *second type* has, like that of the first type, doubled congruence-pencils in the planes α and β , the centres of which are at B_3 and A_3 respectively. But, instead of having five, it has six pairs of congruence-pencils, the vertices of which are all associated principal single points of the quartic, isographic correspondence whence the congruence proceeds, and the planes of which pass, as usual, through the principal lines corresponding to those points.

On the focal surface, these points and planes are singular ones of the kind already described in Art. 23 and elsewhere, and the surface in question touches the planes α and β along the principal cubics which these planes contain. Besides touching α [β], however, along this cubic, which has a double point at A_3 [B_3] and passes through B_3 [A_3], it cuts it along the two tangents to this cubic which can be drawn from B_3 [A_3] to touch the curve elsewhere.

Correlatively, A_3 and B_3 are nodes on the focal surface, at which the cones of contact are of the third class and fourth order. Of each cone, one of the planes α and β is a double, and the other an ordinary tangent plane. The cone whose vertex is A_3 , for instance, touches α along the tangents at the double point A_3 of the principal cubic in α , and it also touches β along the tangent at A_3 to the principal cubic in β . At the same time it cuts the latter plane along the remaining two tangents that can be drawn from A_3 to the cubic just referred to. The last-mentioned generators of the cone of contact at its singular point A_3 lie, indeed, wholly on the focal surface.

On the Airy-Maxwell Solution of the Equations of Equilibrium of an Isotropic Elastic Solid, under Conservative Forces. By
W. J. IBBETSON, M.A., F.R.A.S.

[Read May 13th, 1886.]

Sir G. B. Airy was the first to propose* a very elegant method of solving the equations of stress in two dimensions, the very obvious extension of which to three dimensions is due to Clerk Maxwell.†

* *British Association Report*, Cambridge, 1862, p. 82; and *Phil. Trans.* for 1863, p. 49.

† *Edinburgh Trans.*, Vol. xxvi., p. 31.

Airy himself seems to have regarded the function upon which his solution depends as entirely arbitrary in form; and Clerk Maxwell, after pointing out the inconsistency of Airy's results with the general conditions of strain, passes very lightly over the limitations to which the method is subject, and himself gives a solution which equally fails to satisfy those conditions.

I propose, in the present paper, to start with the general equations of equilibrium in three dimensions; to deduce Maxwell's formulæ for the component stresses; and, by applying the conditions of integrability, to obtain the corresponding expressions for the components of displacement. A similar mode of treatment applied to the equations of plane stress will lead us without difficulty to the most general form of solution, applicable to Airy's case of rectangular beams under gravity. We shall see that the method is available only for a limited class of cases, and it is possible that some of those discussed by Airy are altogether beyond its scope.

If ρ denote the natural density of the body, Ψ the potential of the applied forces per unit mass, and P, Q, R, S, T, U the normal and tangential components of stress, the general equations of equilibrium

$$\text{may be written } \left. \begin{aligned} \frac{d(P+\rho\Psi)}{dx} + \frac{dU}{dy} + \frac{dT}{dz} &= 0 \\ \frac{dU}{dx} + \frac{d(Q+\rho\Psi)}{dy} + \frac{dS}{dz} &= 0 \\ \frac{dT}{dx} + \frac{dS}{dy} + \frac{d(R+\rho\Psi)}{dz} &= 0 \end{aligned} \right\} \dots\dots\dots (1),$$

and it is at once evident, on substitution, that these are satisfied by the assumptions

$$\left. \begin{aligned} P &= \frac{d^2\chi_2}{dy^2} + \frac{d^2\chi_3}{dz^2} - \rho\Psi \\ Q &= \frac{d^2\chi_1}{dz^2} + \frac{d^2\chi_3}{dx^2} - \rho\Psi \\ R &= \frac{d^2\chi_2}{dx^2} + \frac{d^2\chi_1}{dy^2} - \rho\Psi \\ S &= -\frac{d^2\chi_1}{dy\,dz} \\ T &= -\frac{d^2\chi_2}{dz\,dx} \\ U &= -\frac{d^2\chi_3}{dx\,dy} \end{aligned} \right\} \dots\dots\dots (2),$$

where χ_1, χ_2, χ_3 may *so far* be any continuous single-valued functions of position. It is, however, obvious that the *six* stress components, being linear functions of the first derivatives of the *three* independent components of displacement, must satisfy *six* independent differential equations of the second order, in order to insure the possibility of re-integration.

These differential equations are most easily obtained as follows.

Let e, f, g ; a, b, c ; $\theta_1, \theta_2, \theta_3$ be the components of dilatation, shear, and rotation, respectively; then we may show without difficulty that

$$\left. \begin{aligned} 2 \frac{d\theta_1}{dx} &= \frac{db}{dy} - \frac{dc}{dz}, & 2 \frac{d\theta_1}{dy} &= \frac{da}{dz} - 2 \frac{df}{dz}, & 2 \frac{d\theta_1}{dz} &= 2 \frac{dg}{dy} - \frac{da}{dz} \\ 2 \frac{d\theta_2}{dx} &= 2 \frac{de}{dz} - \frac{db}{dx}, & 2 \frac{d\theta_2}{dy} &= \frac{dc}{dz} - \frac{da}{dx}, & 2 \frac{d\theta_2}{dz} &= \frac{db}{dz} - 2 \frac{dg}{dx} \\ 2 \frac{d\theta_3}{dx} &= \frac{dc}{dx} - 2 \frac{de}{dy}, & 2 \frac{d\theta_3}{dy} &= 2 \frac{df}{dx} - \frac{dc}{dy}, & 2 \frac{d\theta_3}{dz} &= \frac{da}{dx} - \frac{db}{dy} \end{aligned} \right\} \dots (3).$$

On eliminating $\theta_1, \theta_2, \theta_3$ from equations (3) by cross differentiation in all possible ways, we obtain

$$\left. \begin{aligned} \frac{d^2g}{dy^2} + \frac{d^2f}{dz^2} &= \frac{d^2a}{dy \, dz} \\ \frac{d^2e}{dz^2} + \frac{d^2g}{dx^2} &= \frac{d^2b}{dz \, dx} \\ \frac{d^2f}{dx^2} + \frac{d^2e}{dy^2} &= \frac{d^2c}{dx \, dy} \\ 2 \left(\frac{d^2e}{dy \, dz} + \frac{d^2a}{dx^2} \right) &= \frac{d}{dx} \left(\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} \right) \\ 2 \left(\frac{d^2f}{dz \, dx} + \frac{d^2b}{dy^2} \right) &= \frac{d}{dy} \left(\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} \right) \\ 2 \left(\frac{d^2g}{dx \, dy} + \frac{d^2c}{dz^2} \right) &= \frac{d}{dz} \left(\frac{da}{dx} + \frac{db}{dy} + \frac{dc}{dz} \right) \end{aligned} \right\} \dots (4).$$

These are the six equations required, and they must be satisfied *identically* by every system of values that can be legitimately assumed for the component strains.

If q denotes Young's modulus, and σ the ratio of lateral contraction

to longitudinal elongation under simple tension,

$$\left. \begin{aligned} e &= P / q - \sigma (Q + R) / q \\ a &= 2 (1 + \sigma) S / q \\ &\&c. \qquad \&c. \end{aligned} \right\}.$$

Thus, on substitution from (2), the values of the strain components, expressed in terms of the χ -functions, are given by

$$\left. \begin{aligned} qe &= \Phi - (1 + \sigma) \left[\nabla^2 \chi_1 + \frac{d^2}{dx^2} (\chi_2 + \chi_3 - \chi_1) \right] \\ qf &= \Phi - (1 + \sigma) \left[\nabla^2 \chi_2 + \frac{d^2}{dy^2} (\chi_3 + \chi_1 - \chi_2) \right] \\ qg &= \Phi - (1 + \sigma) \left[\nabla^2 \chi_3 + \frac{d^2}{dz^2} (\chi_1 + \chi_2 - \chi_3) \right] \\ qa &= -2 (1 + \sigma) \frac{d^2 \chi_1}{dy \, dz} \\ qb &= -2 (1 + \sigma) \frac{d^2 \chi_2}{dz \, dx} \\ qc &= -2 (1 + \sigma) \frac{d^2 \chi_3}{dx \, dy} \end{aligned} \right\} \dots\dots\dots (5),$$

$$\text{where } \Phi = \nabla^2 (\chi_1 + \chi_2 + \chi_3) - \left(\frac{d^2 \chi_1}{dx^2} + \frac{d^2 \chi_2}{dy^2} + \frac{d^2 \chi_3}{dz^2} \right) - (1 - 2\sigma) \rho \Psi \dots (6).$$

Inserting these values in equations (4), they reduce to

$$\left. \begin{aligned} \frac{d^4}{dy^2} \left[\Phi - (1 + \sigma) \nabla^2 \chi_3 \right] + \frac{d^2}{dz^2} \left[\Phi - (1 + \sigma) \nabla^2 \chi_2 \right] &= 0 \\ \frac{d^4}{dz^2} \left[\Phi - (1 + \sigma) \nabla^2 \chi_1 \right] + \frac{d^2}{dx^2} \left[\Phi - (1 + \sigma) \nabla^2 \chi_3 \right] &= 0 \\ \frac{d^4}{dx^2} \left[\Phi - (1 + \sigma) \nabla^2 \chi_2 \right] + \frac{d^2}{dy^2} \left[\Phi - (1 + \sigma) \nabla^2 \chi_1 \right] &= 0 \\ \frac{d^2}{dy \, dz} \left[\Phi - (1 + \sigma) \nabla^2 \chi_1 \right] &= 0 \\ \frac{d^2}{dz \, dx} \left[\Phi - (1 + \sigma) \nabla^2 \chi_2 \right] &= 0 \\ \frac{d^2}{dx \, dy} \left[\Phi - (1 + \sigma) \nabla^2 \chi_3 \right] &= 0 \end{aligned} \right\} \dots\dots\dots (7),$$

thus giving six equations of the fourth order to be satisfied *identically* by χ_1, χ_2, χ_3 . If each of the first three of these be integrated twice,

we obtain the more useful forms

$$\left. \begin{aligned} \frac{d}{dy} \left\{ \int [\Phi - (1 + \sigma) \nabla^2 \chi_3] dz + \phi_1(y) \right\} \\ + \frac{d}{dz} \left\{ \int [\Phi - (1 + \sigma) \nabla^2 \chi_2] dy + \psi_1(z) \right\} &= 0 \\ \frac{d}{dz} \left\{ \int [\Phi - (1 + \sigma) \nabla^2 \chi_1] dx + \phi_2(z) \right\} \\ + \frac{d}{dx} \left\{ \int [\Phi - (1 + \sigma) \nabla^2 \chi_3] dz + \psi_2(x) \right\} &= 0 \\ \frac{d}{dx} \left\{ \int [\Phi - (1 + \sigma) \nabla^2 \chi_2] dy + \phi_3(x) \right\} \\ + \frac{d}{dy} \left\{ \int [\Phi - (1 + \sigma) \nabla^2 \chi_1] dx + \psi_3(y) \right\} &= 0 \end{aligned} \right\} \dots\dots (8),$$

where the complementary functions ϕ, ψ are wholly arbitrary.

We can now integrate the component rotations; for, on substituting from (5) in (3), we have

$$\left. \begin{aligned} \frac{d}{dx} \left[q\theta_1 + (1 + \sigma) \frac{d^2 (\chi_2 - \chi_3)}{dy dz} \right] &= 0 \\ \frac{d}{dy} \left[q\theta_1 + (1 + \sigma) \frac{d^2 (\chi_2 - \chi_3)}{dy dz} \right] &= - \frac{d}{dz} [\Phi - (1 + \sigma) \nabla^2 \chi_3] \\ \frac{d}{dz} \left[q\theta_1 + (1 + \sigma) \frac{d^2 (\chi_2 - \chi_3)}{dy dz} \right] &= + \frac{d}{dy} [\Phi - (1 + \sigma) \nabla^2 \chi_2] \end{aligned} \right\}$$

whence, by the help of (8),

$$\left. \begin{aligned} q\theta_1 + (1 + \sigma) \frac{d^2 (\chi_2 - \chi_3)}{dy dz} &= \frac{d}{dy} \left\{ \int [\Phi - (1 + \sigma) \nabla^2 \chi_3] dz + \phi_1(y) \right\} \\ &= - \frac{d}{dz} \left\{ \int [\Phi - (1 + \sigma) \nabla^2 \chi_1] dy + \psi_1(z) \right\} \\ q\theta_2 + (1 + \sigma) \frac{d^2 (\chi_3 - \chi_1)}{dz dx} &= \frac{d}{dz} \left\{ \int [\Phi - (1 + \sigma) \nabla^2 \chi_1] dx + \phi_2(z) \right\} \\ &= - \frac{d}{dx} \left\{ \int [\Phi - (1 + \sigma) \nabla^2 \chi_3] dz + \psi_2(x) \right\} \\ q\theta_3 + (1 + \sigma) \frac{d^2 (\chi_1 - \chi_2)}{dx dy} &= \frac{d}{dx} \left\{ \int [\Phi - (1 + \sigma) \nabla^2 \chi_2] dy + \phi_3(x) \right\} \\ &= - \frac{d}{dy} \left\{ \int [\Phi - (1 + \sigma) \nabla^2 \chi_1] dx + \psi_3(y) \right\} \end{aligned} \right\} \dots\dots\dots (9).$$

Now, if u, v, w be the component displacements, it is easy to show that

$$\frac{du}{dx} = e, \quad \frac{du}{dy} = \frac{1}{2}c - \theta_3, \quad \frac{du}{dz} = \frac{1}{2}b + \theta_2, \dots\dots\dots(10);$$

and therefore, by (9),

$$\left. \begin{aligned} \frac{d}{dx} \left\{ qu - \int [\Phi - (1 + \sigma) \nabla^2 \chi_1] dx + (1 + \sigma) \frac{d}{dx} (\chi_2 + \chi_3 - \chi_1) \right\} &= 0 \\ \frac{d}{dy} \left\{ qu - \int [\Phi - (1 + \sigma) \nabla^2 \chi_1] dx + (1 + \sigma) \frac{d}{dx} (\chi_2 + \chi_3 - \chi_1) \right\} &= \psi'_3(y) \\ \frac{d}{dz} \left\{ qu - \int [\Phi - (1 + \sigma) \nabla^2 \chi_1] dx + (1 + \sigma) \frac{d}{dx} (\chi_2 + \chi_3 - \chi_1) \right\} &= \phi'_2(z) \end{aligned} \right\},$$

and, finally,

$$\left. \begin{aligned} qu &= \int [\Phi - (1 + \sigma) \nabla^2 \chi_1] dx - (1 + \sigma) \frac{d}{dx} (\chi_2 + \chi_3 - \chi_1) + \psi_3(y) + \phi_2(z) \\ qv &= \int [\Phi - (1 + \sigma) \nabla^2 \chi_2] dy - (1 + \sigma) \frac{d}{dy} (\chi_2 + \chi_1 - \chi_3) + \psi_1(z) + \phi_3(x) \\ qw &= \int [\Phi - (1 + \sigma) \nabla^2 \chi_3] dz - (1 + \sigma) \frac{d}{dz} (\chi_1 + \chi_2 - \chi_3) + \psi_2(x) + \phi_1(y) \end{aligned} \right\} \dots\dots\dots(11).$$

In the case of *Plane Stress* (e.g., in planes perpendicular to Oz), we have $R = S = T = 0$, while P, Q, U , and Ψ are independent of z . On substitution of these quantities in the identical equations (4), the fourth and fifth are satisfied identically, while the others reduce to

$$\left. \begin{aligned} \frac{d^2}{dx^2} (P + Q) &= \frac{d^2}{dy^2} (P + Q) = \frac{d^2}{dx dy} (P + Q) = 0 \\ \frac{d^2 P}{dx^2} + \frac{d^2 Q}{dy^2} + 2 \frac{d^2 U}{dx dy} &= \frac{1}{1 + \sigma} \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} \right) (P + Q) = 0 \end{aligned} \right\} \dots\dots(12).$$

But, by differentiating the first of equations (1) as to x , and the second as to y , and adding the results, we obtain

$$\frac{d^2 P}{dx^2} + \frac{d^2 Q}{dy^2} + 2 \frac{d^2 U}{dx dy} = -\rho \left(\frac{d^2 \Psi}{dx^2} + \frac{d^2 \Psi}{dy^2} \right);$$

and, on substitution in the last of equations (12), we see that

$$\frac{d^2 \Psi}{dx^2} + \frac{d^2 \Psi}{dy^2} = 0 \dots\dots\dots(13).$$

It appears, then, that in all cases of Plane Stress the sum of the principal stresses must be a linear function of the coordinates, and that equilibrium is impossible unless the force potential satisfies (13).

We can now determine the form of the stress functions without difficulty. Let ∇^2 represent the operator $d^2/dx^2 + d^2/dy^2$, and let ψ be the particular integral of the equation

$$\nabla^2\psi = \Psi \dots\dots\dots(14),$$

then (12) and (1) give us

$$\left. \begin{aligned} \nabla^2 \left(P - 2\rho \frac{d^2\psi}{dy^2} \right) &= 0 \\ \nabla^2 \left(Q + 2\rho \frac{d^2\psi}{dy^2} \right) &= 0 \\ \nabla^2 \left(U + 2\rho \frac{d^2\psi}{dx dy} \right) &= 0 \end{aligned} \right\}.$$

The appropriate solution of these equations is obviously

$$\left. \begin{aligned} P &= ax + \beta y + \gamma + 2\rho \frac{d^2\psi}{dy^2} + \eta \\ Q &= ax + \beta y + \gamma - 2\rho \frac{d^2\psi}{dy^2} - \eta \\ U &= -2\rho \frac{d^2\psi}{dx dy} + \xi \end{aligned} \right\},$$

where α, β, γ are arbitrary constants, and ξ, η are any solutions of (13). Substituting these values in (1), we have

$$\left. \begin{aligned} \frac{d}{dx} (\eta + \rho\Psi + ax - \beta y) + \frac{d\xi}{dy} &= 0 \\ \frac{d\xi}{dx} - \frac{d}{dy} (\eta + \rho\Psi + ax - \beta y) &= 0 \end{aligned} \right\},$$

so that ξ and $\eta + \rho\Psi + ax - \beta y$ are conjugate functions of x and y . We may therefore write

$$\left. \begin{aligned} \xi &= -\beta x - \alpha y - \frac{d^2\zeta}{dx dy} \\ \eta &= -\rho\Psi - \frac{d^2\zeta}{dx^2} \end{aligned} \right\},$$

where ζ is any solution whatever of (13), and finally,

$$\left. \begin{aligned} P &= \alpha x + \beta y + \gamma - \rho \Psi + 2\rho \frac{d^2 \psi}{dy^2} + \frac{d^2 \zeta}{dy^2} \\ Q &= \alpha x + \beta y + \gamma - \rho \Psi + 2\rho \frac{d^2 \psi}{dx^2} + \frac{d^2 \zeta}{dx^2} \\ U &= -\beta x - \alpha y - 2\rho \frac{d^2 \psi}{dx dy} - \frac{d^2 \zeta}{dx dy} \end{aligned} \right\} \dots\dots\dots (15).$$

Comparing our result with (2), we see that it is a particular case of the former solution, in which $\chi_1 = \chi_2 = 0$, and

$$\chi_3 = \frac{1}{6} [\alpha (x^3 + 3xy^2) + \beta (3x^2y + y^3) + 3\gamma (x^2 + y^2)] + 2\rho\psi + \zeta.$$

The expressions for the displacements are most easily found by direct integration of (15), which may be written in the form

$$\left. \begin{aligned} &\frac{d}{dx} \left[qu + (1+\sigma) \frac{d}{dx} (2\rho\psi + \zeta) + (1+\sigma) \alpha y^2 \right] \\ &= \frac{d}{dy} \left[qv + (1+\sigma) \frac{d}{dy} (2\rho\psi + \zeta) + (1+\sigma) \beta x^2 \right] \\ &= (1-\sigma)(\alpha x + \beta y + \gamma) + (1+\sigma) \rho \Psi \\ &\frac{d}{dy} \left[qu + (1+\sigma) \frac{d}{dx} (2\rho\psi + \zeta) + (1+\sigma) \alpha y^2 \right] \\ &+ \frac{d}{dx} \left[qv + (1+\sigma) \frac{d}{dy} (2\rho\psi + \zeta) + (1+\sigma) \beta x^2 \right] = 0 \\ &\frac{dw}{dz} = -2\sigma (\alpha x + \beta y + \gamma) \end{aligned} \right\},$$

showing that the expressions in square brackets are conjugate functions of x and y .

Denoting them by ξ' , η' , we have

$$\frac{d\xi'}{dx} = \frac{d\eta'}{dy} = (1-\sigma)(\alpha x + \beta y + \gamma) + (1+\sigma) \rho \Psi;$$

$$\text{whence, if } \frac{d}{dx} \int \Psi dy + \frac{d}{dy} \int \Psi dx = X + Y \dots\dots\dots (16),$$

X being a function of x only, and Y of y only, by reason of (13),

$$\left. \begin{aligned} qu &= (1-\sigma) \left[\frac{1}{2} \alpha (x^2 - y^2) + \beta xy + \gamma x \right] + \sigma \alpha x^2 \\ &- (1+\sigma) \left[\alpha y^2 + \rho \left(2 \frac{d\psi}{dx} + \int Y dy - \int \Psi dx \right) + \frac{d\zeta}{dx} \right] \\ qv &= (1-\sigma) \left[\alpha xy + \frac{1}{2} \beta (y^2 - x^2) + \gamma y \right] + \sigma \beta y^2 \\ &- (1+\sigma) \left[\beta x^2 + \rho \left(2 \frac{d\psi}{dy} + \int X dx - \int \Psi dy \right) + \frac{d\zeta}{dy} \right] \\ qw &= -2\sigma [\alpha x + \beta y + \gamma] \end{aligned} \right\} \dots (17).$$

Equations (17), (16), and (15) present the complete Airian solution of the problem of Plane Stress in its most general form.

In the case of rectangular beams under gravity, if Oy be directed vertically downwards, $\Psi = g y$, and consequently $\psi = \frac{1}{12}g(3x^2y + y^3)$, $X = g\rho x$, $Y = 0$. Thus,

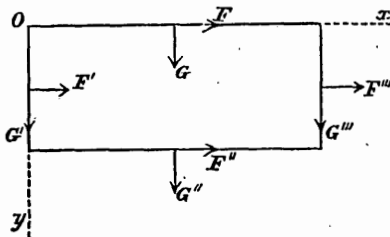
$$\left. \begin{aligned} P &= \alpha x + \beta y + \gamma + \frac{d^2\zeta}{dy^2} \\ Q &= \alpha x + \beta y + \gamma + \frac{d^2\zeta}{dx^2} \\ U &= -(\beta + g\rho)x - \alpha y - \frac{d^2\zeta}{dx dy} \end{aligned} \right\} \dots\dots\dots (18),$$

$$\text{and} \quad \left. \begin{aligned} qu &= (1-\sigma) \left[\frac{1}{2}\alpha (x^2 - y^2) + \beta xy + \gamma x \right] \\ &\quad - (1+\sigma) \left[\alpha y^2 + \frac{d\zeta}{dx} \right] + \sigma \alpha x^2 \\ qv &= (1-\sigma) \left[\alpha xy + \frac{1}{2}\beta (y^2 - x^2) + \gamma y \right] \\ &\quad - (1+\sigma) \left[(\beta + g\rho)x^2 + \frac{d\zeta}{dy} \right] + \sigma \beta x^2 \\ qw &= -2\sigma z [\alpha x + \beta y + \gamma] \end{aligned} \right\} \dots\dots\dots (19).$$

It will be observed that neither Airy's solutions, nor that given by Maxwell at the end of the paper referred to, are of the required form.

The complete solution, which requires the determination of ζ and the arbitrary constants α, β, γ , is easily obtained when the distribution of the surface stress or surface displacement is completely known. We proceed to solve the former problem in general terms.

Let the upper edge of one end of the beam be taken as axis of z , the axis of y being directed vertically downwards, and the axis of x horizontally in the direction of the length of the beam. Let L be the length, and D the vertical depth. Then we are to



suppose that, when $y = 0$ and when $y = D$, Q and U are known for all values of x between and including 0 and L ; and similarly, when $x = 0$, or L , P , and U are known, from $y = 0$ to $y = D$ inclusive. Let these surface stresses be denoted by the letters F and G , as in

Thomson and Tait's notation, and let accents distinguish the different faces of the beam. Then (see figure),

$$\begin{aligned} \text{when } y = 0, \quad U &= -F, \quad Q = -G, \quad P = 2(\alpha x + \gamma) + G, \\ \text{,, } y = D, \quad U &= F'', \quad Q = G'', \quad P = 2(\alpha x + \beta D + \gamma) - G'', \\ \text{,, } x = 0, \quad U &= -G', \quad P = -F', \quad Q = 2(\beta y + \gamma) + F', \\ \text{,, } x = L, \quad U &= G''', \quad P = F''', \quad Q = 2(\alpha L + \beta y + \gamma) - F'''. \end{aligned}$$

Now, it appears from (18) that

$$U + (\beta + g\rho)x + \alpha y \quad \text{and} \quad P - \alpha x - \beta y - \gamma$$

are conjugate functions of x and y ; so that, if U and V be conjugates,

$$P = V + (2\beta + g\rho)y + \gamma.$$

Thus we have

$$\begin{aligned} \text{when } y = 0, \quad U &= -F, \quad V = 2\alpha x + \gamma + G, \\ \text{,, } y = D, \quad U &= F'', \quad V = 2\alpha x + \gamma - g\rho D - G'', \\ \text{,, } x = 0, \quad U &= -G', \quad V = -(2\beta + g\rho)y - \gamma - F', \\ \text{,, } x = L, \quad U &= G''', \quad V = -(2\beta + g\rho)y - \gamma + F'''. \end{aligned}$$

It is known that any function of x is completely represented, for all values of x between and including the limits 0 and L , by *either* of the series

$$\begin{aligned} \phi(x) &= \phi(0) + \frac{x}{L} [\phi(L) - \phi(0)] \\ &+ \frac{2}{L} \sum_{n=1}^{\infty} \sin \frac{i\pi x}{L} \int_0^L \left\{ \phi(p) - \phi(0) - \frac{p}{L} [\phi(L) - \phi(0)] \right\} \sin \frac{i\pi p}{L} dp \\ &\dots\dots\dots(20), \\ &= \frac{1}{L} \int_0^L \phi(p) dp + \frac{2}{L} \sum_{n=1}^{\infty} \cos \frac{i\pi x}{L} \int_0^L \phi(p) \cos \frac{i\pi p}{L} dp \dots\dots\dots(21), \end{aligned}$$

both of which are perfectly determinate when the form of the function ϕ is given. We shall suppose the tangential components of the surface stresses to be expanded in the form (20), and the normal

components in the form (21), giving

$$\left. \begin{aligned} F &= -U(0, 0) + \frac{x}{L} [U(0, 0) - U(L, 0)] + \Sigma \mathcal{F}_i \sin \frac{i\pi x}{L} \\ G &= \mathcal{G}_0 + \Sigma \mathcal{G}_i \cos \frac{i\pi x}{L} \\ F' &= \mathcal{F}'_0 + \Sigma \mathcal{F}'_i \cos \frac{i\pi y}{D} \\ G' &= -U(0, 0) + \frac{y}{D} [U(0, 0) - U(0, D)] + \Sigma \mathcal{G}'_i \sin \frac{i\pi y}{D} \\ F'' &= U(0, D) + \frac{x}{L} [U(L, D) - U(0, D)] + \Sigma \mathcal{F}''_i \sin \frac{i\pi x}{L} \\ G'' &= \mathcal{G}''_0 + \Sigma \mathcal{G}''_i \cos \frac{i\pi x}{L} \\ F''' &= \mathcal{F}'''_0 + \Sigma \mathcal{F}'''_i \cos \frac{i\pi y}{D} \\ G''' &= U(L, 0) + \frac{y}{D} [U(L, D) - U(L, 0)] + \Sigma \mathcal{G}'''_i \sin \frac{i\pi y}{D} \end{aligned} \right\} \dots (22),$$

where the Old English letters denote absolute and determinate constants.

The value of the shearing stress U throughout the beam is then completely represented by the expression

$$\begin{aligned} U &= U(0, 0) + \frac{x}{L} [U(L, 0) - U(0, 0)] + \frac{y}{D} [U(0, D) - U(0, 0)] \\ &\quad + \frac{xy}{LD} [U(L, D) + U(0, 0) - U(0, D) - U(L, 0)] \\ &\quad + \Sigma \left[\mathcal{F}''_i \sinh \frac{i\pi y}{L} - \mathcal{F}_i \sinh \frac{i\pi (D-y)}{L} \right] \sin \frac{i\pi x}{L} / \sinh \frac{i\pi D}{L} \\ &\quad + \Sigma \left[\mathcal{G}'''_i \sinh \frac{i\pi x}{D} - \mathcal{G}'_i \sinh \frac{i\pi (L-x)}{D} \right] \sin \frac{i\pi y}{D} / \sinh \frac{i\pi L}{D} \\ &\quad \dots \dots \dots (23), \end{aligned}$$

for this satisfies $d^2U/dx^2 + d^2U/dy^2 = 0$, and has the required values over all the boundaries. From (23) we easily deduce

$$\begin{aligned} V &= C + \frac{y}{L} [U(L, 0) - U(0, 0)] - \frac{x}{D} [U(0, D) - U(0, 0)] \\ &\quad + \frac{y^2 - x^2}{2LD} [U(L, D) + U(0, 0) - U(0, D) - U(L, 0)] \end{aligned}$$

$$\begin{aligned}
& + \mathfrak{F}_i'' \cosh \frac{i\pi y}{L} + \mathfrak{F}_i \cosh \frac{i\pi (D-y)}{L} \Big] \cos \frac{i\pi x}{L} \Big/ \sinh \frac{i\pi D}{L} \\
& - \mathfrak{G}_i''' \cosh \frac{i\pi x}{D} + \mathfrak{G}_i' \cosh \frac{i\pi (L-x)}{D} \Big] \cos \frac{i\pi y}{D} \Big/ \sinh \frac{i\pi L}{D} \\
& \dots\dots\dots(24),
\end{aligned}$$

where C is an arbitrary constant. We will postpone for the present the investigation of the relations that must exist between the components of the surface stresses, in order that this form of V may assume the required values at the boundaries; and proceed at once to determine ζ from the equations

$$\left. \begin{aligned} \frac{d^2 \zeta}{dx dy} &= -U - (\beta + g\rho) x - \alpha y \\ \frac{d^2 \zeta}{dx^2} &= -\frac{d^2 \zeta}{dy^2} = -V - (\beta + g\rho) y + \alpha x \end{aligned} \right\}.$$

Substituting the values of U and V , and integrating,

$$\begin{aligned}
\frac{d\zeta}{dx} &= A - Cx - yU(0, 0) + \frac{x^2 - y^2}{2D} [\alpha D + U(0, D) - U(0, 0)] \\
& - \frac{xy}{L} [(\beta + g\rho) L + U(L, 0) - U(0, 0)] \\
& + \frac{x^3 - 3xy^2}{6LD} [U(L, D) + U(0, 0) - U(0, D) - U(L, 0)] \\
& - \frac{L}{\pi} \mathfrak{F}_i' \cosh \frac{i\pi y}{L} + \mathfrak{F}_i \cosh \frac{i\pi (D-y)}{L} \Big] \sin \frac{i\pi x}{L} \Big/ \sinh \frac{i\pi D}{L} \\
& + \frac{D}{\pi} \mathfrak{G}_i' \sinh \frac{i\pi x}{D} - \mathfrak{G}_i \sinh \frac{i\pi (L-x)}{D} \Big] \cos \frac{i\pi y}{D} \Big/ \sinh \frac{i\pi L}{D},
\end{aligned}$$

$$\begin{aligned}
\frac{d\zeta}{dy} &= B + Cy - xU(0, 0) - \frac{xy}{D} [\alpha D + U(0, D) - U(0, 0)] \\
& + \frac{y^3 - x^3}{2L} [(\beta + g\rho) L + U(L, 0) - U(0, 0)] \\
& + \frac{y^3 - 3x^2y}{6LD} [U(L, D) + U(0, 0) - U(0, D) - U(L, 0)] \\
& + \frac{L}{\pi} \mathfrak{F}_i'' \sinh \frac{i\pi y}{L} - \mathfrak{F}_i' \sinh \frac{i\pi (D-y)}{L} \Big] \cos \frac{i\pi x}{L} \Big/ \sinh \frac{i\pi D}{L} \\
& - \frac{D}{\pi} \mathfrak{G}_i''' \cosh \frac{i\pi x}{D} + \mathfrak{G}_i' \cosh \frac{i\pi (L-x)}{D} \Big] \sin \frac{i\pi y}{D} \Big/ \sinh \frac{i\pi L}{D},
\end{aligned}$$

where A and B are arbitrary constants which may be so determined as to fix any required point of the beam.

The second integration gives

$$\begin{aligned} \zeta = & Ax + By + \frac{1}{2}C(y^2 - x^2) - xyU(0, 0) \\ & + \frac{x^3 - 3xy^2}{6D} [\alpha D + U(0, D) - U(0, 0)] \\ & + \frac{y^3 - 3x^2y}{6L} [(\beta + g\rho)L + U(L, 0) - U(0, 0)] \\ & + \frac{x^4 - 6x^2y^2 + y^4}{24LD} [U(L, D) - U(L, 0) - U(0, D) + U(0, 0)] \\ & + \frac{L^3}{\pi^2} \sum_{i=1}^{\infty} \frac{1}{i^2} \left[\mathcal{F}_i'' \cosh \frac{i\pi y}{L} + \mathcal{F}_i \cosh \frac{i\pi(D-y)}{L} \right] \cos \frac{i\pi x}{L} \bigg/ \sinh \frac{i\pi D}{L} \\ & + \frac{D^3}{\pi^2} \sum_{i=1}^{\infty} \frac{1}{i^2} \left[\mathcal{G}_i'' \cosh \frac{i\pi x}{D} + \mathcal{G}_i \cosh \frac{i\pi(L-x)}{D} \right] \cos \frac{i\pi y}{D} \bigg/ \sinh \frac{i\pi L}{D} \\ & \dots\dots\dots(25). \end{aligned}$$

It only remains to determine the four constants α, β, γ, C . This is easily done by substituting in the equation $P + Q = 2\alpha x + 2\beta y + 2\gamma$ the coordinates $(0, 0), (L, 0), (0, D)$ successively, and in the equation $P - V = (2\beta + g\rho)y + \gamma$ the coordinates $(0, 0)$. We thus find

$$\left. \begin{aligned} \alpha &= \frac{1}{2L} [\mathcal{f}_0'' + \mathcal{f}_0' + \Sigma (\mathcal{f}_i'' + \mathcal{f}_i' + 2\mathcal{g}_{2i-1})] \\ \beta &= \frac{1}{2D} [\mathcal{g}_0'' + \mathcal{g}_0' + \Sigma (\mathcal{g}_i'' + \mathcal{g}_i' + 2\mathcal{f}_{2i-1})] \\ \gamma &= -\frac{1}{2} [\mathcal{g}_0 + \mathcal{f}_0' + \Sigma (\mathcal{g}_i + \mathcal{f}_i')] \\ C &= \frac{1}{2} \left\{ \mathcal{g}_0 - \mathcal{f}_0' + \Sigma \left[\mathcal{g}_i - \mathcal{f}_i' - 2 \left(\mathcal{F}_i'' + \mathcal{F}_i \cosh \frac{i\pi D}{L} \right) \bigg/ \sinh \frac{i\pi D}{L} \right. \right. \\ & \quad \left. \left. + 2 \left(\mathcal{G}_i'' + \mathcal{G}_i \cosh \frac{i\pi L}{D} \right) \bigg/ \sinh \frac{i\pi L}{D} \right] \right\} \end{aligned} \right\} \dots(26).$$

Equations (25) and (26) present the complete solution for all cases of equilibrium in which the stress is entirely plane, and in magnitude independent of z . In order that this may be the case, certain identical relations must exist between the components of the surface stresses. These relations will of course be identical with the conditions that the form (24) of V , after substitution of the value of C given by (26), may give to the normal

stress components their assigned values over the bounding surfaces: e.g., when $y = 0$,

$$\begin{aligned} C - \gamma - g_0 - \frac{x}{D} [2aD + U(0, D) - U(0, 0)] \\ - \frac{x^3}{2LD} [U(L, D) + U(0, 0) - U(0, D) - U(L, 0)] \\ - \Sigma \left[\mathfrak{G}_i''' \cosh \frac{i\pi x}{D} + \mathfrak{G}_i' \cosh \frac{i\pi (L-x)}{D} \right] / \sinh \frac{i\pi L}{D} \\ = \Sigma \left\{ g_i - \left[\mathfrak{F}_i'' + \mathfrak{F}_i \cosh \frac{i\pi D}{L} \right] / \sinh \frac{i\pi D}{L} \right\} \cos \frac{i\pi x}{L}. \end{aligned}$$

Throwing the other three equations into a similar form, expanding the functions on the left-hand side by formula (21), and equating constant terms and coefficients of $\cos i\pi x/L$ or $\cos i\pi y/D$ in each, we have twelve equations of condition to be satisfied identically by the 24 coefficients (of orders 0, $2i$, and $2i-1$) in the expressions (22) for the surface tractions. The fact that the coefficients of shearing-stress appear in these equations *in series* makes the inverse problem—of determining distributions of surface traction competent to maintain a plane stress uniformly distributed in parallel planes—a difficult one, and I have not yet succeeded in obtaining consistent solutions for the cases of a beam supported by one end only, or symmetrically by both ends, with its upper and under surfaces free from stress, or uniformly loaded. I am inclined to believe that the stress in these cases is not of the kind that we have discussed, but the question requires further consideration.*

* I am glad to express my indebtedness to Professor Greenhill for the reference to Clerk Maxwell's previous work in this direction, and to Professor J. J. Thomson for suggestions in connection with additions that have been made to this paper since it was read before the Society.